Pro's and contras of Fey. integral
1. Pro's in the simplest model – oscillator
2. $\mathcal{L} = d\varphi * d\bar\varphi + \lambda \varphi^2 \bar\varphi^2$ for $\dim X = 4$

Higgs model

cont' of Fey Approach

a) why Fey. def. of local observable should be modified

b) How problem of $\int < \mathcal{O}(p)\, \mathcal{O}(p') >$, $p, p' \in X \times X$ divergence is solved

$\boxed{\text{if}}$ we use definition of local observable coming from functorial approach to QFT

It happens that Higgs in $d=4$ is the simplest.

Feynman's approach is outdated and should be replaced by a new approach. (!)

Functorial approach has $\underline{NO}$ UV divergences, Fey. appr. has them, but they are artefacts of Fey approach.

1) Harm. oscillator

$$S = \int \left\{ \left(\frac{dX}{dt}\right)^2 - \omega^2 X^2 \right\} dt \qquad X(t) = \sum_{n \in \mathbb{Z}} X_n \exp\left(i \frac{t}{T} n\right)$$

$$S = T X_0^2 \omega^2 + \sum_{n>0} |X_n|^2 T \left(\frac{n^2}{T^2} - \omega^2\right)$$

$$\int dX_0 \prod_{n>0} dX_n d\bar X_n \cdot \mu(T) \exp \frac{i}{\hbar} S$$

$$I = \mu(T) \cdot \prod_{n>0} \frac{T}{(n^2 - \omega^2 T^2)} .$$

what does it mean?

This product does not exist in math.

Infinite products in math

$$\ln \underline{I} = \ln \mu(T) + \sum_{n>0} \left(\ln T - \ln(n^2 - \omega^2 T^2)\right)$$

divergent seria!

No math. meaning $\rightarrow$ $\ln(n^2-\omega^2T^2) = \ln n^2 - \ln\left(1-\frac{\omega^2T^2}{n^2}\right)$

$$\sum_{n>0} \left(\ln T - \ln n^2\right)$$

something divergent but a function of T only

There is also $\mu(T)$ that we do not know from the first priciples of Fey. approach! $\underline{\int \mathcal{D}X}$ $\rightarrow$ known only up to mult. const

$$\mu(T) + \sum_{n>0} \left(\ln T - \ln n^2\right) =$$

$$= \tilde{\mu}(T)$$ want it to be finite

$$\ln \underline{I} = \ln \tilde{\mu}(T) - \sum_n \ln\left(1 - \frac{\omega^2T^2}{n^2}\right)$$

This is a convergent seria!

$$\underline{I} = \prod_{n>0} \frac{1}{\left(1 - \frac{\omega^2T^2}{n^2}\right)} \cdot \ln \tilde{\mu}(T) =$$

$2\pi = 1$

$$\frac{\sin \frac{\omega T}{2}}{T} \leftarrow \text{math.}$$

$$\underline{I} = \frac{1}{\sin\frac{\omega T}{2}} \underbrace{T \cdot \tilde{\mu}(T)}_{\tilde{\tilde{\mu}}(T)}$$

Result of F.I. after putting all ∞'s into $\mu(T)$

$$\underline{I} = \frac{1}{\sin\frac{\omega T}{2}} \cdot \tilde{\tilde{\mu}}(T)$$

Compare it with expected

$\hbar = 1$ for simplicity

$$\mathrm{Tr}\, e^{iTH} = \sum_n e^{iTE_n}$$

Here E_n are eigenvalues of

$$H = -\left(\frac{\partial}{\partial x}\right)^2 + \omega^2 x^2$$

are known to be $E_n = \omega\left(n+\frac{1}{2}\right)$

$$\sum_{n=0}^{+\infty} e^{iT\frac{\omega}{2}}\left(e^{iTn\omega}\right) = \frac{e^{iT\frac{\omega}{2}}}{1-e^{iT\omega}} =$$

$$= \frac{1}{e^{-\frac{iT\omega}{2}} - e^{\frac{iT\omega}{2}}} = \frac{2}{\sin\left(\frac{T\omega}{2}\right)}$$

Pro of F.I is that it really gives a correct answer for Harm. oscillator.

Contr. → $\mu(T)$ is divergent

To make e^{iTH} convergent consider imaginary $T = i\beta$ → No problem in Funct. approach $\sin \to \mathrm{sh}$

Still there are UV peculiarities in F.I.

