2021-10-06 Kähler geometry

Aubin's approach. for $k=1$ (16) (1)

to overcome the difficulty of (2).

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-t\varphi + F} \quad \text{———} (*_t)$$

- $S := \{ t \in [0,1] \mid (*_t) \text{ has a solution} \}$.
- Enough to show $S = [0,1]$, as $*_1 = *$
- $0 \in S$ by Yau. So $S \neq \phi$.
- S is open (implicit function theorem).
- Difficulty is showing closedness of S.

  But again, by Yau's work, it is sufficient to show the C^0-estimate.

Chen - Donaldson - Sun, Tian

For Fano mfd M. (i.e. $c_1(M) > 0$
$k=1$ case)

$\exists$ KE $\Longleftrightarrow$ K-stability

They used a continuity method using "cone angle" as a parameter.
(explained later).

Twisted Kähler-Einstein metric.

Let M be a compact Kähler manifold of dim m.

Let ω be a fixed Kähler form.

$[\omega]$ Kähler class (fixed).

$\omega_\varphi = \omega + i\partial\bar\partial\varphi$, $\varphi \in C^\infty(M)$

another Kähler form in $[\omega]$.

Let $F \in C^\infty(M)$, given.

Solve the following complex Monge-Ampère equation.

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{-\lambda\varphi + F} \qquad \varphi_{i\bar j} = \frac{\partial^2\varphi}{\partial z^i \partial \bar z^j}$$

$\lambda = -1, 0$ or 1.

on any compact Kähler manifold M.

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Theorem (Yau $\lambda = 0, -1$, Aubin $\lambda = -1$)
1976 – 1977

$\overbrace{\omega \wedge \cdots \wedge \omega}^{m} = \omega^m$

On any compact Kähler manifold

for any $F \in C^\infty(M)$ $\left(\text{with } \int_M e^F \omega^n = \int_M \omega^n \text{ for } \lambda = 0 \right)$

there exists a unique solution $\varphi \in C^\infty(M)$ of the equation for $\lambda = -1$ and $\lambda = 0$.

(requiring $\int \varphi\, \omega^n = 0$ when $\lambda = 0$).

For $\lambda = 1$, there is no general existence result.

(Except for some speci situation for Fano manifold related to Yau-Tian-Donaldson conjecture. Major progress by Kewei Zhang recently)

The equation is equivalent to solving the following problem.

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Given a closed $(1,1)$-form α representing $c_1(M) - \lambda[\omega]$, does there exist a Kähler form $\omega_\varphi \in [\omega]$ such that

$$\mathrm{Ric}(\omega_\varphi) - \lambda \omega_\varphi = \alpha \quad ? \qquad \cdots\cdots (1)$$

(ω_φ is called a twisted KE metric.)

To see this equation, let $F \in C^\infty(M)$ satisfying

$$\mathrm{Ric}(\omega) - \lambda \omega = \alpha + i\partial\bar\partial F \qquad \cdots\cdots (2)$$

(2) $-$ (1) gives

$i^m \det(g_{i\bar j} + \varphi_{i\bar j})\, dz^1 \wedge \cdots \wedge d\bar z^m$

$$i\partial\bar\partial \log \frac{\omega_\varphi^m}{\omega^m} = i\partial\bar\partial(-\lambda\varphi + F)$$

$i^m \det(g_{i\bar j})\, dz^1 \wedge d\bar z^1 \wedge \cdots \wedge dz^m \wedge d\bar z^m$

$$\frac{\det(g_{i\bar j} + \varphi_{i\bar j})}{\det(g_{i\bar j})} = e^{-\lambda\varphi + F + \mathrm{const}}$$

F

$$= e^{-\lambda\varphi + F}$$

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Cor (Yau, Aubin)

For $\lambda = -1, 0$, the twisted K-E equation can be always solved.

Cor (Calabi conjecture 1976, Yau)

On any compact Kähler mfd with fixed Kähler class, given any real closed (1,1)-form α representing $c_1(M)$, there exists a unique Kähler metric ω in the fixed Kähler class s.t.

$$\mathrm{Ric}(\omega) = \alpha.$$

Rem $\alpha = 0$, $\lambda = 1$ $\longrightarrow$ Yau-Tian-Donaldson conj solved for Fano mfd.

Theorem (twisted KE case $\lambda = 1$, Berman-Boucksom-Jonsson (2015/2020 version algebraic case, Kewei Zhang 2020 transcendental case)

Let (M, ω) be a compact Kähler manifold and α be a closed 2-form representing

$c_1(M) - [\omega]$. Assume $\alpha \geq 0$

(so that $c_1(M) = [\omega] + [\alpha] > 0$
so M Fano)

Then we have

(1) If $\delta([\omega]) > 1$ then $\exists \omega_\varphi \in [\omega]$
such that $Ric(\omega_\varphi) = \omega_\varphi + \alpha$

(2) If $\exists \omega_\varphi \in [\omega]$ (resp. $\exists! \omega_\varphi \in [\omega]$)
s.t. $Ric(\omega_\varphi) = \omega_\varphi + \alpha$ then
$\delta([\omega]) \geq 1$ (resp $\delta([\omega_\varphi]) > 1$).

The def of $\delta([\omega])$ is as follows.

$L \in N^1(X)_{\mathbb{R}}$ be a big $\mathbb{R}$-line bundle
in Néron-Severi space $(X = M)$

$$\delta(L) = \inf_F \frac{A(F)}{S_L(F)}$$

where inf is taken over all prime divisors
F over X, i.e.

$\mu: Y \to X$ is any surjective birational
morphism, F is a prime divisor in Y.

$A(F) =$ log discrepancy of F

$$= 1 + \mathrm{coeff}_F\,(K_Y - \mu^*(K_X + L))$$

$$S_L(F) = \frac{1}{\mathrm{vol}(L)} \int_0^\infty \mathrm{vol}\,(L - xF)\,dx$$

For an ample line bdle L

$$\chi(L^k) = \int_M \mathrm{ch}\, L^k\, \mathrm{Todd}_M$$

$$= \int_M \frac{1}{m!} (k\, \underbrace{c_1(L)}_{\omega})^m + \text{lower order term in } k.$$

$\|$

$\dim H^0(L^k)$ Kodaira vanishing for large k

$$\mathrm{vol}(M,\omega) = \int_M \omega^m = \lim_{k\to\infty} \frac{\dim H^0(L^k)}{k^m/m!} =: \mathrm{vol}(L)$$

Using this, we define

$$\mathrm{vol}\,(L - xF) = \lim_{k\to\infty} \frac{\dim H^0(kL - \lfloor kx \rfloor F)}{k^m/m!}$$

$\lfloor kx \rfloor$ = least integer n s.t. $kx \geq n$

($\lim_{k\to\infty}$ coincides with $\limsup_{k\to\infty}$. well known)