Non-compact manifolds with positive scalar curvature

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Outline of talk

- **Main goal**: Let M^n be an open manifold, $\kappa \geq 0$, under what kind of conditions do we have $\inf_M R_g \leq \kappa$, g is any **complete** Riemannian metric on M^n.

1. I will discuss certain non-compact manifolds admitting no complete positive scalar curvature (PSC) metrics ($\Longrightarrow \inf_M R_g \leq 0$) or no complete and uniformly positive scalar curvature (UPSC) metrics ($\Longleftrightarrow \inf_M R_g \leq 0$);
2. I will discuss various Llarull type theorems on certain complete and non-compact manifolds ($\Longleftrightarrow \inf_M R_g \leq \kappa$, for some $\kappa > 0$, need some geometric normalizations on g);

- The talk is based on my joint works with T.Hao, Y.Sun, R.Wu, J.Wang and J.Zhu

Some basic facts of scalar curvature

- Scalar curvature

1. $R = \sum_{i,j} g^{ij} R_{ij}$;

2. Let $B_r(p)$ be a small geodesic ball with radius r and centre $p,$ then

$$Vol(B_r(p)) = \omega_n r^n (1 - \frac{R(p)}{6(n+2)} r^2 + \cdots)$$

If $R > 0$, then the volume of a small geodesic ball is smaller than that of ball with the same radius in $\mathbf{R}^n$.

3. All compact differentiable manifolds with dimension at least 3 admits a smooth metric of negative scalar curvature.

4. Not all manifolds admit PSC metrics. What kind of manifolds admit/ cannot admit PSC metrics?

Manifolds admitting no PSC metrics

- Some typical results on global effects of scalar curvature.

1. **Theorem (Schoen-Yau, 1980):** $\mathbb{T}^n$, $3 \leq n \leq 10$, admits no PSC metric

2. **Theorem (Gromov-Lawson, 1983):** All compact Cartan-Hadamard(CH) manifolds admit no PSC metric

A manifold M is said to be CH manifold if it admits non-positive sectional curvature.

3. **Theorem(Llarull, 1998):** Let $(\mathbb{S}^n, g_0)$ be the standard unit sphere, if $g \geq g_0$, and $R_g \geq R_{g_0} = n(n-1)$, then $g = g_0$.

- **Observations behind those typical examples:**

1. Nonexistence of PSC metrics depends heavily on the topology of the underlying manifolds;
2. A compact manifold admitting no PSC metrics usually has "complicated" topology; for instance, higher genus in 2-dim case;
3. One cannot enlarge the metric in all directions and increasing the scalar curvature simultaneously. More specifically:

- **Observation (Gromov, 2023):** Let (M^n, g) be a compact Riemannian manifold, (N^n, g_0) be a compact manifold with constant sectional curvature κ, $f: M^n \to N^n$ be $1-$Lipschitz and non-zero degree map, then $\inf_M R_g \leq C_n \cdot inj(\tilde{N}, \tilde{g}_0)^{-2}$, here $(\tilde{N}, \tilde{g}_0)$ is the universal covering space of (N^n, g_0).

- **Example 1**: Let $(\mathbb{S}^n, g_0)$ be the standard unit sphere, if $g \geq g_0 \Longrightarrow id: (\mathbf{S}^n, g) \to (\mathbf{S}^n, g_0)$ is $1-$Lipschitz $\Longrightarrow \inf_M R_g \leq R_{g_0} = n(n-1)$.

- **Example 2**: If (N^n, g_0) is a compact CH manifold, M^n is any compact manifold with $f: M^n \to N^n$ be a non-zero degree map, then M^n admits no PSC metrics.

- Suppose g is a PSC metric on M^n, then we may choose $\lambda > 0$ with $f: (M^n, \lambda^2 g) \to (N^n, g_0)$ is 1-Lipschtiz, $inj(\tilde{N}, \tilde{g}_0) = \infty$, we see that $\inf_M R_g \leq C_n \cdot inj(\tilde{N}, \tilde{g}_0)^{-2} = 0$, contradiction.

- **SYS(Schoen-Yau-Schick) manifold**: Let M^n be a compact manifold, $\beta_1, \cdots, \beta_{n-2} \in H^1(M, Z)$, if $[M] \cap (\beta_1 \smile \cdots \smile \beta_{n-2}) \in H_2(M, Z)$ is not contained in the image of Hurewicz map $Hu : \pi_2(M) \to H_2(M, Z)$.

- **Example**: $\mathbb{T}^n$, $\Sigma^2 \times \mathbb{T}^{n-2}$ with $g(\Sigma) \geq 1$ are SYS manifolds.

$$[\Sigma^2 \times \mathbf{T}^{n-2}] \cap (d\theta_1 \smile \cdots \smile d\theta_{n-2}) = [\mathbf{\Sigma}^2]$$

- **Theorem**(Schoen-Yau1980; Schick1998): Let M^n be a compact SYS manifold, $3 \leq n \leq 10$, then M^n admits no PSC metrics.