Remark on quantization of gravity:

$$\int \mathcal{D}g \, e^{\frac{i}{\hbar} S_{HE}(g)}$$

metric.

some measure on a infinite dimensional space of metrics. Note, that it is not just

$\prod dg_{\mu\nu}$ we need to keep invariance w. respect to general coord. transformations:

$$\cancel{\int \mathcal{D}g} \longrightarrow \int \frac{\mathcal{D}g}{\mathrm{Diff}} \leftarrow \text{curved space}$$

Like $\mathbb{C}^2/\mathbb{C}^*$ is $\mathbb{C}P^1$

curved.

F.I. is not a very nice prescription...

2) Problems in Higgs model on R^4

$$S' = \int_{R^4} d\varphi * d\bar\varphi + \lambda \varphi^2 \bar\varphi^2 .$$

Task:

$$\int \mathcal{D}\varphi \, e^{\frac{i}{\hbar} S(\varphi)} \varphi(x_1)\, \varphi(x_2)\, \bar\varphi(y_1)\, \bar\varphi(y_2) = F'(x_1, x_2, y_1, y_2, \lambda, \hbar)$$

a) if $x_i \neq y_j$ and $\lambda = 0$ it is not a problem

$$F = \frac{1}{|x_1 - y_2|^2}\frac{1}{|x_2 - y_1|^2} + \underline{\frac{1}{|x_1 - y_1|^2}\frac{1}{|x_2 - y_2|^2}}$$

Now, inspect $x_1 = y_1$

From F. perspective – nothing wrong

$$\int \mathcal{D}\varphi \, e^{\frac{i}{\hbar} S(\varphi, \bar\varphi)} \varphi\bar\varphi(x_1)\, \varphi(x_2)\, \bar\varphi(y_2)$$

$\varphi\bar\varphi$ is just a function (local function)

actually result diverges !!!

Something is wrong in Feynman def. of local observables → they are <u>NOT</u>

just $F'(\varphi, \partial\varphi, \dots)(x)$ | why this is a problem?

First def. in λ =

$$\int \mathcal{D}\varphi \, e^{\frac{i}{\hbar}\int d\varphi * d\bar\varphi} \int_{x \in R^4} \varphi^2\bar\varphi^2(x)\, \varphi(x_1) \dots$$

This is divergent.

– two loops

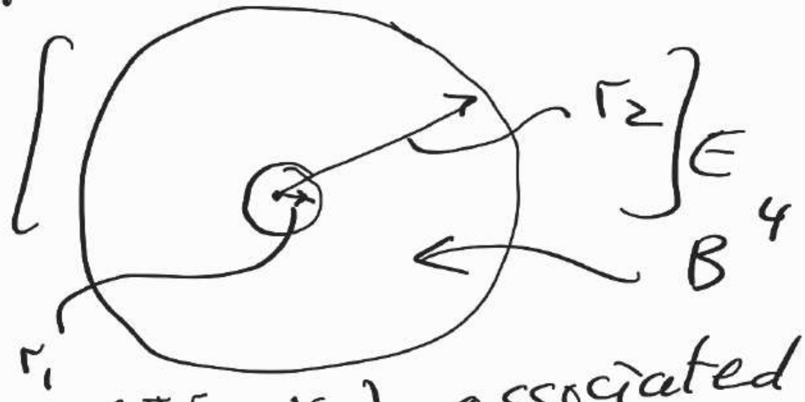
problem is that $G(x,x)$ does not exist

It is not just a constant:

– this also does not exist !

Thus, F.I. (prescription) was to be modified:

Functorial approach comes — local observables are a derived object

Fun 

$F = \mathrm{Hom}(V_1, V_2)$ associated to spheres of radii r_1 and r_2 respectively

Local observable is a state-valued function $\Psi_{r_1} \in V_1$, such that

$\exists \lim\limits_{r_1 \to 0} F\Psi_{r_1}$. such limit would depend on a state-valued function Ψ_{r_1}; different Ψ_{r_1} and $\overline{\Psi}_{r_1}$ can give equal limits → we identify them.

Analogy: in dif. geometry tangent vector is an equivalence class of maps $\gamma : [0,1] \to X$

$\gamma(0) = x_0$, and $\gamma \sim \tilde{\gamma}$ if

$$\left(\frac{d}{dt}\tilde{\gamma}^* f - \frac{d}{dt}\gamma^* f\right)\bigg|_{t=0} = 0, \quad f \in \mathrm{Fun}(X)$$

Ψ_r are analogues of γ

local observable come from a limit of a sphere collapsing to a point.

statement 1.

