Locally Chern homogeneous manifolds

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Outline

This talk is dedicated to the special occasion of Professor Yau's 75th birthday.

Happy birthday, Professor Yau!

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- I apologize to those who have listened to our report on this topic already! I will add some discussion on recent development to avoid total duplication.
- We are interested in understanding a special type of locally homogeneous Hermitian manifolds, whose Chern connection is an Ambrose-Singer connection (namely, has parallel torsion and curvature).
- Recall that a *Hermitian manifold* is complex manifold M equipped with a Riemannian metric $g = \langle , \rangle$ that is compatible with the almost complex structure J, namely, $\langle Jx, Jy \rangle = \langle x, y \rangle$ for any tangent vectors x and y.

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- There exists a unique connection D on M that is *metric* (i.e., $Dg = 0$) and *torsion-free:*

$$T^D(x, y) := D_x y - D_y x - [x, y] \, = \, 0, \qquad \forall \, x, y.$$

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- If D is compatible with the almost complex structure (i.e., $DJ = 0$), then g is called a *Kähler metric.* In this case, Riemannian geometry fits perfectly with the complex analytic properties.

- But for a given compact complex manifold M, the existence of a Kähler metric would impose strong topological restrictions on M, by Hodge theory, for instance, so the 'majority' of compact complex manifolds do not admit Kähler metrics.

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- On (M, g), there always exists a connection ∇^c with $\nabla^c g = 0$, $\nabla^c J = 0$, and its torsion T^c has no $(1,1)$-part:

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- ∇^c is called the *Chern connection* of (M, g). It is the unique metric connection with $(\nabla^c)^{(0,1)} = \overline{\partial}$. It also uniquely exists on any holomorphic vector bundle.

- Besides Levi-Civita and Chern, there is another canonical connection ∇^b on (M, g) that is well-studied, that's the *Bismut connection* (aka *Strominger connection*).

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$$\langle T^b(x, y), z\rangle = -\langle T^b(x, z), y\rangle, \quad \forall\, x, y, z.$$

- ∇^b serves as a 'bridge' between D and ∇^c: its Riemannian geometry is not as good as D, but 'better' than ∇^c, in the sense that T^b is 'simpler' than T^c (e.g., ∇^b has the same set of geodesics as D), in the mean time, its 'complex geometry' is not as good as ∇^c (e.g., ∇^b is not compatible with the complex structure: $(\nabla^b)^{(0,1)} \neq \overline{\partial}$), but better than D (e.g., $\nabla^b J = 0$). ∇^b is the most favorite connection for physicists.

- In summary, on a given Hermitian manifold (M^n, g), if g is Kähler, then $D = \nabla^c = \nabla^b$, but when g is not Kähler, these three connections are mutually distinct. In this talk, we will focus on ∇^c.

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- Let (M^n, g) be a complete Riemannian manifold. A metric (i.e., $\nabla g = 0$) connection ∇ is said to *Ambrose-Singer,* if its torsion T and curvature R are both parallel under ∇ (i.e., $\nabla T = 0$ and $\nabla R = 0$).

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- **Theorem (Ambrose-Singer, 1958):** Let (M^n, g) be a complete, simply-connected Riemannian manifold. Then it is homogeneous (namely, its isometry group acts transitively) if and only if it admits a metric connection which is Ambrose-Singer.

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- A point $b \in F(M)$ is in the form $b = (x; \varepsilon_1, \ldots, \varepsilon_n)$ where $x = \pi(b) \in M$ and $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is an orthonormal basis of $T_x M$.

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- The frame bundle $F(M)$ naturally admits a global tangent frame $\{E_i, E_{jk}\}$, where $1 \le i \le n$ and $1 \le j < k \le n$, so that $E_i(b) = \varepsilon_i$ and $E_{jk} \in V = \ker(d\pi)$.

