# Lecture 11. The Khovanov homology

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**杂诗三首 · 其二**

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The present lecture is devoted to a crucial breakthrough in knot theory made in the very end of 20-th century: the Khovanov homology.

Khovanov proposed the following idea: to generalise the notion of Kauffman's bracket using some formal complexes and their homology groups.
First, we give a slight modification of the Jones polynomial and Kauffman bracket due to Khovanov. The (unnormalised) Jones polynomial is the graded Euler characteristic of the Khovanov complex [7]. The version of the Jones polynomial used here differs slightly from that proposed below. They become the same after a suitable variable change.
The Khovanov homology is the first instance of the idea of categorification: instead of a polynomial invariant (which can be treated just as a set of coefficients) one associates with any knot diagram a cell complex whose (graded) homology groups are invariant under Reidemeister moves and whose (graded) Euler characteristic is the Jones polynomial.

The main feature of the categorification is its functoriality. Having just an (invariant) number, say, 5, it is impossible to map this number to another number, say, 3; however, a 5-dimensional space can be mapped to a three-dimensional space. Hence, we get a much deeper invariant structure that just (graded) Euler chararcteristics which are just numbers (polynomials).

After the categorification, people invented further enhancements of knot invariants, say, spectrification and Khovanov homotopy where with a knot diagram we associate some topological space (spectrum) whose homotopy type is invariant under Reidemeister moves[18]. This gives rise to a much deeper structure of invariants than the set of polynomial coefficients.

Another idea of categorification came simultaneously in the work by Ozsvath and Szabo: they categorified the Alexander polynomial[19]. The main difference was that the Khovanov homology was defined in a purely combinatorial manner, whence the Ozsvath-Szabo homology (that they called Heegaard-Floer homology) had origins in geometry.

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# Kauffman bracket

The axioms for the Kauffman bracket will be the following:

1. The Kauffman bracket of the empty set (zero–component link) equals 1.
2. $\langle L \sqcup \bigcirc \rangle = (q + q^{-1})\langle L \rangle$.
3. For any three diagrams $L =$ , $L_A =$ , $L_B =$  of unoriented links, we have

$$\langle L \rangle = \langle L_A \rangle - q\langle L_B \rangle.$$

Denote the state A of a vertex to be the 0–smoothing, and the state B to be the 1–smoothing. If the vertices are numbered, then each way of smoothing for all crossings of the diagram is thought to be a vertex of the n–dimensional cube $\{0, 1\}^{\mathcal{X}}$, where $\mathcal{X}$ is the set of vertices of the diagram.

# Kauffman bracket

Let the diagram L have $n_+$ positive crossings and $n_-$ negative crossings; denote the sum $n_+ + n_-$ by n (that is the total number of crossings).
Denote the unnormalised Jones polynomial by

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ + 2n_-} \langle L \rangle.$$

Let the Jones polynomial (denoted now by J, according to [3]) be defined as follows:

$$J(L) = \frac{\hat{J}(L)}{q + q^{-1}}.$$

Thus,

$$J(L) = (-1)^{n_-} q^{n_+ - 2n_-} \sum_s -q^{\beta(s)} (q + q^{-1})^{\gamma(s)-1}.$$

## Remark 1.1

This normalised polynomial J differs from the Jones–Kauffman polynomial by a simple variable change: $a = \sqrt{(-q^{-1})}$. Namely,

$$(-a)^{-3(n_+-n_-)}\sum_s a^{\alpha(s)-\beta(s)}(-a^2-a^{-2})^{\gamma(s)-1}$$

$$= (-1)^n a^{-3(n_+-n_-)}\sum_s a^{-2\beta(s)+n}\cdot(q+q^{-1})^{\gamma(s)-1}$$

$$= (-1)^n a^{4n_- -2n_+}\sum_s (-q)^{\beta(s)}(q+q^{-1})^{\gamma(s)-1}$$

$$= (-1)^{n_-}q^{n_+-2n_-}\sum_s(-q)^{\beta(s)}(q+q^{-1})^{\gamma(s)-1}.$$

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Khovanov's categorification idea is to replace polynomials by graded vector spaces with some "graded dimension". This makes the Jones polynomial a homological object. On the other hand, the graded dimension allows us to consider the invariant to be constructed as a polynomial in two variables.
We shall construct a "Khovanov bracket" (unnormalised complex that plays the same role for the Khovanov complex as the Kauffman bracket for the Jones polynomial). This will be denoted by double square brackets.