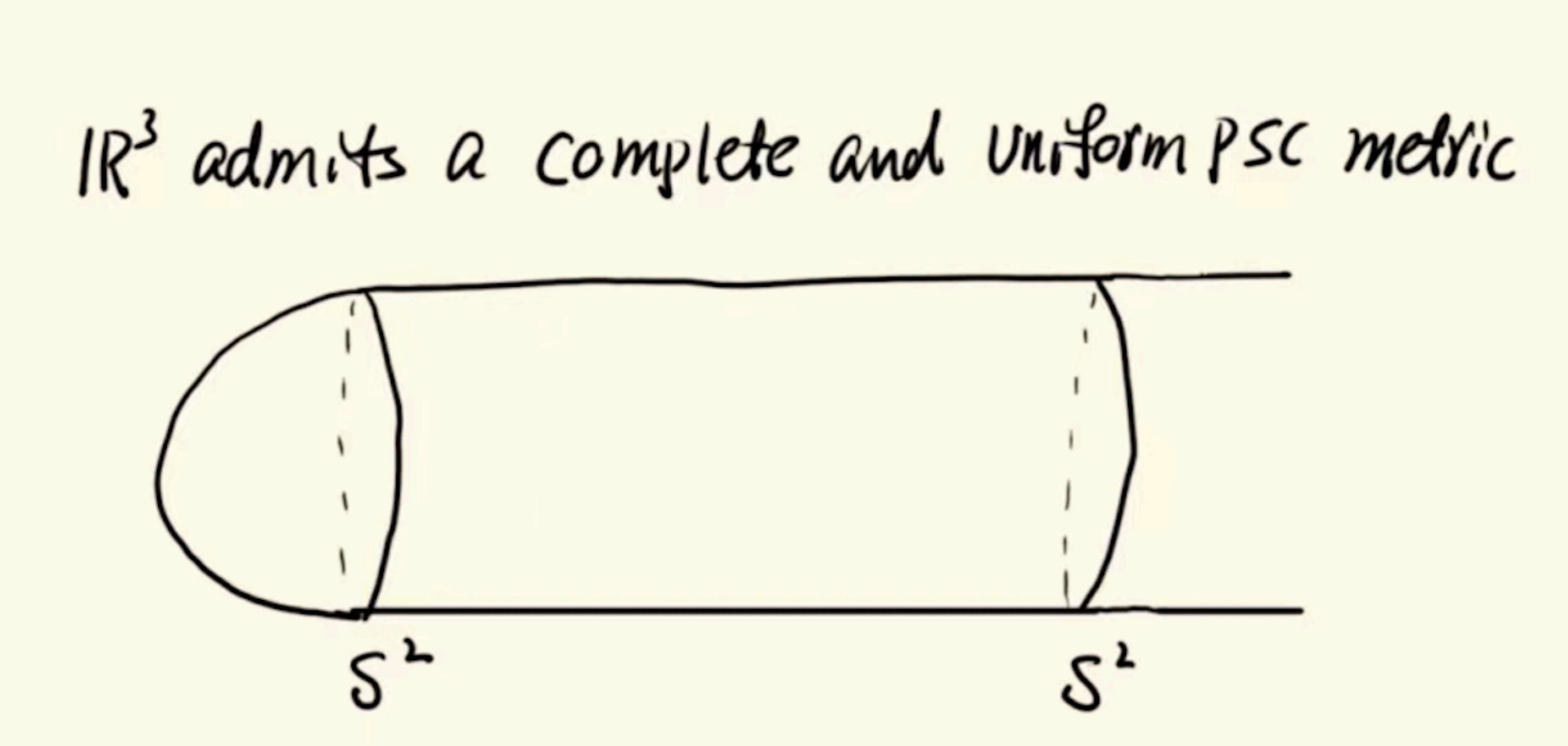
- **How about non-compact cases?**
- Some examples:

1. $\mathbb{T}^{n-1} \times \mathbb{R}$ admits no complete PSC metrics$\Longrightarrow \inf_{\mathbb{T}^{n-1}\times\mathbb{R}} R_g \le 0.$

2. $\mathbb{T}^{n-2} \times \mathbb{R}^2$ admits complete PSC metrics but no complete and uniformly PSC metrics$\Longleftrightarrow \inf_{\mathbb{T}^{n-2}\times\mathbb{R}^2} R_g \le 0.$

   paraboloid $(\mathbb{R}^2, g)$, $g = dx_1^2 + dx_2^2 + 4a^2(x_1 dx_1 + x_2 dx_2)^2$, with $R_g > 0$

3. $\mathbb{T}^{n-3} \times \mathbb{R}^3$ admits complete and uniformly PSC (UPSC)metrics。



- All above manifolds are CH manifolds.
- Noncompact CH manifolds may carry complete UPSC metrics

- **Some known results:**
- **Theorem** (Gromov & Lawson 1983): Let $\mathbb{T}^k \subset \mathbb{T}^n$ be linear subtorus of $\mathbb{T}^n$, $0 \leq k < n$, then $\mathbb{T}^n \backslash \mathbb{T}^k$ admits no complete PSC metrics.
- They proved more general result : any Λ^2-enlargeable manifold admits no complete PSC metrics.
- **Theorem**(Lesourd, Unger & Yau 2020; Chodosh & Li 2020): For $3 \leq n \leq 10$, and any open manifold M^n, then $\mathbb{T}^n \# M^n$ admits no complete PSC metrics.
- **Theorem**(S.Chen 2022): For $3 \leq n \leq 10$, any compact SYS manifold N^n and any open manifold M^n, then $N^n \# M^n$ admits no complete PSC metrics.
- **Theorem**(Chen, Chu & Zhu 2023): For $n \in \{3,4,5\}$, any compact aspherical manifold N^n and any open manifold M^n, then $N^n \# M^n$ admits no complete PSC metrics.

