BV differentials and Derived Lagrangian intersections on moduli Spaces of Surfaces in Fano and CY threefolds

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Motivation : S - duality Conjecture.

Assumptions

- X : Smooth, proj, CY

- $\mathrm{Pic}(X)$ generated by an ample divisor L

- For fixed $k \in \mathbb{Z}_{>0}$ ⇝ let $H \in |kL|$

let $l :=$ generator of $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$

let $i, n \in \mathbb{Z}$ fixed

let $ch^{*}(i,n) := (0, H, H^2/2 - il, \chi(\mathcal{O}_H) - H \cdot td_2(X) - n)$

fixed Chern character

$M(X; ch^*) := \{$moduli space of Gieseker-semistable sheaves $\mathcal{F} \in Coh(X) \mid ch(\mathcal{F}) = ch^* \}$

when Stability = semistability

$M(X; ch^*)$ has Perfect obstruction theory

$\overset{\shortparallel}{M}$

$$\mathbb{E}^\bullet_M \xrightarrow{\phi} \mathbb{L}^\bullet_M$$

$h^0(\phi)$ isom

$h^{-1}(\phi)$ epimor

$h^i(\phi) = 0 \quad \forall\, i \neq -1, 0$

$$rk(E^\bullet)\big|_{p \in M} = \dim(h^0(E^\bullet)) - \dim(h^{-1}(E^\bullet)) = \text{vdim}\, M$$

$$= \dim(TM|_p) - \dim(Ob_M|_p)$$

$$= \dim(Ext^1(\mathcal{F},\mathcal{F})) - \dim(Ext^2(\mathcal{F},\mathcal{F})) = 0$$

Serre duality.

$$E^{\bullet}_{M} \xrightarrow[\text{Behrend-Fantechi}]{\text{induces}} [M]^{vir} \in A_{0}(M)$$

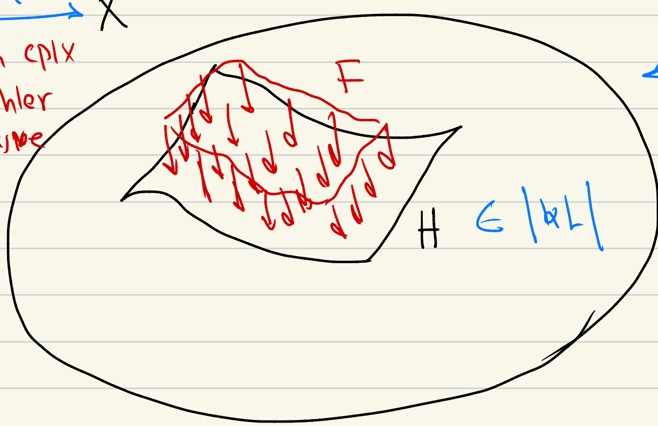
vdim(M)

$$DT(X; ch^{*}) = \int_{[M]^{vir}} 1 \in \mathbb{Z}$$

It is an intersection number

Counts deformation invariant systems

$X' \xleftrightarrow[\text{both cplx and kähler structure}]{\text{infinitesimal def}^n} X$



Rough Picture:

$\mathcal{E}$ → Suitable bundle.

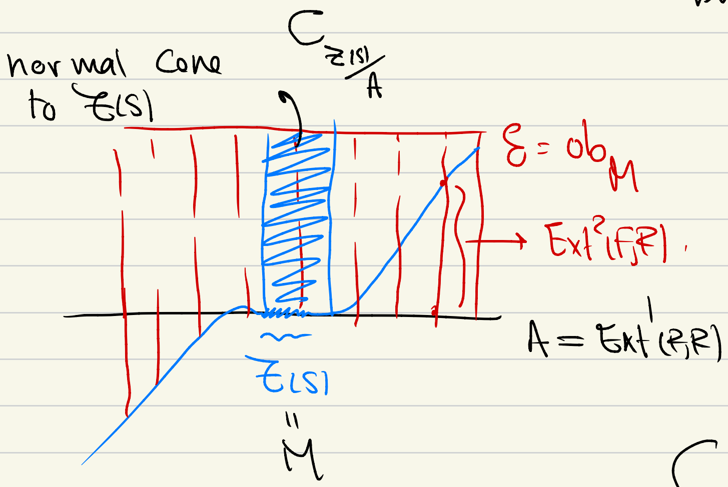
$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ Z(s) = M \arrow[r, hook] & A \arrow[u, bend right, "s"'] \end{tikzcd}$$