Let us start with the basic definitions and introduce the notation.

Let a linear space $\mathrm{M}$ (or a free module $\mathrm{M}$ over a ring $\mathcal{R}$) have a preferred quantum grading $\mathrm{q}$. Then one has the following decomposition $\mathrm{M} = \bigoplus_{\mathrm{i}} \mathrm{M}_{\mathrm{i}}$, where $\mathrm{M}_{\mathrm{i}}$ is the homogeneous component of grading $\mathrm{i}$. By the graded dimension of the space $\mathrm{M}$ we mean the polynomial $\mathrm{qdim}\,\mathrm{M} = \sum_{\mathrm{i}} \mathrm{q}^{\mathrm{i}} \dim \mathrm{M}_{\mathrm{i}}$.

For such complexes there are naturally defined operations of the height shift $\mathcal{C} \mapsto \mathcal{C}[k]$ and the grading shift $\mathcal{C} \mapsto \mathcal{C}\{l\}$ defined according to the following rules: $(\mathcal{C}[k])^{i,j} = \mathcal{C}^{i-k,j}$; $(\mathcal{C}\{l\})^{i,j} = \mathcal{C}^{i,j-l}$. In the first case, together with chains, all differentials are shifted accordingly (i.e. the differential $\partial_i$, which was acting from $\mathcal{C}^{i,*}$ to $\mathcal{C}^{i+1,*}$, will now act in the same way from $\mathcal{C}^{i-k,*}$ to $\mathcal{C}^{i+1-k,*}$). By the graded Euler characteristic of the complex $\mathcal{C}^{i,j}$ we mean the alternating sum of the graded dimensions of the chain spaces, or, which is the same, the graded dimension of the homology groups. For chain spaces, we have:

$$\chi_q(\mathcal{C}^{i,j}) = \sum_i (-1)^i \mathrm{qdim}\, \mathcal{C}^i = \sum_{i,j} (-1)^i q^j \dim \mathcal{C}^{i,j}.$$

For such complexes, for every bigraded dimension $(i,j)$ there is the (co)homology group $H^{ij}(\mathcal{C})$ which is defined as the quotient module of the corresponding module of cycles by the submodule of boundaries.

**Definition 1.2**

Two graded (respectively, bigraded) complexes $\mathcal{C}$ and $\mathcal{C}'$ are called quasi-isotopic, if there exist two bigrading preserving maps $\mathrm{f}\colon \mathcal{C} \to \mathcal{C}'$, $\mathrm{g}\colon \mathcal{C}' \to \mathcal{C}$ together with a map u decreasing the height by one and preserving the second grading if such exists, such that $\mathrm{f} \circ \mathrm{g} = \mathrm{Id}_{\mathcal{C}'}$, and $\mathrm{g} \circ \mathrm{f} - \mathrm{Id}_{\mathcal{C}} = \mathrm{d} \circ \mathrm{u} + \mathrm{u} \circ \mathrm{d}$. Here, $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $\mathrm{Id}_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}'$ denote the corresponding identity maps.

Homology groups of quasi-isomorphic complexes are isomorphic.

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Let $L, n$ and $n_\pm$ be defined as before. Let $\mathcal{X}$ be the set of all crossings of $L$. Let $V$ be the graded vector space generated by two basis elements $v_\pm$ of degrees $\pm 1$, respectively. Thus, $\mathrm{qdim} V = q + q^{-1}$.

## Definition 2.1

By bifurcation cube we understand the cube $\{0,1\}^{\mathcal{X}}$ where each vertex is assigned the number of circles (as in the state cube), and each edge indicates which circles bifurcate when passing from a state to an adjacent one. The height of a state (a vertex of the cube) is the number of B-smoothings.

We orient the edges of the cube as the sum of coordinates increases (i.e. from an A-smoothing to a B-smoothing).
We also call it $1 = V_+, X = V_-$.