- In 2023, in his Four lectures on scalar curvature, M. Gromov proposed the following conjecture:

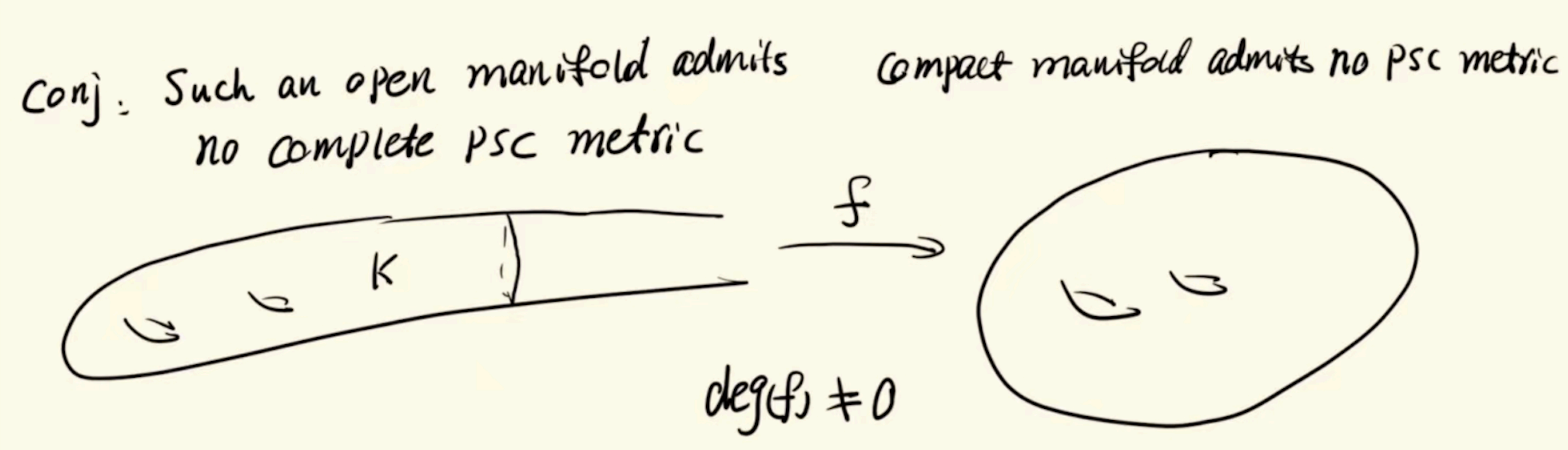
Non-compact Domination Conjecture 11☹. *If a compact orientable n-manifold (or pseudomanifold) X_0 can't be dominated (with maps of degree 1) by compact manifolds with $Sc > 0$, then it can't be dominated by complete manifolds with $Sc > 0$.*

- Let X be a compact manifold, we say X dominates X_0 if there is $f : X \to X_0$ with $deg(f) \neq 0$.

- Let X be an open manifold, we say X dominates X_0 if there is $f : X \to X_0$ and $K \subset\subset X$ with $f|_{X \setminus K} = const.$, $deg(f) \neq 0$.

- **Remark on domination:**

1. For compact manifolds X, X_0, X dominates X_0, I.e., $f : X \to X_0$ with $deg(f) \neq 0$ $\Longrightarrow$the topology of X looks like that of X_0 in certain sense.
2. For an open manifold X dominates compact X_0, I.e., $f : X \to X_0$ and $K \subset\subset X$ with $f|_{X \backslash K} = const.$, $deg(f) \neq 0$. $\Longrightarrow$the topology of K looks like that of X_0 in certain sense.



- It seems natural to believe the answer to the following special case of Gromov's conjecture should be affirmative.

- **A special case of Gromov's conjecture:**

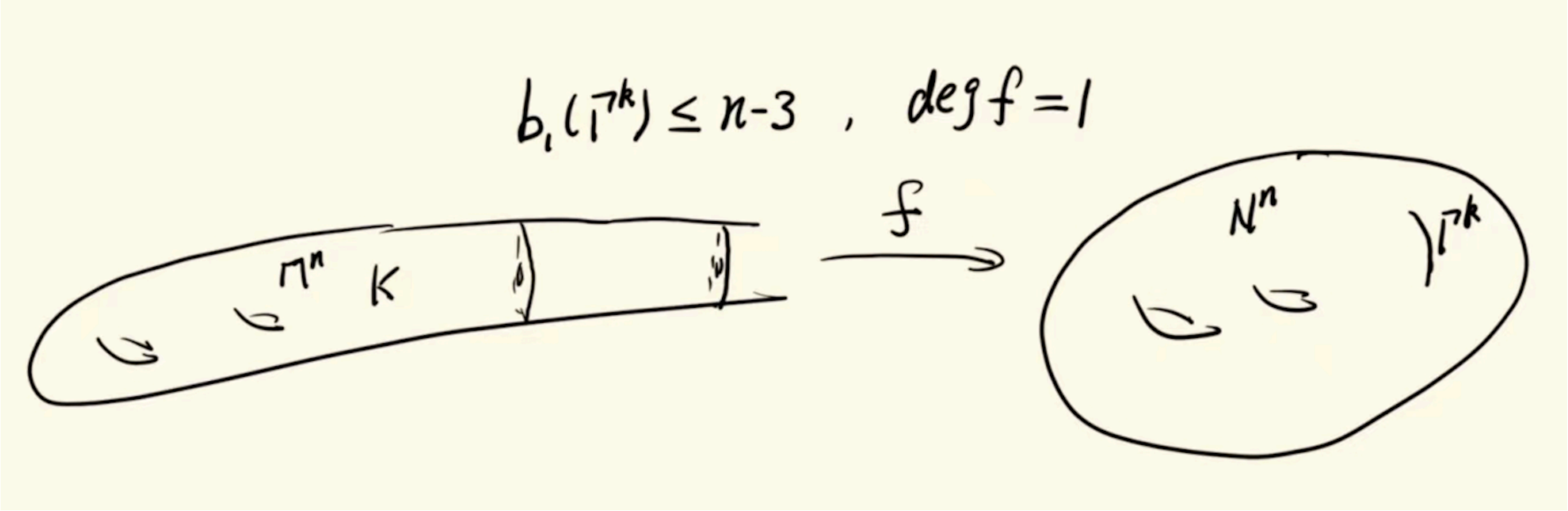
Let X_0 be a compact SYS manifold, X is an open manifold, $K \subset \subset X$, $f|_{X \backslash K} = const$. $f: X \to X_0$ with $deg(f) = 1$, then X admits no complete PSC metrics.