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- Using this natural frame as orthonormal frame, we get a Riemannian metric $\hat{g}$ on $F(M)$, and π becomes a Riemannian submersion.

- If H is a subgroup of the isometry group $I(M)$ acting transitively on M, then for any fixed $b = (x; \varepsilon_1, \ldots, \varepsilon_n)$ in $F(M)$, we have a smooth map $\Psi_b : H \to F(M)$

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- Let $e \in H$ be the unit element. Then $\Psi_b(e) = eb = b$, so one can define distributions in $F(M)$:

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- Here $\perp$ is with respect to $\hat{g}$. That way we get a subbundle $Q \subseteq TF(M)$ and $TF(M) = Q \oplus V$. As is well-known, metric connections on M correspond to horizontal distributions in $F(M)$, so the above Q gives us a metric connection ∇^Q.

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- Conversely, if ∇ is an Ambrose-Singer metric connection on M, corresponding a horizontal distribution Q in $F(M)$, then fix any $b_0 \in F(M)$ and denote by $H \subseteq F(M)$ the subset of points which can be connected to b_0 by piecewise smooth horizontal paths.

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- The Ambrose-Singer condition leads to the fact that H is a Lie group, acting transitively on M as isometries. Hence M is a homogeneous Riemannian manifold.

- The proof also shows that the set $\mathcal{AS}$ of all Ambrose-Singer metric connections on M are in one one correspondence with the conjugacy classes of connected Lie subgroups H of $I(M)$ which acts transitively on M, which may or may not be unique.

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- Clearly, the Levi-Civita connection will be Ambrose-Singer if and only if M is (locally) symmetric. So for non-symmetric locally homogeneous spaces, $\mathcal{AS}$ does not contain the Levi-Civita connection. When will it be unique?
- Also, the Ambrose-Singer connection corresponding to the identity component $I_0(M)$ is uniquely determined, which will be called the *canonical Ambrose-Singer connection*. What kind of geometric and algebraic properties will the canonical Ambrose-Singer connection possess?

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- Let $\widetilde{F}(M) \subseteq F(M)$ be the subbundle of all unitary frames of M, namely, points in $F(M)$ of the form $b = (x; \varepsilon_1, \ldots, \varepsilon_n, J\varepsilon_1, \ldots, J\varepsilon_n)$. Following the same line of argument by Ambrose and Singer, Sekigawa proved that

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- **Theorem (Sekigawa, 1978):** A complete, simply-connected Hermitian manifold (M^n, g) is homogeneous (namely, the group of holomorphic isometries acting transitively on M) if and only if there is a Hermitian (meaning that $\nabla g = 0$, $\nabla J = 0$) connection ∇ on M which is Ambrose-Singer (namely, its torsion and curvature are both parallel under ∇).

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- Similar questions can be phrased for the Bismut connection, or t-Gauduchon connection which is defined as $(1-\frac{t}{2})\nabla^c + \frac{t}{2}\nabla^b$ for any $t \in \mathbb{R}$.
- Example 1: When g is Kähler, the Ambrose-Singer condition means that M is locally Hermitian symmetric. So any compact (locally) Hermitian symmetric space is a CAS manifold.

- Example 1 (cont.) A Hermitian symmetric space is a Hermitian manifold whose underlying Riemannian manifold is symmetric. Hermitian symmetric spaces are products of $\mathbb{C}^n$ with irreducible factors of compact or non-compact type: there are four classic sequences, and two pairs of exceptional ones.