With every vertex $\alpha$ of the bifurcation cube $\{0,1\}^{\mathcal{X}}$ we associate the graded vector space $V_\alpha(L) = V^{\otimes k}\{r\}$, where k (formerly $\gamma$) is the number of circles in the smoothing of L corresponding to $\alpha$ and r is the height $|\alpha| = \sum_i \alpha_i$ of $\alpha$ (so that qdim $V_\alpha(L)$ is the polynomial that appears at the vertex $\alpha$ in the cube). Now, let the r–th chain group $[[L]]^r$ be the direct sum of all vector spaces at height r, that is $\oplus_{\alpha:\ |\alpha|=r} V_\alpha(L)$.

Let us forget for a moment that [[L]] is not endowed with a differential, and hence, is not a complex. Set $\mathcal{C} := [[L]][-n_-]\{n_+ - 2n_-\}$.

**Remark 2.2**

It is easy to show that for a complex C the graded dimension $\chi_q(C)$ equals the alternating sum of the graded dimensions of its chain groups. This is quite analogous to the case of the usual Euler characteristics.

Thus, we can calculate the graded Euler characteristic of C (taking into account only its graded chains); the differential will be introduced later.

**Theorem 2.3**

The graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalised Jones polynomial $\hat{J}$ of L.

Proof. This theorem is almost trivial. One should just take the alternating sum of graded dimensions of chain groups and mention that $\mathrm{qdim}(V^{\oplus n}) = n\,\mathrm{qdim}(V)$. The remaining part follows straightforwardly. □

Now, let us prove that the Khovanov complex is indeed a complex. So, let us introduce the differentials for it. First, we set all $[[L]]^r$ to be the direct sums of the vector spaces appearing in the vertices of the cube with precisely r coordinates equal to 1.
The edges of the cube $\{0,1\}^{\mathcal{X}}$ can be labelled by sequences in $\{0,1,*\}$ of length n having precisely one $*$. This means that the edge connects two vertices, obtained from this sequence by replacing $*$ with one or zero.

### Definition 2.4

The height $|\xi|$ of the edge $\xi$ is defined to be the height of its tail (the end having smaller height).

Thus, if the maps for the edges are called $\mathrm{d}_\xi$, then we get $\mathrm{d}^\mathrm{r} = \sum_{\{|\xi|=\mathrm{r}\}} (-1)^\xi \mathrm{d}_\xi$.

**Definition 2.5**

The cube with partial differentials $\mathrm{d}_\xi$ going along edges in the coordinate increasing direction is called commutative, if each two-dimensional face of this cube is a commutative diagram and anticommutative, if each two-dimensional face is an anticommutative diagram.

Now, we have to explain the sign $(-1)^\xi$ and to define the edge maps $\mathrm{d}_\xi$. Indeed, in order to get a "good" differential operator d, such that $\mathrm{d} \circ \mathrm{d} = 0$, it suffices to show that all square faces of the cube anticommute.
This can be done in the following way. First, we make all faces commutative, and then we multiply each $\mathrm{d}_\xi$ by $(-1)^\xi = (-1)^{\sum_{\mathrm{i}<\mathrm{j}} \xi_\mathrm{i}}$, where j is the position of $*$ in $\xi$.

**Exercise 2.6**

Show that such coefficients really make any commutative cube anticommutative.

Thus, we should find maps that can make our cube commute. Each edge represents some switch of the state for our diagram at some vertex. So, this means either dividing one cycle into two cycles, or joining two cycles together. In these cases, we shall use the comultiplication $\Delta$ and multiplication $m$ maps defined as follows. The map $m : V \otimes V \to V$:

$$\begin{cases} 1 \otimes X \mapsto X, 1 \otimes 1 \mapsto 1, \\ X \otimes 1 \mapsto X, X \otimes X \mapsto 0. \end{cases} \tag{1}$$

The map $\Delta : V \to V \otimes V$

$$\begin{cases} 1 \mapsto 1 \otimes X + X \otimes 1 \\ X \mapsto X \otimes X. \end{cases} \tag{2}$$

Because of the degree shifts, our maps $m$ and $\Delta$ are chosen to have degree $(-1)$.

# A 1+1-dimensional TQFT

By means of the comultiplication $\Delta$ and the multiplication m we can obtain a mapping from cobordisms between oriented 1-manifolds to vector spaces (or commutative modules) with $\Delta$ and m. More precisely, let us consider a cobordism W with 1-dimensional boundaries $M, N$. Note that M and N are disjoint unions of copies of $S^1$. If M (N) is a disjoint union of m (n) copies of $S^1$, then

$$M \mapsto V^{\otimes m} \text{ and } N \mapsto V^{\otimes n}.$$

Each saddle point gives the comultiplication $\Delta$ and the multiplication m as described in Fig. 1.