- **Remark**

1. The above conjecture implies LUY, CL, S. Chen 's theorems. As there always exists $f: X_0 \# N \to X_0$ with $deg(f) = 1$ for any N.
2. Let $\beta_1, \cdots, \beta_{n-2} \in H^1(X_0, Z)$ be as in the definition of SYS manifold, WLOG, we may always assume those $\beta_1, \cdots, \beta_{n-2}$ is linear dependent around $p \in X_0$.

- **Theorem 1**(Shi, Wang, Wu & Zhu, 2024): Let N^n be a compact SYS manifold, $3 \leq n \leq 10$, $\Gamma^k \subset N^n$ be a compact submanifold of N^n, M^n be an open manifold, let $f : M^n \to N^n$ be a continuous map and $f : M^n \to N^n \backslash \Gamma^k$ is proper or $f : M^n \backslash K \to \Gamma^k$ for some compact domain K of M^n, and $deg(f) = 1$, if

1. $b_1(\Gamma) \leq n - 3$, then M^n admits no completely PSC metric ($\Longrightarrow \inf_M R_g \leq 0$)
2. Γ^k is spherical, $codim(\Gamma) \geq 3$, then M^n admits no completely uniformly PSC metric($\Longleftrightarrow$ $\inf_M R_g \leq 0$).



- **Remark:**

1. The same conclusion is still true if $\beta_1, \cdots, \beta_{n-2} \in H^1(N, Z)$ is linear dependent on Γ^k. Take Γ^k be a point of N^n, the above theorem give an affirmative answer to the special case of Gromov's conjecture.

2. More generally, we can define a notion so called open SYS manifold, and we can show those open SYS manifolds carry no complete PSC metrics with dimension $3 \leq n \leq 10$.

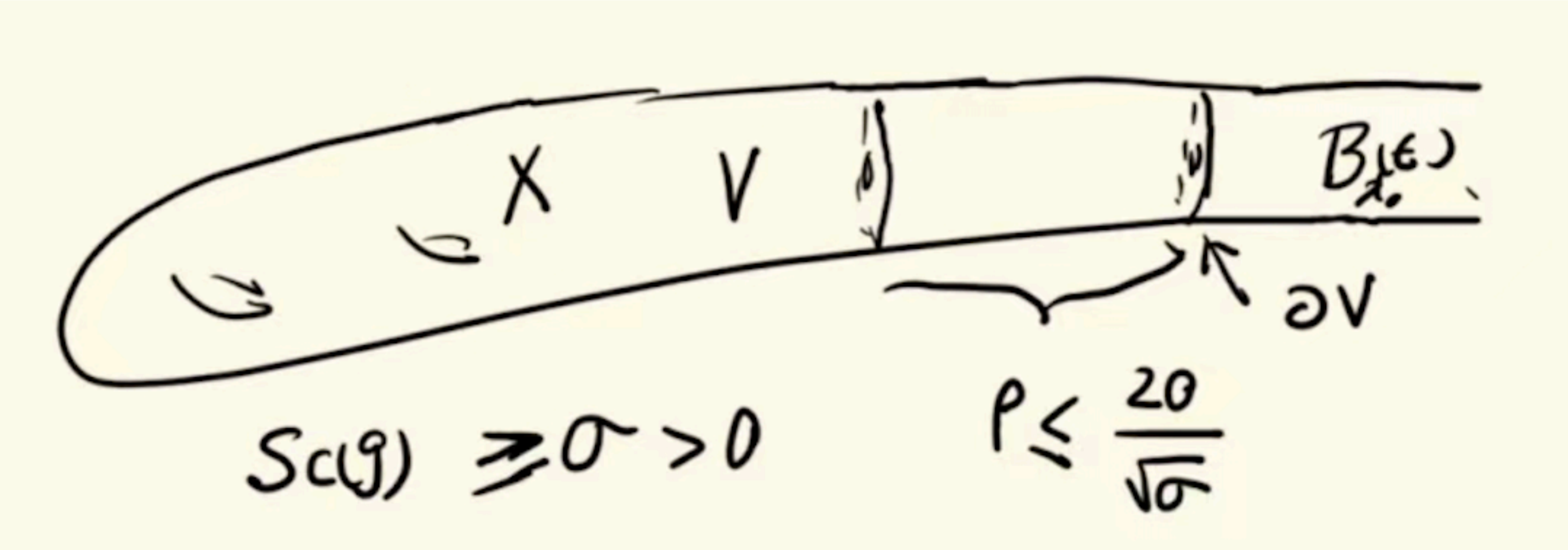
3. M^n in Theorem 1 and $N \times \mathbf{R}$ are an open SYS manifolds If N is a closed SYS manifold