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- Let $V_{m\times n} \cong \mathbb{C}^{mn}$ be the space of all $m \times n$ complex matrices. The four classic sequences of irreducible Hermitian symmetric spaces of non-compact type (also known as bounded symmetric domains) are:

$$\begin{aligned}
D^1_{m,n} &= \{z \in V_{m\times n} \mid I_m - zz^* > 0\} \\
D^2_{n} &= \{z \in V_{n\times n} \mid {}^t z = z,\ I_n - zz^* > 0\} \\
D^3_{n} &= \{z \in V_{n\times n} \mid {}^t z = -z,\ I_n - zz^* > 0\} \\
D^4_{n} &= \{z \in V_{1\times n} = \mathbb{C}^n \mid |z| < 1, |{}^t z z|^2 + 1 - 2|z|^2 > 0\}
\end{aligned}$$

- Their compact dual are: $D^1_{m,n} \longleftrightarrow Gr(m, m+n)$, $D^4_n \longleftrightarrow \mathbb{Q}^n \subseteq \mathbb{CP}^{n+1}$, while D^2_n and D^3_n are totally geodesic subspaces in $D^1_{n,n}$, dual respectively to $P^+_n = Sp(n)/U(n)$ and $P^-_n = SO(2n)/U(n)$, which are totally geodesic submanifolds in $Gr(n, 2n)$.

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- Hua L.K., *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains,* American Mathematical Society, Providence, RI, 1963.
- two exceptional bounded symmetric domains: D^5_{16} and D^6_{27}.

- Example 2: Assume that (M^n, g) is a compact Chern flat manifold. By Boothby's Theorem (1958), the Chern connection ∇^c will have parallel torsion, hence is an Ambrose-Singer connection, and $M = G/\Gamma$, where G is a complex Lie group (equipped with a compatible left-invariant metric), and Γ is a discrete subgroup of the automorphism group of G.

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- Example (Iwasawa manifold): Consider the complex Lie group:

$$G(\mathbb{C}) = \{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{C}\},$$

$$M^3 = G(\mathbb{C})/G(\mathbb{Z} + i\mathbb{Z}).$$

$$\omega = i\{dx \wedge \overline{dx} + dz \wedge \overline{dz} + (dy - xdz) \wedge \overline{(dy - xdz)}\}.$$

- So what the aforementioned question asks is really the following: are there any CAS manifold other than locally Hermitian symmetric spaces or Chern flat ones (or their product)?

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- A famous and important result in homogeneous Riemannian manifold theory is the following:

  Theorem (Alekseevskii-Kimelfeld, 1975): For any complete homogeneous Riemannian manifold, if it is Ricci flat, then it is flat.

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- Note that any compact quotient of $M_1 \times M_2$ as above is obviously a *CAS* manifold. So the above can be viewed as a rigidity result for *CAS* structure.

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- The proof contains three parts: a) extract the Kähler de Rham factors; b) the non-Kähler factors are Chern Ricci flat; c) an algebraic analogue to Alekseevskii-Kimelfeld's theorem: an irreducible symmetric holonomy system is flat if it is Ricci flat.

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- The third part is to establish an algebraic analogue to the Alekseevskii-Kimelfeld Theorem, using Simons' holonomy system (Annals, 1962), who used it to give an intrinsic proof of Berger's holonomy theorem.

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- The second part is by the construction of a generic holomorphic symplectic form which leads to the Ricci flatness on the non-Kähler de Rham factors.
- The third part is to establish an algebraic analogue to the Alekseevskii-Kimelfeld Theorem, using Simons' holonomy system (Annals, 1962), who used it to give an intrinsic proof of Berger's holonomy theorem.
- In the following we will briefly sketch the outline arguments for each of these three parts.

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- Let (M^n, g) be a *CAS* manifold. Denote by T and R the torsion and curvature of the Chern connection ∇^c. Under any unitary frame $\{e_i\}$ of type $(1,0)$ complex tangent vectors, their only non-trivial components are:

$$T(e_i, e_j) = \sum_{k=1}^{n} T_{ij}^k e_k, \qquad R_{e_i,\bar{e}_j} e_k = \sum_{\ell=1}^{n} R_{i\bar{j}k\bar{\ell}} e_\ell.$$