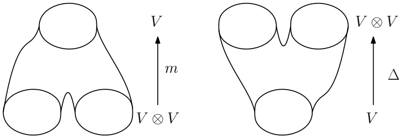


Figure 1: Saddles and the comultiplication $\Delta$ and the multiplication m

# A 1+1-dimensional TQFT: continued

For example, for a given cobordism from $S^1 \sqcup S^1$ to $S^1$ one can obtain morphism from $V \otimes V$ to $V$

$$V \otimes V \xrightarrow{\Delta \otimes Id} V \otimes V \otimes V \xrightarrow{Id \otimes m} V \otimes V \xrightarrow{m} V.$$

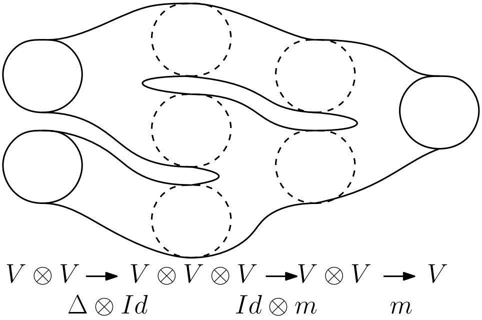


Figure 2: A cobordism from $S^1 \sqcup S^1$ to $S^1$ and the corresponding morphism from $V \otimes V$ to $V$

# A 1+1-dimensional TQFT: continued

That is, we obtain a functor from a category $\mathcal{C}_{1+1}$ to a category of vector spaces with morphisms $\Delta$ and m, where $\mathcal{C}_{1+1}$ is the category whose objects are closed, oriented 1-manifolds where a morphism $M \to N$ is an oriented surface W with $\partial W = M \sqcup N$.

More generally, a monoidal fuctor from $\mathcal{C}_{1+1}$ to the category of R-modules is called a 1+1-dimensional TQFT.

Now, the only thing to check is that the faces of our cube for $d_\xi$ (without $\pm 1$ coefficients) commute. This follows from a routine verification.

The most interesting fact here is the invariance of all homology groups of the Khovanov complex under all Reidemeister moves. Let us speak about this in more detail.

For a link diagram L, denote by Kh(L) the expression

$$\sum_r q^r \mathrm{qdim} \mathcal{H}^r(L).$$

**Remark 2.7**

When we wish to emphasise the field F or a commutative ring R, we write $Kh_F(L)$ or $Kh_R(L)$.

Our main example is the Khovanov homology over $\mathbb{Z}$.

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### Theorem 2.8 (Main theorem)

The graded dimensions of the homology groups $\mathcal{H}^r(L)$ are links invariants, hence Kh(L) is a link invariant polynomial (in the variables $t, q$) that gives the unnormalised Jones' polynomial being evaluated at $t = -1$.

Proof.
We shall restrict ourselves only to three versions of the Reidemeister moves (one of $\Omega_1$, one of $\Omega_2$, and one of $\Omega_3$). The other cases can be reduced to those we are going to consider.
In the case of the Kauffman bracket and the Jones polynomial, the invariance can be proved by reducing the Kauffman bracket of the "complicated case" of the move by using the rule $(\langle L\rangle = \langle L_A\rangle - q\langle L_B\rangle)$. Here we will do almost the same, but since we deal with complexes and homology rather than with polynomials, we must interpret it in another language. Namely, we are going to use the following "cancellation principle".

# Proof of Theorem 2.8: continued

Let $\mathcal{C}$ be a chain complex and let $\mathcal{C}' \subset \mathcal{C}$ be a subchain complex of $\mathcal{C}$. Then the following two statements hold.

**Lemma 2.9 (Cancellation principle)**

1. If $\mathcal{C}'$ is acyclic then $\mathrm{H}(\mathcal{C}) = \mathrm{H}(\mathcal{C}/\mathcal{C}')$.
2. If $\mathcal{C}/\mathcal{C}'$ is acyclic (has no homology) then $\mathrm{H}(\mathcal{C}) = \mathrm{H}(\mathcal{C}')$.