4. The main arguments is to use minimal surface techniques together with careful topological analysis

- In 2018 and his GAFA paper, Gromov proposed the following conjecture

CONJECTURE **D'**. Let X be closed n-manifold, such that X minus a point admits *no complete metric with* $Sc > 0$.
Let V be obtained by removing a small open n-ball from X, i.e. $V = X \setminus B_{x_0}(\varepsilon)$, and let g be a metric on V with $Sc(g) \geq \sigma > 0$. If the ρ-neighbourhood with respect to g of the boundary sphere $S^{n-1} = \partial V = \partial B_{x_0}(\varepsilon)$ is homeomorphic to $S^{n-1} \times [0,1]$, then

$$\rho \leq \frac{20}{\sqrt{\sigma}}.$$



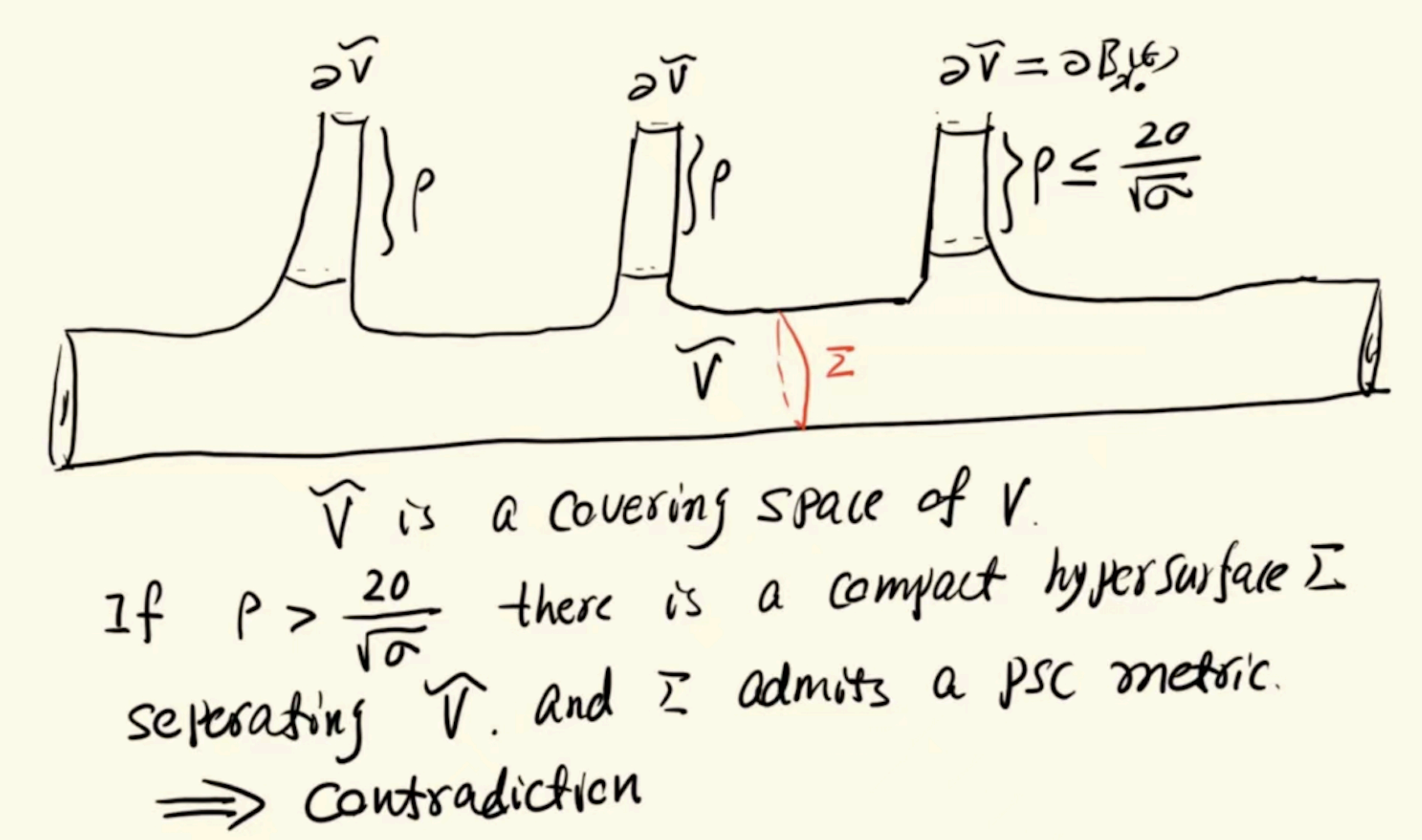
- (If X is a SYS-manifold, then metrics g with $Sc(g) \geq \sigma$ on V do satisfy this inequality as it follows by Schoen–Yau's kind of argument adapted to manifolds with boundaries as in section 11.6.

- Key observation: Let $\beta \in H^1(X, Z)$ be as in the definition of SYS manifold, then $\beta|_{\partial B_{x_0}(\epsilon)} = 0$ as $b_1(\partial B_{x_0}(\epsilon)) = 0$.