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- For $p \in M$, write $V_p = T_p^{1,0}M$, and denote by W_p the subspace generated by $T(X, Y)$ for all $X, Y \in V_p$, the *image distribution* of T, and by N_p the orthogonal complement of W_p, so $V = W \oplus N$. Also denote by $K_p \subseteq N_p$ the *kernel* of T:

$$K_p = \{X \in N_p \mid T(X, Y) = 0, \ \forall\, Y \in V_p\}.$$

- Since $\nabla^c T = 0$, the spaces W_p, N_p, K_p have constant dimensions for all $p \in M$, and form distributions on M, and one can show that they are all parallel under ∇^c. Under the Ambrose-Singer assumption, analysis based on the structure equations and Bianchi identities leads to the following:

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- **Proposition:** On a *CAS* manifold, then $K \oplus \overline{K}$ is parallel under the Levi-Civita connection D, and forms the complex tangent bundle of all the Kähler de Rham factors of M.

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- In particular, a *CAS* manifold is free of Kähler de Rham factors if and only if the Chern torsion has trivial kernel.

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- On each L_j there exists a parallel $(2,0)$-symplectic form which can be identified with the standard symplectic form on $\mathbb{C}^{2n_j}$ where $2n_j$ is the complex dimension of L_j.
- As a consequence, M is Chern Ricci flat.

- §5. **On symmetric holonomy system**

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- Recall that Simons' holonomy system is a triple (V, R, G), where V is a real vector space of dimension n equipped with an inner product, R an algebraic curvature operator on V obeying all the symmetry properties of a Riemannian curvature tensor including the first Bianchi identity, and G is a compact connected subgroup of $SO(n)$ so that its Lie algebra $\mathfrak{g}$ contains R_{xy} for any $x, y \in V$.

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- A holonomy system (V, R, G) is said to be *irreducible* if G acts irreducibly on V and it is said to be *symmetric* if for any $\gamma \in G$, $\gamma(R) = R$, namely, $R_{\gamma(x),\gamma(y)}\gamma(z) = \gamma(R_{x,y}z)$ for any $x, y, z \in V$ and any $\gamma \in G$.

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- Let (M^n, g) be a Riemannian manifold, ∇ a metric connection on M with curvature R. Fix $p \in M$ and let γ be a path from q to p. Also denote by γ the parallel transport along it. The group $G \subseteq SO(T_pM) \cong SO(n)$ generated by γ for all closed paths from q to p is the holonomy group of ∇. Denote by $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g} \subseteq \mathfrak{so}(T_pM) \cong \mathfrak{so}(n)$.

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- **Theorem (Ambrose-Singer):** Let ∇ be a metric connection on a Riemannian manifold (M^n, g), and denote by R the curvature tensor of ∇. Then for any fixed $p \in M$, the holonomy algebra $\mathfrak{g}$ is generated by $\gamma(R^q_{x,y}) := \gamma \circ R_{x,y} \circ \gamma^{-1}$, for any $q \in M$, any $x, y \in T_qM$, and any path γ from q to p.

- In particular, for a *CAS* manifold (M^n, g), let R and G be the curvature and the restricted holonomy group of the Chern connection ∇^c, respectively, and let $V = T_pM$ where $p \in M$ is a fixed point. Then (V, R, G) forms a holonomy system in Simons' sense, and it is actually symmetric since $\nabla^c R = 0$. By Weyl's complete reducibility theorem, one may restrict to subbundles thus assume that the system is also irreducible.

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- The key point in the third step is the following algebraic analogue of Alekseevskii-Kimelfeld Theorem (which is a statement about the Levi-Civita connection thus cannot be applied here):

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- The key point in the third step is the following algebraic analogue of Alekseevskii-Kimelfeld Theorem (which is a statement about the Levi-Civita connection thus cannot be applied here):

- **Theorem:** Let (V, R, G) be an irreducible symmetric holonomy system. If R is Ricci flat, then $R = 0$.