Both statements follow straightforwardly from the following exact sequence:

$$\cdots \to \mathrm{H}^{\mathrm{r}}(\mathcal{C}') \to \mathrm{H}^{\mathrm{r}}(\mathcal{C}) \to \mathrm{H}^{\mathrm{r}}(\mathcal{C}/\mathcal{C}') \to \dots$$

associated with the short exact sequence

$$0 \to \mathcal{C}' \to \mathcal{C} \to \mathcal{C}/\mathcal{C}' \to 0.$$

# Proof of Theorem 2.8: continued

Now, let us prove the invariance of $\mathrm{Kh}(\cdot)$ under the three Reidemeister moves.

Invariance under $\Omega_1$.

Consider the three diagrams , , and .

While computing $\mathcal{H}(\mathrm{P})$, we encounter the complex

$\mathcal{C} = [[$  $]] = \Big( [[$  $]] \xrightarrow{\mathrm{m}} [[$  $]]\{1\} \Big)$.

This means that the total n–dimensional cube for  is split into two $(n-1)$-dimensional cubes, corresponding to the two smoothed diagrams (one of them is shifted); the differentials between these two cubes are all represented via m by definition.

# Proof of Theorem 2.8: continued

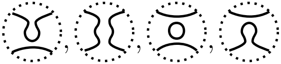
As we can easily see, all chains in  where the small circle ∘ is 1 ($v_+$), “kill” all cycles in  according to our differential, because 1 ($v_+$) plays the role of the unit element in V with respect to the multiplication m. Thus, the only homology groups we can have lie in [[]] when the small circle is marked by X ($v_-$). It is easy to see, that after the necessary normalisation, these homology groups precisely coincide with those of [[]].

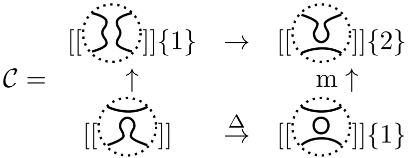
The case of the other curl  can be considered analogously.

# Proof of Theorem 2.8: continued

In the case of $\Omega_2$, we shall consider the one case. In this case, the

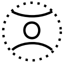
[[]] will be represented in the terms of brackets of

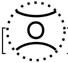
 and differentials between them:



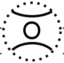

Thus, we have four cubes of codimension two and we know what the differentials in these small cubes look like: we may catch the cohomology elements in terms of these differentials. So, we only have to check whether they really represent homology groups in the big cube.

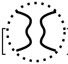
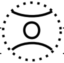
# Proof of Theorem 2.8: continued

The lower–left part of the diagram contains the diagram  (more precisely, all states corresponding to this local state).
Observation 1. It is easy to see that the terms of this state cannot give rise to homology groups of the complex: their differentials have non-trivial projection to $\{1\}$.

Observation 2. All terms corresponding to $[[$$\{1\}]]$ are not boundaries of terms corresponding to $[[$$]]$: the differential of each term from $[[$$]]$ also has an impact on $[[$$]]\{1\}$.

# Proof of Theorem 2.8: continued

Observation 3. The complex [[]]$\{1\}_1 \xrightarrow{m}$ [[]]$\{2\}$ is acyclic.

Observation 4. Each boundary element x in [[]]$\{2\}$ coming from an element $z \in$ [[]]$\{1\}$ has a unique compensating element in $y \in$ [[]]$\{1\}$ such that $\partial y = \partial z = x$. This follows from observation 3. Thus, there exists a y in this complex such that $\partial y = x$.

# Proof of Theorem 2.8: continued

Taking into account observations 2 and 4 we conclude that all homology groups containing elements from  are in one–to–one correspondence with homology groups of the complex C[[]].

It is easy to check that the complex $\mathcal{C}$ has no other homology groups (this follows from observations 1 and 3; the proof is left for the reader).

This results in the invariants of homology groups up to height and degree shifts. Taking into account the normalisation constants, we obtain the invariance of the Khovanov complex under the second Reidemeister move $\Omega_2$.

The invariance proof for the other cases of $\Omega_2$ is quite analogous to the case considered above. The direct calculation via $\Omega_2$ does not work, thus we have to use the cancellation method described above.

In the case of the third Reidemeister move $\Omega_3$ the situation is more difficult than the similar one for the case of the Kauffman polynomial. In this case we have the following local pictures; see Fig. 3.