- Given a function μ on a Riemannian manifold (M^n, g), a $\mu-$bubble is a boundary of a minimizer (and a critical point) of the functional

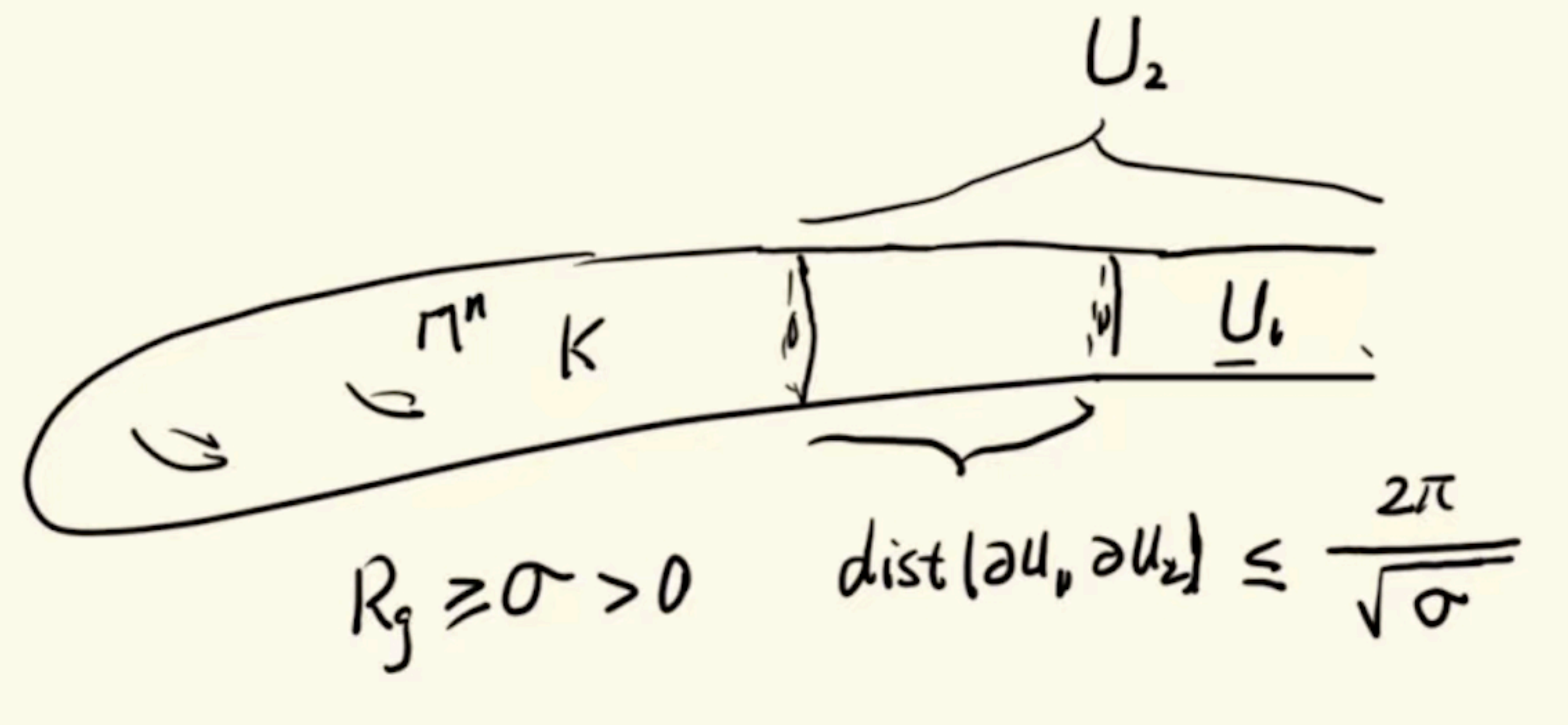
$$\Omega \mapsto vol_{n-1}(\partial\Omega) - \int_\Omega \mu dv_g$$

defined for suitable subsets $\Omega \subset M$;



- **Theorem** (Shi, Wang, Wu & Zhu, 2024): Let (M^n, g) be a compact SYS manifold with $3 \leq n \leq 10$, $\Gamma \subset M, Codim(\Gamma) \geq 3$, if Γ is spherical, I.e. $\pi_2(\Gamma) \to H_2(\Gamma, Z)$ is surjective, and $R_g \geq \sigma > 0$, then for any two tubular neighborhood U_1, U_2 of Γ, $dist(\partial U_1, \partial U_2) \leq \frac{2\pi}{\sqrt{\sigma}}$, especially $M \backslash \Gamma$ does not admit a complete uniformly positive scalar curvature(UPSC)metric.

  $\Longleftrightarrow \inf_{M\backslash\Gamma} R_g \leq 0$ for any complete g



- **Remark:** 1. if $\Gamma = \{p\}$ then it is automatically spherical;

  2. One difficulty: $\beta_i|_\Gamma$ may not equal to zero, more careful analysis is needed.

- **Summary**:

1. We can define open SYS manifold, those manifolds admit no complete PSC metrics $\Longrightarrow \inf_M R_g \leq 0$;
2. Let M^n a closed SYS manifold then $M \times \mathbf{R}$ and $M^n \backslash \Gamma^k$ are an open SYS manifolds provided $b_1(\Gamma^k) \leq n-3$;
3. Let M^n a closed SYS manifold, $\Gamma^k \subset M^n$ is spherical and $n-k \geq 3$, then $\inf_{M \backslash \Gamma} R_g \leq 0$,

Llarull type theorems on complete Riemannian manifolds

- **Aim:** Under some geometric normalizations on $g \Longrightarrow \inf_M R_g \leq \kappa$, for some $\kappa > 0$.