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- In summary, we have shown that for compact Hermitian manifolds, if the Chern connection is Ambrose-Singer, then its universal cover must be the product of Hermitian symmetric spaces with complex Lie groups (Chern flat). In other words there are no 'non-trivial' examples. Also, as a consequence, the AK type theorem holds for the Chern connection, namely:

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- In summary, we have shown that for compact Hermitian manifolds, if the Chern connection is Ambrose-Singer, then its universal cover must be the product of Hermitian symmetric spaces with complex Lie groups (Chern flat). In other words there are no 'non-trivial' examples. Also, as a consequence, the AK type theorem holds for the Chern connection, namely:

- **The Main Corollary:** Given a complete Hermitian manifold, if its Chern connection is Ambrose-Singer and Ricci flat, then its is (Chern) flat.

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- **Question:** If a compact (locally homogeneous) Hermitian manifold has vanishing *first, second and third* Chern Ricci, then must it be Chern flat?

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- Next we consider similar questions for the Bismut connection ∇^b and the t-Gauduchon connection $\nabla^{(t)}$ for $t \neq 0, 2$.

- §6. **Bismut Ambrose-Singer connections**

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- Now let us switch from Chern to Bismut, and consider the *BAS* manifolds, which means compact (or complete) Hermitian manifolds (M^n, g) whose Bismut connection has parallel torsion and curvature.

- §6. **Bismut Ambrose-Singer connections**
- Now let us switch from Chern to Bismut, and consider the *BAS* manifolds, which means compact (or complete) Hermitian manifolds (M^n, g) whose Bismut connection has parallel torsion and curvature.
- When g is Kähler, it is locally Hermitian symmetric spaces. It is well-known that *Bismut flat* manifolds have parallel torsion thus are BAS. Bismut flat manifolds are quotients of Samelson spaces: (G, J, g) where G is a Lie group equipped with a bi-invariant metric g, and J a left-invariant complex structure compatible with g. Such spaces are known to be Bismut flat since the work of Samelson and Pittie in 1950s and 1980s, and the converse was shown by Wang-Yang-Z in 2020.

- Besides locally Hermitian symmetric spaces and Bismut flat manifolds (the 'trivial' examples), there are actually 'lots' of other BAS manifolds, starting in complex dim 3.

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- Their universal covers form a subset of *naturally reductive homogeneous spaces*. The latter was studied by Agricola, Cleyton, Ferreira, Friedrich, Kowalski, Moroianu, Schoenmann, Storm, Swann, Tricerri, Vanhecke and others. Classification was obtained in low dimensions (≤ 8), but in general dim it might not be feasible.

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- **Question:** Can one characterise (or in some sense classify) all BAS manifolds? If a BAS manifold has vanishing (first and third) Bismut Ricci, then must it be Bismut flat?

- Towards the second question above, we have the following partial result:

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- **Proposition:** For $n = 2$, BAS surfaces are exactly Vaisman surfaces with constant scalar curvature, or equivalently locally Hermitian homogeneous Vaisman surfaces. Vaisman surfaces are fully classified by Belgun. Such a surface is Bismut Ricci flat only if it is Bismut flat (isosceles Hopf surfaces).

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- Note that for a Hermitian connection, the Ricci curvature (as a metric connection) is also called the third Ricci. There are also first and second Ricci. In general the three Ricci are not equal (for BAS manifolds, the first and second Ricci are always equal).

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- The question is related to their theory of generalized Riemannian geometry and it corresponds to special types of generalized Einstein structures. They showed that it is true in dim ≤ 4. But in dim 5 or higher, Podestà and Raffero constructed counterexamples.
- To be more precise, Podestà and Raffero constructed an explicit sequence $M_{p,q}$ of compact homogeneous 5-manifolds with 3-form H, such that the metric connection ∇^H with torsion H is Ricci flat but not flat. Here $p \geq q$: relatively prime positive integers.