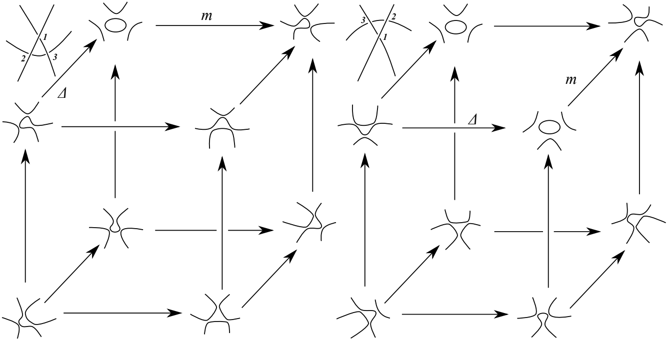


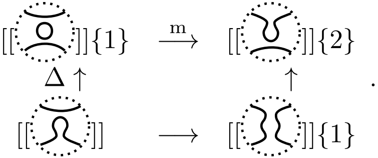

Figure 3: Behaviour of Khovanov's complex under $\Omega_3$

# Proof of Theorem 2.8: continued

Let us recall the invariance proof for the Jones one–variable polynomial under $\Omega_3$. First, we smooth one crossing and then we see that this invariance follows from the invariance under $\Omega_2$. We are going to do something similar: we consider our three–dimensional cubes and take their top layers that differ by a move $\Omega_2$ (bottom layers of these cubes coincide).

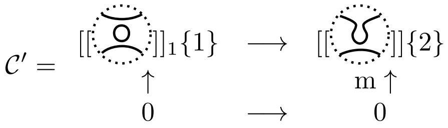
If we consider the situation that occurs while performing the move $\Omega_2$, we have the following complex.

The initial complex $\mathcal{C}$ looks like

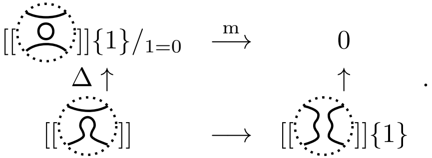


# Proof of Theorem 2.8: continued

This complex contains the subcomplex $\mathcal{C}'$ that looks as follows




The acyclicity of the complex $\mathcal{C}'$ is obvious.
After factorising the complex $\mathcal{C}$ by $\mathcal{C}'$, we obtain the complex



# Proof of Theorem 2.8: continued

Now, if we consider the special case of the top layer shown in Fig. 3, we see that the complex $\mathcal{C}'$ contains a subcomplex

$$\mathcal{C}''' = \begin{array}{ccc} \beta & \longrightarrow & 0 \\ \Delta \uparrow & \overset{\tau = d_{*0}\Delta^{-1}}{\searrow} & \uparrow \\ \alpha & \xrightarrow{d_{*0}} & \tau\beta, \end{array}$$

which is acyclic because $\Delta$ is an isomorphic map.
After this, we see that

$$(\mathcal{C}/\mathcal{C}')/\mathcal{C}''' = \begin{array}{ccc} \beta & \longrightarrow & 0 \\ \uparrow & \searrow & \uparrow \\ 0 & \longrightarrow & \gamma. \end{array}$$

# Proof of Theorem 2.8: continued

**Remark 2.10**

Here the arrow $\tau$ is not a differential. In the sequel, the diagonal arrow like $\beta = \tau\beta$ means that we identify two elements of the cube (arrows do not represent differentials).

# Proof of Theorem 2.8: continued

By the cancellation principle, we can perform this operation (factorising by C′ and C‴ defined for the top layers of the 3–cube) for the two cubes shown in Fig. 3 (only to the top layers of them). The resulting cubes are shown in Fig. 4.