- **Observation**: For certain compact manifolds with PSC metrics, one cannot increase the scalar curvature and enlarge the manifold in all directions simultaneously.

- **Example**: let $(\mathbb{S}^2, g_0)$ be the standard unit sphere, if $g \geq g_0$, and $R_g \geq 2 = R_{g_0}$, then $g = g_0$.
- Proof: Gauss-Bonnet Theorem $\Longrightarrow \int_{\mathbb{S}^2} R_g d\mu_g = \int_{\mathbb{S}^2} R_{g_0} d\mu_{g_0} = 8\pi$

  $\Longrightarrow g = g_0$.

- **Model space:** $(\mathbb{S}^n, g_0)$

- **Theorem**(Llarull, 1998): Let $(\mathbb{S}^n, g_0)$ be the standard unit sphere, if $g \geq g_0$, and $R_g \geq R_{g_0}$, then $g = g_0 \Longrightarrow \inf_{\mathbb{S}^n} R_g \leq n(n-1)$ for all $g \geq g_0$

- **Remark**:

1. $id : (\mathbb{S}^n, g) \mapsto (\mathbb{S}^n, g_0)$ is $1-$Lipschitz with degree 1 served to be a geometric normalization;

2. $(\mathbb{S}^n, g_0)$ can be replaced by any strictly convex hypersurfaces in $\mathbb{R}^n$;

3. Llarull's Theorem was proved by Dirac operator originally, some lower dimensional cases can be handled by μ-bubble arguments;

4. $1-$Lipschitz condition can be relaxed to area—non increasing;

5. $(\mathbb{S}^n, g_0)$ is $\delta-$gap length extremal for $\delta \geq 0$.

- **Model spaces:** $\mathbb{S}^k \times N^{n-k}$ with N^{n-k} being enlargeable, for instance, $N^{n-k} = \mathbb{T}^{n-k}, \mathbb{R}^{n-k} \ldots$

- **Problem 1**: Let (M^n, g) be compact orientable Riemannian manifold with $R_g \geq k(k-1)$, we assume that there is a non-zero degree and 1 -Lipschitz map $f : M^n \mapsto \mathbb{S}^k \times \mathbb{T}^{n-k}$, is (M^n, g) locally isometric to $\mathbb{S}^k \times \mathbb{T}^{n-k}$?

- **Problem 2**: Let (M^n, g) be complete orientable Riemannian manifold with $R_g \geq k(k-1)$, we assume that there is a non-zero degree and 1-Lipschitz map $f : M^n \mapsto \mathbb{S}^k \times \mathbb{R}^{n-k}$, is (M^n, g) is locally isometric to $\mathbb{S}^k \times \mathbb{R}^{n-k}$?

- **Remark: 1**. There would be no such f provided $R_g > k(k-1)$, once answers to above problems are affirmative;

  2. There are three types of model space:

$$f : M^n \mapsto \mathbf{S}^n \text{ with } R \geq n(n-1);$$

$$f : M^n \mapsto \mathbf{S}^{n-1} \times \mathbf{R} \text{ with } R \geq (n-2)(n-1);$$

$$f : M^n \mapsto \mathbf{S}^k \times \mathbf{R}^{n-k} \text{ with } n-k \geq 2 \text{ and } R \geq k(k-1)$$

- **Theorem** (W.Zhang, 2020): let (M^n, g) be noncompact and complete spin manifold, $f : M^n \mapsto \mathbf{S}^n$ with $f|_{M\backslash K} = const$. and $deg(f) \neq 0$, area non-increasing and $R_g \geq n(n-1)$ on $Supp(df)$ if n is even ($R_g > n(n-1)$ on $Supp(df)$ if n is odd), then $\inf_M R_g < 0$.
- **Corollary:** let (M^n, g) be noncompact and complete SPIN manifold with $R_g > n(n-1)$, then there is no $f : M^n \mapsto \mathbf{S}^n$ with $f|_{M\backslash K} = const$. and $deg(f) \neq 0$. area non-increasing .

- **Problem**: Is it possible to show the same result without spin assumption?
- **Difficulty**: For higher dim case, the manifold may not be spin, hence Dirac operator cannot be used directly; Gauss-Bonnet formula cannot be used on its $\mu-$bubble either.

- Recently, T. Has, Y.Shi & Y.Sun proved the following results
- **Theorem 1**(HSS, 2023): Let (M^n, g) be n-dimensional compact orientable Riemannian manifold with $R_g \geq 6, 4 \leq n \leq 10,$ we assume that there is an non-zero degree and 1 -Lipschitz map $f : M^n \mapsto \mathbb{S}^3 \times \mathbb{T}^{n-3}$, then (M^n, g) is locally isometric to $\mathbb{S}^3 \times \mathbb{T}^{n-3}$. Moreover, for any $x \in \mathbb{T}^{n-3}$, $P \circ f(\cdot, x) : \mathbb{S}^3 \mapsto \mathbb{S}^3$ is isometric, here P denotes the standard projection of $P : \mathbb{S}^3 \times \mathbb{T}^{n-3} \mapsto \mathbb{S}^3$.
- Under the assumptions in Theorem1, we $\inf_M R_g \leq 6$

- **Problem 2**: Let (M^n, g) be complete orientable Riemannian manifold with $R_g \geq 6,\ 4 \leq n \leq 7$, we assume that there is a non-zero degree and 1-Lipschitz map $f: M^n \mapsto \mathbb{S}^3 \times \mathbb{R}^{n-3}$, is (M^n, g) is locally isometric to $\mathbb{S}^3 \times \mathbb{R}^{n-3}$?