Smooth.

ugly; nonsmooth
multi component
with jumping dims

Local Picture

locally $A = \mathrm{Ext}^1(F,F)$ tangent bundle



$$DT = C_{Z(s) \subset A} \cap 0_{Ob_M} = \int_{[M]^{vir}} 1.$$

$Z(s) = M$

If stability $\neq$ semistability

$\Longrightarrow$ Joyce–Song defined generalized DT

invt.; $\overline{DT}(X; ch^*) \in \mathbb{Q}$

Generating function of $\overline{DT}(X, ch^*)$

$$Z_i^H(q) = \sum_{n \in \mathbb{Z}} \overline{DT}(X; ch^*(i,n))\, q^n$$

Remark

tensoring by $\mathcal{O}(\pm L)$ induces isom

on $M(X, ch^*(i,n)) \Longrightarrow Z_i^H(q)$ and $Z_{i+kL}^H(q)$

only differ by a shift in power of q.

S-duality Conjecture ; Gaiotto - Strominger - Yin

Gaiotto - Yin

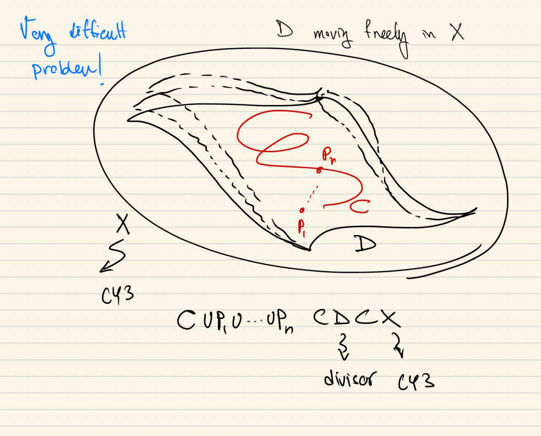
The vector of generating Series $\sum(\dots)$ $\sum(\dots)$

$$\left(q^{a_i} Z_i^{H}(q)\right)_{i=0}^{kL^3-1} = \left(q^{a_0} Z_0^{H}(q), q^{a_1} Z_1^{H}(q), \dots\right)$$

is a holomorphic vector valued modular form

of weight $\frac{-3}{2}$!!!

$$a_i = \frac{(2i+H^3)^2}{8H^3} - \frac{H^3}{8} - \frac{\chi(H)}{24} \in \mathbb{Q}$$



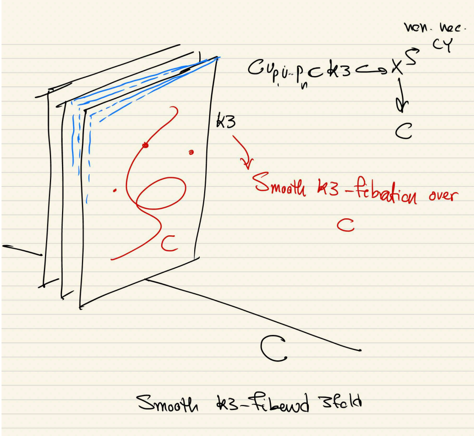
Counting curves on surfaces in Calabi-Yau threefolds, (with Amin Gholampour and Richard P. Thomas), Mathematische Annalen, Volume 360, Issue 1-2, pp 67-78 (2014), arXiv:1309.0051.

Obtain modular partition function? Almost!

$$Z = \sum_{\beta, n} = \blacksquare + \blacksquare + \square + \square + \square + \cdots \blacksquare + \square + \blacksquare + \cdots$$

To prove modularity we need:

1. Generalize from rank 1, ideal sheaf counting to rank one general sheaf counting
2. Compute higher rank sheaf counting from rank 1 sheaf ocunting (Feyzbakhsh, Thomas)



Donaldson-Thomas Invariants of 2-Dimensional sheaves inside threefolds and modular forms, (with Amin Gholampour), Advances in Mathematics, Vol. 326, No. 21, p. 79-107 arXiv:1309.0050.