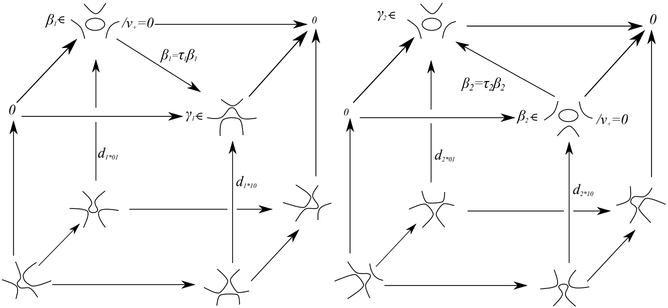



Figure 4: Invariance under $\Omega_3$

# Proof of Theorem 2.8: continued

Now, these two complexes are in fact isomorphic via the map $\mathfrak{Y}$ which keeps the bottom layers shown in Fig. 4. in their place and transposes the top layers by mapping the pair $(\beta_1, \gamma_1)$ to the pair $\beta_2, \gamma_2$.
The fact that $\mathfrak{Y}$ is really an isomorphism of spaces is obvious. To show that it is really an isomorphism of complexes, we need to know that it commutes with the edge maps. In this case, only the vertical edges require a proof. The proof of this fact, namely that $\tau_1 \circ \mathrm{d}_{1*01} = \mathrm{d}_{2*01}$ and $\mathrm{d}_{1*10} = \tau_2 \circ \mathrm{d}_{2*10}$, is left as an exercise.$\square$

### Definition 2.11

Let us call by the height $h(Kh(K))$ of the Khovanov polynomial of a link $K$ the difference between the leading and lowest non-zero quantum gradings of non-zero terms of Khovanov polynomial of $K$.

The height of the Khovanov polynomial justifies the estimates coming from the span of the Kauffman bracket polynomial. The latter is responsible for non-cancellability of the leading and lowest terms in the decomposition

$$\langle \bar{L} \rangle = \Sigma_s a^{\alpha(s)-\beta(s)}(-a^2 - a^{-2})^{\gamma(s)-1},$$

where the sum is taken over all states $s$ of the diagram $\bar{L}$, at the same time chains of the Khovanov complex are in natural one-to-one correspondence with monomials of the bracket multiplied by $(-a^2 - a^{-2})$.

By construction it is clear that

$$h(Kh(K)) - 2 \geq \frac{\mathrm{span}\langle K\rangle}{2}.$$

As we have said before, the Khovanov polynomial (with rational homology groups) is strictly stronger than the Jones polynomial. The example of two knots for which the Jones polynomial coincides and Khovanov's homology groups do not, is shown in Fig. 5.

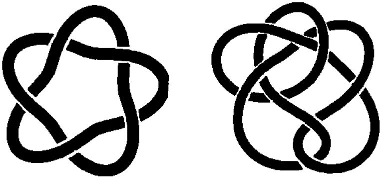


Figure 5: Khovanov's $\mathbb{Q}$–homology groups are stronger than the Jones polynomial

**Exercise 2.12**

Perform the calculation check for this example.

# Contents

We shall describe a slightly different approach to calculating (more precisely, to estimating) the Khovanov homology, thanks to which some properties of the Khovanov homology became clearer.
Let us formulate the lemma from the theory of algebraic complexes, we shall follow S.Wehrli [12].

### Lemma 3.1

Let $C_0$ and $C_1$ be graded complexes and $C_i = A_i \oplus B_i$, where the complexes $B_i$ have zero homology. Let $w\colon C_0 \to C_1$ be a map of chains preserving the grading, and let $w_{AA}\colon A_0 \to A_1$ be a “part” of the map w; i.e. the composition of the map w with the evident projection and embedding. Let A be a cone of the map $w_{AA}$, C be a cone of the map w, and B be contractible complex of type $B_0 \oplus B_1[1]$. Then the complexes C and $A \oplus B$ have the same homology.

The proof of this theorem is purely algebraic, it does not concern the "internal" structure of differentials in the complexes $A_i$ and $B_i$. The lemma is a key point in the proof of Theorem 3.2 about the spanning tree for the Khovanov complex.
The main idea of constructing the spanning tree leading to the proof of the theorem is the same as the Thistlethwaite idea which he used for constructing the spanning tree of the Kauffman bracket polynomial: It is necessary to take the bifurcation cube and split it into small subcubes corresponding to states from the set $\mathcal{V}_1$ of the states with one circle. After that we have to consider the Khovanov homology for each of these subcubes; i.e. the copies of the homology groups of the unknot and apply Lemma 3.1 to them repeatedly. We should apply this lemma at each splitting of the cube into two parts.