$e^{\int d\varphi * d\varphi}$ has two formulations, as F.I (analogue of Harm oscillator) and as a functorial QFT (analogue of e^{tH}) [we will prove it later]

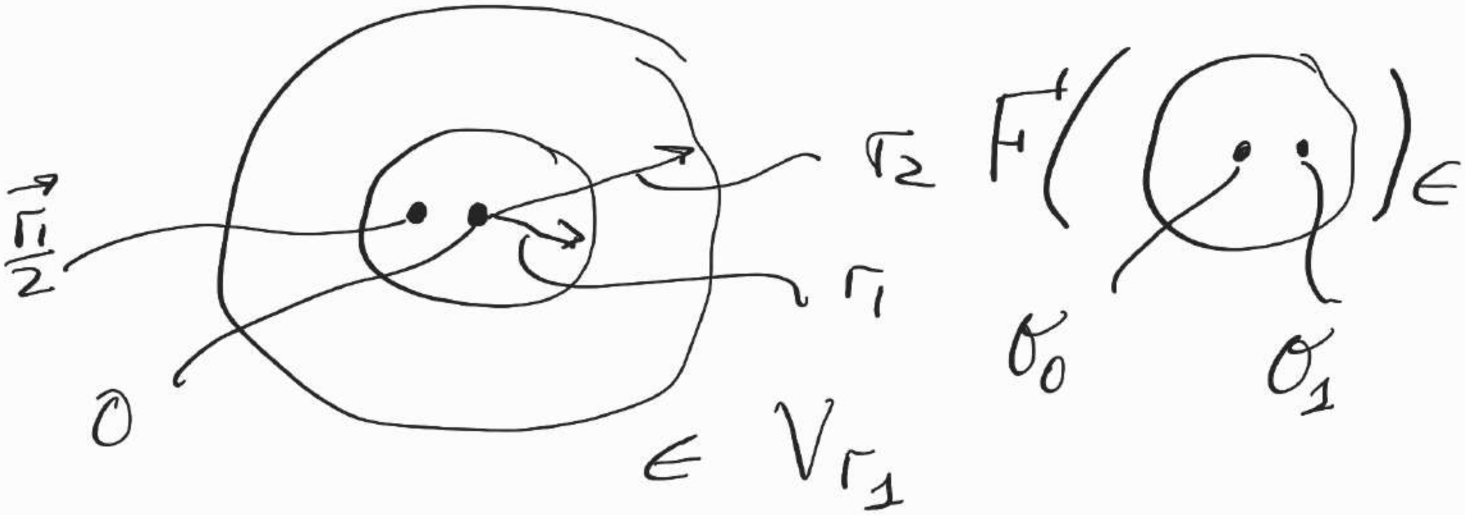
statement 2:

$\varphi(x)$ in F.I is also an observable in functorial QFT (prove it later)

Construction in functorial QFT (OPE - historical name)

Given two local observables one can construct infinitely many new observables by the following procedure.

And we will see the actual meaning of $\varphi\bar{\varphi}$!



I will try

$F\Big($  $\Big)$ as φ_{r_1}; is it a good vector-valued function of r_1?

It may happen, that when $r_1 \to 0$ there is a limit:

<u>Example:</u> $O_0 = \varphi$ $\quad O_1 = \varphi$

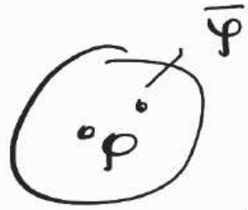
$\varphi(0)\,\varphi(\frac{r_1}{2})$ has no singularity as $r_1 \to 0$, so

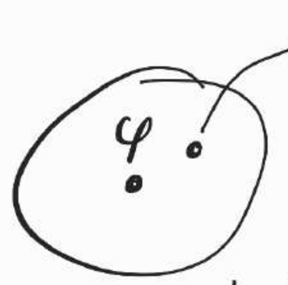
the state valued function has a limit that corresponds to $\varphi^2(x)$ — well-defined even in Fey. picture!

Another exampl:

$O_0 = \varphi \quad O_1 = \bar{\varphi}$

$F\Big($  $\Big) \sim \frac{1}{r_1^2}$ + reg. terms

$F\Big($  $\Big)$ is not a function, corresponding to local observable, however (if we subtract singularity) then

$\psi_{r_1} = F\Big($  $\Big) - \frac{1}{r_1^2} F\Big($  $\Big)$

has a limit when $r_1^2 \to 0$

Such ψ_{r_1} is called in literature

$:\varphi\bar{\varphi}:$ One problem solved

$\varphi\bar{\varphi}$ is a bad object $:\varphi\bar{\varphi}:$ is a good object.