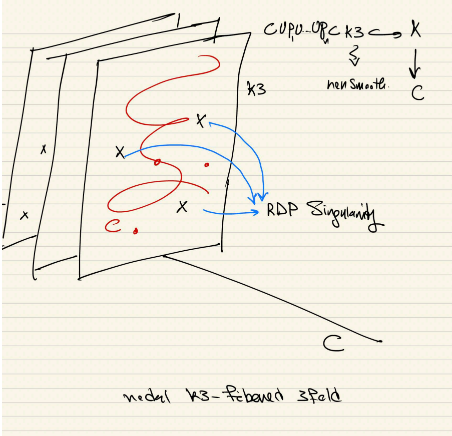
$$= (\text{Göettche invariants} \to \textbf{Modular}) \cdot (\text{Noether-Lefschetz numbers} \to \textbf{Modular};\ \text{Borcherds})$$

$$Z(X, q) = \frac{\Phi^{\widetilde{\pi}}(q) - kv_0}{2\eta(q)^{24}},$$

where

$$\Phi^{\widetilde{\pi}}(q) = \sum_{d=0}^{\ell-1} \Phi_d^{\widetilde{\pi}}(q) v_d \in \mathbb{C}[[q^{1/2\ell}]] \otimes \mathbb{C}[\mathbb{Z}/4\ell\mathbb{Z}]$$

$$\Phi_d^{\widetilde{\pi}}(q) = q^{1+d^2/2\ell} \sum_{h \in \mathbb{Z}} NL_{h,d}^{\widetilde{\pi}} q^{-h},$$



Stable pairs on nodal K3 brations, (with Amin Gholampour and Yukinu Toda), International Mathematical Research Notices, Vol. 2017, No. 00, pp. 1-50, arXiv:1308.4722.

$$\sum_{h=0}^{\infty} \sum_{n=1-h}^{\infty} (-1)^{n+2h-1} \chi(P_n(S,h)) y^n q^h =$$

$$-\left(\sqrt{-y} - \frac{1}{\sqrt{-y}}\right)^{-2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2}.$$

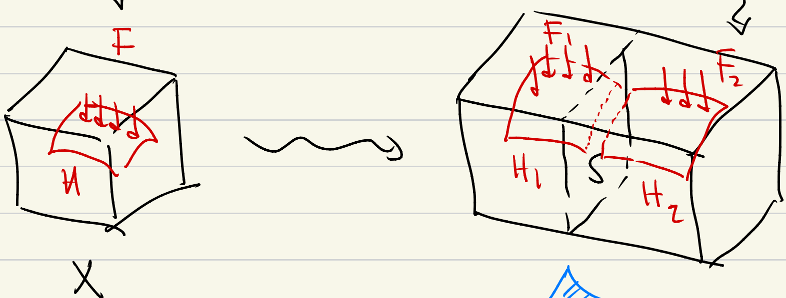
How to Compute $DT(X; ch^*)$?

$\longrightarrow$ Degenerate $X \rightsquigarrow Y_1 \cup_S Y_2$

Normal Crossing

$ch(F) = ch^*$ $\qquad$ $ch(F_1) = ch_1^*$ $\qquad$ $ch(F_2) = ch_2^*$

Dream

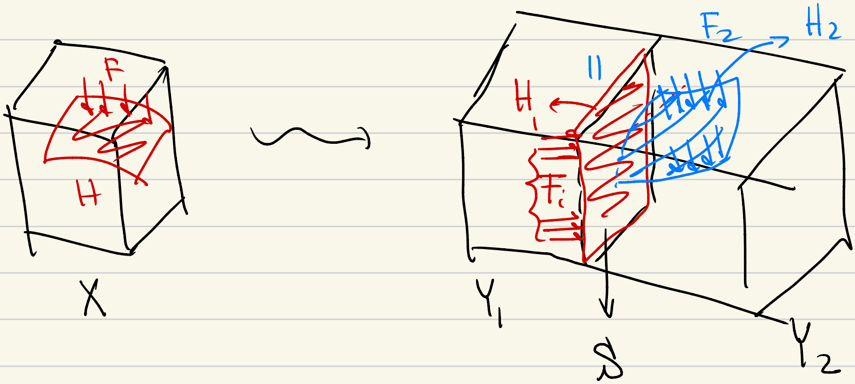


Require F_1 and F_2 to meet S transversely and glue.