- Noncompact situations are much more complicated than those of compact cases.
- **Observation**: Llarull type theorems are not true on $\mathbf{R}^n$ for $n \geq 2$.

- **Example**: Let $\Sigma^n, n \geq 2$, be a paraboloid of revolution in $\mathbf{R}^{n+1}$ which is also a graph of $\mathbf{R}^n \subset \mathbf{R}^{n+1}$, $P: \Sigma^n \mapsto \mathbf{R}^n$ denotes the restriction of the standard projection in $\mathbf{R}^{n+1}$ to $\mathbf{R}^n$ is $1-$Lipschitz and $R_\Sigma > n(n+1)$.

- **Example**: Let $\Sigma^n, n \geq 2$, be a paraboloid of revolution in $\mathbf{R}^{n+1}$which is also a graph of $\mathbf{R}^n \subset \mathbf{R}^{n+1}, P : \Sigma^n \mapsto \mathbf{R}^n$ denotes the restriction of the standard projection. Let $(M^{n+m}, g) = \mathbf{S}^m \times \Sigma^n, m \geq 2,$ note that its scalar curvature $R_g > m(m+1)$. Then $f := (id, P) : M^{n+m} \mapsto \mathbf{S}^m \times \mathbf{R}^n$ is a proper and 1-Lipschitz map with non-zero degree.

- **Observation**: there is no proper and 1-Lipschitz map $f : (\mathbf{R}^n, g) \mapsto (\mathbf{R}^n, g_{euc})$ with non-zero degree, here $R_g \geq \delta > 0$.

- By the similar arguments and together with Theorem 1, we get:

- **Theorem 2** (HSS, 2023): For any $\delta > 0$, let (M^n, g), $5 \leq n \leq 7$, be an noncompact orientable and complete Riemannian manifolds with scalar curvature $R_g \geq 6 + \delta$. Then there is no proper and 1-Lipschitz map $f : M^n \mapsto \mathbb{S}^3 \times \mathbb{R}^{n-3}$ with non-zero degree.

- Under the assumptions in Theorem 2, we have $\inf_M R_g \leq 6 + \delta$, for any $\delta > 0$

- **Observation**: any compact perturbation of $\mathbf{R}^n$, $3 \le n \le 7$, with $R \ge 0$ is trivial.

- By the similar arguments and together with Theorem 1, we get:

- **Theorem 3** (HSS, 2023): Let (M^n, g), $4 \le n \le 7$, be an noncompact orientable and complete Riemannian manifolds with scalar curvature$R_g \ge 6$, $f: M^n \mapsto \mathbb{S}^3 \times \mathbb{R}^{n-3}$ be a proper and 1-Lipschitz map with non-zero degree. Then (M^n, g) is isometric to $\mathbb{S}^3 \times \mathbb{R}^{n-3}$ provided f is isometric outside a compact domain of M^n.

$\Longleftrightarrow$ one cannot do compact perturbations on $\mathbb{S}^3 \times \mathbb{R}^{n-3}$ to get (M^n, g) with $R_g \ge 6$, $f: M^n \mapsto \mathbb{S}^3 \times \mathbb{R}^{n-3}$ being a proper and 1-Lipschitz map and $\deg(f) \ne 0$

- Situations are complete different if we take $\mathbb{S}^3 \times \mathbf{R}$ as the model space.
- **Theorem 4** (HSS, 2023)Let (M^4, g)be an noncompact orientable and complete Riemannian manifolds with scalar curvature $R_g \geq 6$, $f: M^4 \mapsto \mathbb{S}^3 \times \mathbb{R}$ be a proper and 1 -Lipschitz map with non-zero degree. Then (M^4, g) is isometric to $\mathbb{S}^3 \times \mathbb{R}$ provided (M^4, g) is geometric bounded, i.e. $\sup_M \|Rm\| < \infty$ and its injective radius $inj(M^n, g) > 0$.
- **Theorem 5** (HSS, 2023) Let (M^4, g) be an noncompact orientable and complete Riemannian manifolds with scalar curvature $R_g > 6$, then there is no proper and 1 -Lipschitz map $f: M^4 \mapsto \mathbb{S}^3 \times \mathbb{R}$ with non-zero degree.
- Under the assumptions in Theorem5, we have $\inf_M R_g \leq 6$

A metric g_0 on Y is *ε-gap length extremal* if *no* $g \geq g_0$ on Y satisfies

$$Sc(g) - Sc(g_0) > \varepsilon.$$

Then g_0 is called *gap length extremal* if it is ε-gap length extremal for all $\varepsilon > 0$ (0-gap extremal=extremal).

- P. 152, M.Gromov: A Dozen Problems, Questions and Conjectures About Positive Scalar Curvature ;
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D$_3$. **Question.** Does gap extremality is always stable under $Y \rightsquigarrow Y \times \mathbb{R}^m$? (Beware of $dim(Y) = 4$.)

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D_3. Question. Does gap extremality is always stable under $Y \rightsquigarrow Y \times \mathbb{R}^m$? (Beware of $dim(Y) = 4$.)

- **Conclusion**: For any $\delta > 0, \ \ 1 \leq m \leq 4$, the $\delta-$gap length extremity of $(\mathbf{S}^3, g_0)$ is stable under above sense.

Thank you for attention