Invariants of Knots and 3-manifolds, Representation Theory and Quantum Field Theory.

The goal of this course is an introduction to theory of invariants of knots and to closely related subjects such as representation theory of quantum groups and topological quantum field theory. We will start with a brief reminder of what are braids, knots, tangled graphs and 3-manifolds. Then invariants of knots will be constructed out of solutions to the Yang-Baxter equation. Examples of such construction are Jones, HOMFLY and Kauffman polynomials. After this a functorial description of these invariants will be given following the work of Turaev and myself. For this we will make a detour into representation theory of Hopf algebras. After this we will discuss how these invariants of knots are related to quantum field theory.

Tentative schedule by weeks:

(1) Braids, knots, links, tangled graphs, a framing. Knots as closures of braids, Markov moves. Diagram of a knot, Redemeister moves. Later, if time permit, we will discuss knots and tangled graphs in an arbitrary 3-manifold.
(2) Representations of braid groups. Yang-Baxter equation. Invariants of knots and traces of braid group representations. Examples of R-matrices (corresponding to vector representation of $U_q(sl_n)$).
(3) State sum invariants of knots and links. Examples: HOMFLY and Jones polynomials as state sums. Skein relations, skein definition of invariants, now q^n can be any number.
(4) (2-3 weeks) Categories: Abelian, monoidal, braided monoidal. Examples: 1) the category of framed tangles. 2) the catgeory of vector spaces. 3) categories of modules over algebras, bialgebras, Hopf algebras, quasitriangular Hopf algebras. Properties of quasitriangular Hopf algebras.
(5) Invariants of framed tangles as a functor from the category of colored framed tangles of the category of modules over a quasitriangular Hopf algebra.
(6) $U_q(sl_2)$ example.
(7) Functorial framework of quantum field theory: a functor from the category of cobordisms to the category of vector spaces.
(8) The idea of a path integral. Why it defines a functor. How define a path integral as a mathematical object? 1) Analytical approach: probabilistic (Wiener integral), analytical (constructive field theory), perturbative (define it as a formal power series, mimicking the computation of an integral by a steepest descent method).
(9) Chern-Simons classical field theory. (closed manifolds)
(10) Its path integral quantization. Invariants of 3-manifolds with links in them. Perturbation theory.
(11) The semiclassical hypothesis. The importance of the theory for space times with boundary.
(12) What did not fit into this course.

This schedule will most likely change, depending on how fast the material will go. Most likely it will be slower.

Prerequisits Basic familiarity with manifolds, topology, and algebra will be useful.

Some useful references are below. This list is is far from being complete but it gives an idea about the subject. I will give more references during the class.

References

[1] D. Freed, Lectures on Topological Quantum Field Theory, https://web.ma.utexas.edu/users/dafr/OldTQFTLectures.pdf.

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[4] A. Kirillov, B. Bakalov, Lectures on tensor categories and modular functor, AMS, 2001, ISBN-13: 978-0821826867 ISBN-10: 0821826867.

[5] T. Kohno, "Conformal Field Theory and Topology", Translations of Mathematical Monographs, Iwanami Series in Modern Mathematics, AMD 2002.

[6] T. Ohtsuki, "Problems on invariants of knots and 3-manifolds". https://arxiv.org/pdf/math/0406190.pdf.

[7] N. Reshetikhin, "Quasitriangular Hopf algebras and invariants of links". (Russian) Algebra i Analiz 1 (1989), no. 2, 169–188; translation in Leningrad Math. J. 1 (1990), no. 2, 491-513.

[8] N. Reshetikhin, V. Turaev, "Ribbon graphs and their invariants derived from quantum groups", Comm. Math. Phys. 127 (1990), no. 1, 1–26.

[9] N. Reshetikhin, V. Turaev, "Invariants of 3-manifolds via link polynomials and quantum groups." Inventiones mathematicae 103.1 (1991): 547597

[10] E. Witten, Quantum field theory and the Jones polynomial, Communications in Mathematical Physics volume 121, pages 351399 (1989)