$$F_i \perp S \overset{\text{Homological}}{\underset{\text{Constraint}}{\Longleftrightarrow}} \mathrm{Tor}_1^{O_{Y_i}}(F_i, O_S) = 0$$

Reality $\mathrm{Tor}_1^{\mathcal{O}_{Y_i}}(F_i, \mathcal{O}_S)=0$ is an open

Condition in $M(F_i^{\perp}, ch_i^*) \overset{\circ}{\subset} M(F_i, ch_i^*)$



Jun Li / Baosen Wu

Need to come up with Compactifications

$$\overline{M(F_1^{\perp}, ch_1^*)} \quad \text{and} \quad \overline{M(F_2^{\perp}, ch_2^*)}$$

These must parameterize

transverse F_1 & F_2 in the

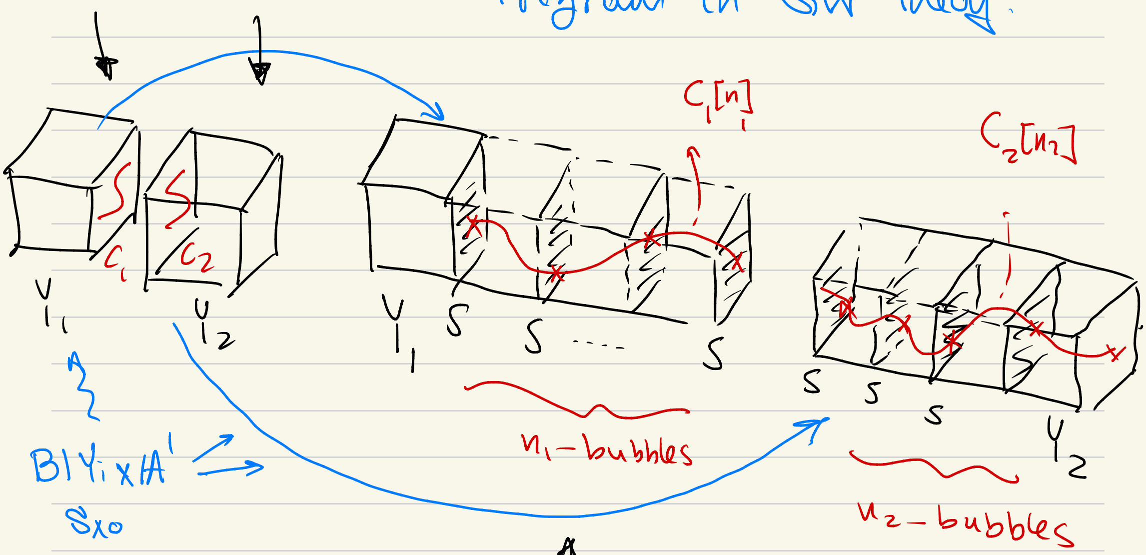
limit.

Li–Wu if F_i are ideal sheaves
or Pandharipande–Thomas
Stable pairs. (PT)