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- Recall that CYT, which stands for *Calabi-Yau with torsion*, means that R^b has zero first Ricci, or equivalently, the restricted holonomy group of the Bismut connection is contained in $SU(n)$.
- The conjecture was previously proved by Yau-Zhao-Z in dimension 3, and was confirmed by Zhao-Z in general dimensions recently. Note that the situation is slightly different as BKL metrics may not be locally homogeneous, but on the other hand BKL is rather restrictive and all three Bismut Ricci are equal.

- By solution to the AOUV Conjecture (Zhao-Z), Bismut Kähler-like (BKL) is equivalent to $\nabla^b T^b = 0$ plus pluriclosedness (SKT). BKL metrics always have parallel torsion but may not have parallel curvature. So BKL is not contained in BAS.

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- As in the Chern or Bismut case, one can ask the following:
- **Question:** For any $t \neq 0, 2$, can one classify all compact t-GAS manifolds? If such a manifold has vanishing t-Gauduchon Ricci, then must it be t-Gauduchon flat (hence Kähler by the result of Lafuente-Stanfield)?

- Here we used the beautiful result:

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- In fact their theorem is much stronger: t-Gauduchon flat can be relaxed to t-Gauduchon Kähler-like, meaning that the curvature tensor of $\nabla^{(t)}$ obeys all Kähler symmetries. Also, they showed that the result remains valid when the compactness assumption is dropped, provided $t \neq \frac{2}{3}, \frac{4}{5}$.

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- For any $t \neq 0, 2$, there are examples of compact, non-Kähler (hence non t-Gauduchon flat) t-GAS manifolds.
- Such manifolds seem to be highly restrictive, so it might be possible to fully classify them?
- **Proposition:** For any $t \neq 0, 2$, any compact t-GAS manifold is always balanced, and if it has vanishing first and third t-Gauduchon Ricci, then it is Kähler.

- Let us consider examples of compact BAS threefolds (M^3, g) which are *balanced:* $d(\omega^{n-1}) = 0$. Assume that g is non-Kähler. Then according to the rank r_B of the B-tensor: $B_{i\bar{j}} = \sum_{r,s=1}^{n} T^j_{rs}\overline{T^i_{rs}}$, either $r_B = 3$ and $M = SO(3,\mathbb{C})/\Gamma$ is Chern flat, or $r_B = 1$ and M^3 is a Fano threefold with index 4 or 2 (del Pezzo threefolds: 7 types fully classified by Fujita), it turns out that M must be $\mathbb{P}(T_{\mathbb{P}^2})$, the flag threefold, and g must be the Wallach metric which is the Kähler-Einstein metric minus the a globally $(1,1)$-form, corresponding to the null-correlation bundle which is the generating section of $H^0(M^3, \Omega \otimes K^{-\frac{1}{2}}) \cong \mathbb{C}$. The case $r_B = 2$ (the middle type) contains most of the balanced examples.

- For balanced BAS threefold (M^3, g) of middle type, denote the complex structure by J. Then either M^3 or a double cover of it will admit another complex structure I so that (M, g, I) is a Hermitian threefold that is Vaisman. Also, $IJ = JI$, and both Hermitian threefolds share the same Bismut connection (thus they are both BAS).

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- In particular, there is a family of solvmanifolds that fall into this class: $A_{u,a}$ with structure equations (where φ is a unitary left-invariant coframe):

$$A_{u,a}: \begin{cases} d\varphi_1 = 0 \\ d\varphi_2 = \varphi_2 \wedge (u\varphi_1 - \overline{u}\,\overline{\varphi}_1) + ia\varphi_2 \wedge (\varphi_3 + \overline{\varphi}_3) \\ d\varphi_3 = \varphi_1 \wedge \overline{\varphi}_1 - \varphi_2 \wedge \overline{\varphi}_2 \end{cases}$$

where a is a real number and u is a complex number.

Thank You!