Later on we will play more with this construction

$x_2 \neq y_2$

$$\langle :\varphi\bar{\varphi}:(x)\ \varphi(x_2)\ \bar{\varphi}(y_2)\rangle = \frac{1}{|x-y_2|^2\,|x-x_2|^2}$$

just one term

why?

$$\lim_{y_1 \to x_1} \langle \varphi(x_1)\, \bar\varphi(y_1)\ \ \varphi(x_2)\, \bar\varphi(y_2) \rangle - \langle \frac{1}{|x_1-y_1|^2}\ \ \varphi(x_2)\, \bar\varphi(y_2) \rangle =$$

$$= \lim_{y_1 \to x_1} \frac{1}{|x_1-y_2|^2} \frac{1}{|x_2-y_1|^2}$$

Similarly, we may do it for

$$\varphi^2 \bar\varphi^2 \longrightarrow :\varphi^2 \bar\varphi^2:$$

wrong

Now;

$$\int \mathcal{D}\varphi\, e^{\int d\varphi * d\bar\varphi + \lambda \varphi^2 \bar\varphi^2}$$

just wrong!

wrong

$$\int \mathcal{D}\varphi\, e^{\int d\varphi * d\bar\varphi + \lambda :\varphi^2 \bar\varphi^2:}$$

correct

has a chance to be correct

but it is correct only in first order in λ

Deformation in the second order in λ is an iteration of the def. in the first order!

$$\int \mathcal{D}\varphi \exp \int d\varphi * d\bar\varphi + \lambda_1 :\varphi^2 \bar\varphi^2: + \lambda_2 :\varphi^2 \bar\varphi^2:$$

$\Rightarrow$ term linear in λ_1 and λ_2

wrong!

What is true is the following procedure:

1) Deform theory by $:\varphi^2\bar{\varphi}^2:$ in the first order in λ_1

2) Recalculate local observables in deformed theory

3) Use new local observable to deform further!

<u>Analogy.</u>

Def. of the point x_0 to the second order is

1) transport v to the point $(x_0 + v_{x_0}\lambda_1)$, and then get $x_0 + v_{x_0}\lambda_1 + \lambda_2 \, v(x_0+\lambda_1 v)$

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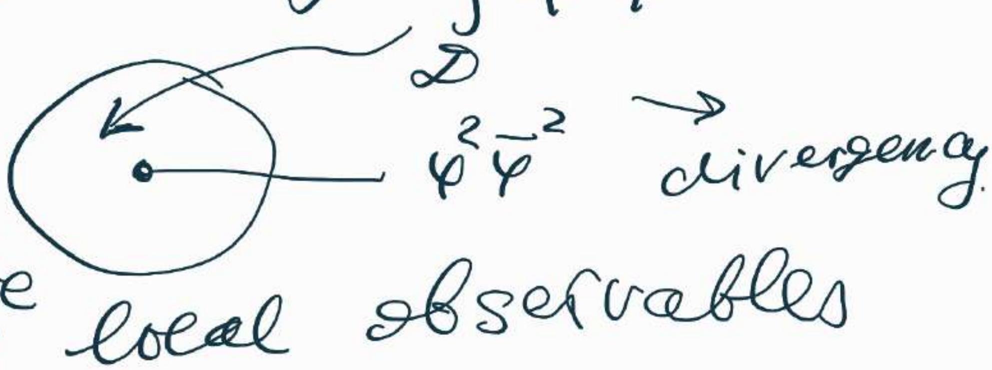
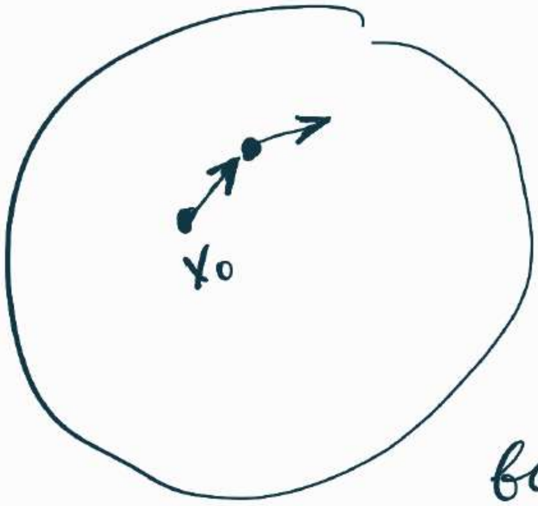
phenomena:

, $:\varphi^2\bar{\varphi}^2:$ is not a local observable in deformed theory, really $\int \varphi^2\bar{\varphi}^2$



Recalculate space of local observables



what is a trajectory up to the second order?

I would like to consider base of deformation as $\mathbb{C}[[\lambda_1, \lambda_2]]$ and consider term in front of $\lambda_1 \lambda_2$

Later you may say that we take $\lambda_1 = \lambda_2 = \lambda$ and term $\lambda_1 \lambda_2 \to \lambda^2$