then expanded degenerations give Compactification

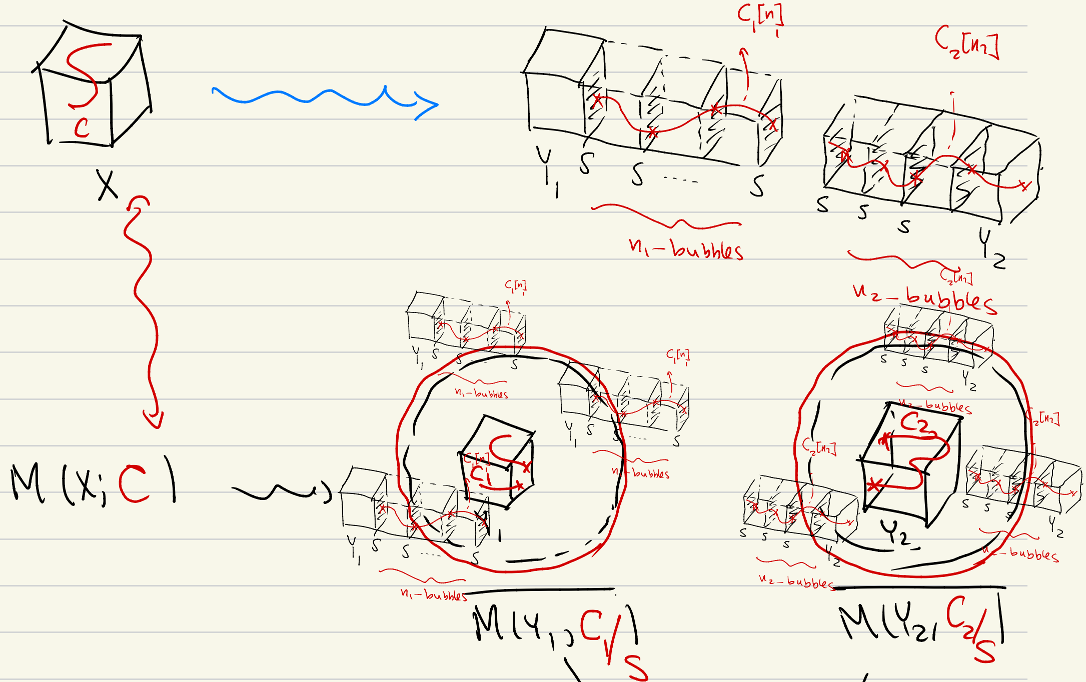
Similar to Jun Li's degeneration Program in GW theory.

Bad Scenario



Recall from GW Theory

in the limit!



$$
\begin{tikzcd}
M(Y_1, C_1/S) \arrow[dr, "r_1"] & & M(Y_2, C_2/S) \arrow[dl, "r_2"'] \\
\mathrm{Hilb}^n(S) \arrow[r, "\Delta"] & \mathrm{Hilb}^n(S) \times \mathrm{Hilb}^n(S) &
\end{tikzcd}
$$

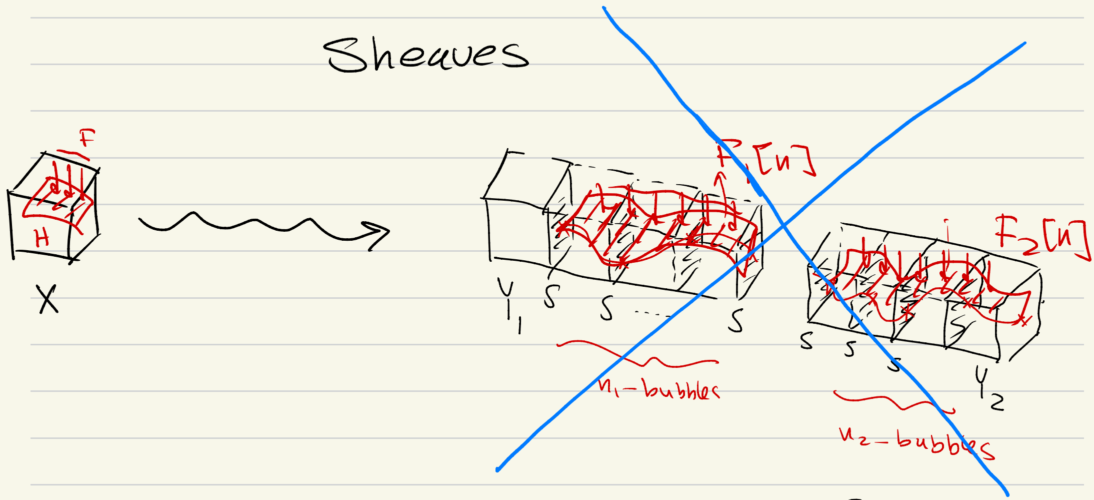
$$\Downarrow$$

$$[M(X;C)]^{vir} \overset{\text{def'n}}{\underset{\text{invariance}}{=}} \sum_{C = C_1 \cup C_2} \Delta^! \left([M(Y_1, C_1/S)]^{vir} \times [M(Y_2, C_2/S)]^{vir} \right)$$

Works for ideal Sheaves and PT pairs

Not for general Coherent

Sheaves



Due to Gieseker stability

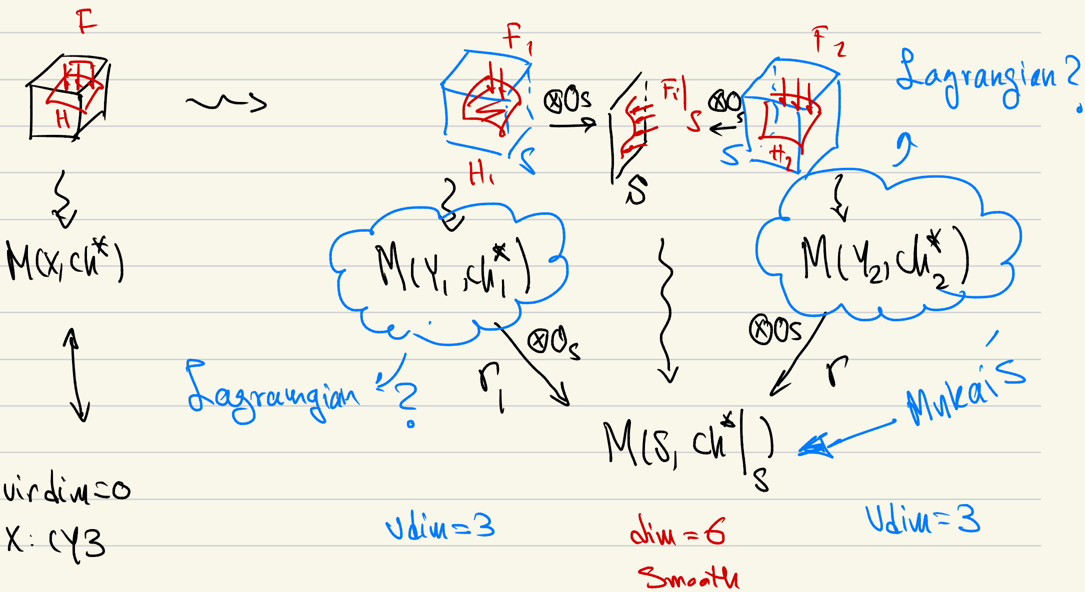
issues.

$F_1[n_1]$ or $F_2[n_2]$

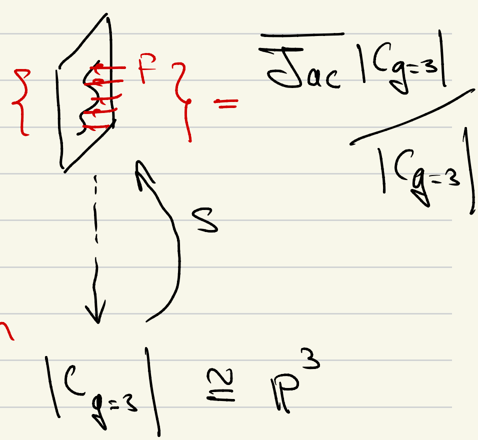
can be destabilized

Remedy use derived intersection theory.

and compute categorification of DT invts



Assuming H_i are hyperplane sections



Lagrangian fibration

Theorem (Baranovsky, Katzarkov, Kontsevich, S–)

Let $r_i : M(Y_i, ch^*_i) \longrightarrow M(S, ch^*|_S)$ denote the derived restriction morphism; given at the level of points by $F_i \longrightarrow F_i \otimes^L \mathcal{O}_S \quad \forall F_i \in M(Y_i, ch^*_i)$

Then r_i satisfies the condition of inducing a Lagrangian structure

$$\boxed{\Theta_{r_i} = \Pi^\circ_{r_i} \xrightarrow{\cong} \mathbb{L}^\bullet_{M(Y_i, ch^*_i)}[-1]}$$

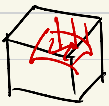
is a quasi-isomorphism of perfect complexes

Lemma. The induced (−1)-shifted symplectic form on derived fiber product $M(Y_1, ch^*_1) \times^{DR}_{M(S)} M(Y_2, ch^*_2)$ agrees with the canonical structure on derived intersection of two Lagrangians $\mathcal{O}^\bullet_{M(Y_1)} \otimes^{\mathcal{L}}_{\mathcal{O}_{M(S)}} \mathcal{O}^\bullet_{M(Y_2)}$

Categorified DT invts from derived Lagrangian

intersection.

Need to show shifted symplectic structures are invt in degenerating family.

$\{$  $\}$

X

$\shortparallel$

$M(X, ch^*)$

$\shortparallel$

$d\,\mathrm{Crit}(f)^{t\neq 0}_{Dr}$

$Y_1 \qquad Y_2$

$M(Y_1, ch_1^*) \underset{M(S)}{\overset{DR}{\times}} M(Y_2, ch_2^\vee)$

$\shortparallel$

$d\,\mathrm{Crit}(f)^{t=0}_{Dr}$

deformation

invariance

of derived structure

$t\neq 0$ $\qquad$ $t=0$ $\qquad$ $\mathbb{A}^1$

Let $\mathbb{P} = \mathrm{tot}\left(X \rightsquigarrow Y_1 \cup_S Y_2\right)$ Fano 4fold.

Need to Show $\rightsquigarrow$ All derived structure is induced from ambient space!!!

Theorem (Baranovski, Katzarkov, Kontsevich, S–)

Define Koszul algebras

$$A^{\bullet}_{\mathbb{P}} = \left(\mathrm{Sym}^{\bullet}(U[1] \oplus W_2^{\vee}[1]) \otimes \mathrm{Sym}^{\bullet}(U^{\vee} \oplus W_1^{\vee}), d_{\check{\ell}_{\mathbb{P}}}\right)$$

$$A^{\bullet}_{X} = \left(\Lambda^{\bullet}(U[1]) \otimes \mathrm{Sym}^{\bullet}(U^{\vee}), d_{\check{\ell}_X}\right)$$

where over points F in moduli space

$$U = \mathrm{Ext}^1(F,F) \qquad W_1 = \mathrm{Ext}^0_X(F, F \otimes K^{\vee}_{\mathbb{P}}) \qquad W_2 = \mathrm{Ext}^1_X(F, F \otimes K^{\vee}_{\mathbb{P}})$$

$$U^{\vee} = \mathrm{Ext}^2(F,F) \qquad W_1^{\vee} = \mathrm{Ext}^3_X(F \otimes K^{\vee}_{\mathbb{P}}, F) \qquad W_2^{\vee} = \mathrm{Ext}^2_X(F \otimes K^{\vee}_{\mathbb{P}}, F)$$

these see ambient $\mathbb{P}$.

① The dg-algebra A°_X is isomorphic to the Koszul algebra of df (i.e. the dg algebra of functions on the derived critical locus of f)

over fibers of $\mathbb{P}$.

$$f := \sum_{k \geq 2} \frac{1}{(k+1)!} f_{k+1} \in \mathrm{Sym}^{*}(\check{U})$$

over $\mathbb{P}$

② Let $g : U \otimes W_1 \otimes W_2^{\vee} \to \mathbb{C}$ a function linear in last two arguments then the products

$$\mathrm{Sym}^k(U) \to U^{\vee} \text{ and } \mathrm{Sym}^{k-1}(U) \otimes W_1 \to W_2 \quad k \geq 2 \text{ can be}$$

chosen to describe the canonical L-∞ structure on

$U^{\circ}_{\mathbb{P}} = \mathrm{Ext}^{\bullet}_{\mathbb{P}}(F,F)$ therefore existence of g implies existence of f.

★★

③ The dg algebra A°_X is quasi-isom to $d\mathrm{Crit}(f+g)$ on $U + W_2^{\vee} + W_1$

Theorem (Baranovsky, Katzarkov, Kontsevich, S-)

① Let $s \in H^0(\mathbb{P}, K_{\mathbb{P}}^{\vee})$. The periodic cyclic homology $HP(s)$

of the matrix factorization category $D(s)$ is isomorphic

to cohomology (in Zariski topology)

$$H\left(W, \Omega^{\bullet}_{X}(u)\right), -d(f(s)+g(s)) + u\, d_{DR}$$

these are categorified DT invts (Gunningham, Safronov)

② For a good degeneration $X \rightsquigarrow Y_1 \cup_S Y_2$ there is

a flat connection on vector bundle over $\mathbb{A}^1$ with the

fiber $HP(s(t))$ $t \in \mathbb{A}^1$ (this implies def'n invariance

of categorified DT)

$$H^{\bullet}_{ap}(\mathbb{P}Y, B^{\bullet}_{ir}(\Pi^{\bullet}_{h})) \overset{Pay}{\cong} Y^{\bullet}_{au}$$