Let us describe this construction in more detail. We shall consider a non-normalised Khovanov complex of a link. In what follows we should take the “common normalising factor” out; i.e. shift the height and the grading.
Let K be a link diagram. Let us consider its non-normalised bifurcation cube [[K]] with the differential $\partial$. Enumerate all crossings of K and we shall split the cube [[K]] successively into cubes according to Thistlethwaite’s scheme. Namely, in the first step we investigate whether the first crossing is splitting (we call a crossing splitting, if under deleting the corresponding vertex from the diagram it becomes not connected) and, if it is not splitting, we pass to considering two cubes obtained from [[K]] by fixing the first coordinate.

These two cubes represent non-normalised Khovanov complexes for the diagrams $K_0$ and $K_1$ obtained from K by smoothings of type A and B. The Khovanov complex (unnormalised) for $K_i$ has some set of homology groups; if we consider $K_0$ and $K_1$ as non-separated complexes but compound parts of the Khovanov complex corresponding to K, we get some new differentials corresponding to passing from $K_0$ to $K_1$. Lemma 3.1 asserts that the initial (non-normalised) Khovanov complex for the diagram K has the same homology as the complex made only from homology of the complexes $K_0$ and $K_1$ (and as well as some acyclic part).

Further, we apply the second step: we consider the complexes $K_0$ and $K_1$ (as consistent parts of the new complex the homology of which coincides with the Khovanov homology of the link K) and investigate whether the corresponding diagrams split in the second crossing. If some of them (say, $K_0$) do not split, then we reconstruct the complex $K_0$ and get the complex of type $(K_{00} \to K_{01}) \oplus \langle \text{acyclic part} \rangle$.
We continue the process until we reach a diagram with all crossings smoothed. Each of these diagrams represents the unknot; therefore, we conclude that the Khovanov homology can be calculated with the help of a complex consisting of the Khovanov homology of the unknot. In terms of formula it looks like as follows:

**Theorem 3.2**

The non-normalised Khovanov complex of a link diagram K is isomorphic to some complex whose chain group looks like

$$\bigoplus_{s\in\mathcal{V}_1} \mathcal{A}[\beta(s)+w(K_s)]\{\beta(s)+2w(K_s)\}, \tag{3}$$

where $\mathcal{A}$ is the homology group of the unknot.

Later on, we shall use also the phrase Wehrli's complex, by bearing in mind the complex which is quasi-isomtopic to the Khovanov complex, the existence of the latter is given by Theorem 3.2.

# Contents

The Khovanov theory of knots described earlier in this chapter is not unique when considering what one can get with the help of the Kauffman model and the (anti)commutative state cube. The present section is devoted to a generalisation of the Khovanov theory which uses Frobenius extensions.

Frobenius extensions

Let $\mathcal{R}$, $\mathcal{A}$ be commutative rings, and let $\iota\colon \mathcal{R} \to \mathcal{A}$ be an embedding of the commutative rings such that $\iota(1) = 1$. The restriction functor taking $\mathcal{A}$-modules to $\mathcal{R}$-modules has right and left adjoint functors: the induction functor $\mathrm{Ind}(\mathrm{M}) = \mathcal{A} \otimes_{\mathcal{R}} \mathrm{M}$ and the coinduction functor $\mathrm{CoInd}(\mathrm{M}) = \mathrm{Hom}_{\mathcal{R}}(\mathcal{A}, \mathrm{M})$. One says that $\iota$ is a Frobenius mapping, if the induction functor coincides with the coinduction functor. Equivalently: the embedding $\iota$ is Frobenius if the restriction functor has a 3-sided dual functor. In this case one says also that the ring $\mathcal{A}$ is a Frobenius extension over $\mathcal{R}$ by means of $\iota$.

The following proposition takes place.

**Proposition 4.1 ([13])**

The embedding $\iota$ is Frobenius if there exist a mapping $\mathcal{A}$-bimodules $\Delta\colon \mathcal{A} \mapsto \mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}$ and a mapping $\mathcal{R}$-modules $\varepsilon\colon \mathcal{A} \mapsto \mathcal{R}$ such that $\Delta$ is a coassociative and commutative multiplication, herewith $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}$.

A Frobenius extension with a choice $\varepsilon$ and $\Delta$ is denoted by $\mathcal{F} = (\mathcal{R}, \mathcal{A}, \varepsilon, \Delta)$ and called a Frobenius system, [13].

Frobenius extensions are convenient for constructing the Khovanov homology theory for the following reasons. In the module $\mathcal{A}$ defined over the ring $\mathcal{R}$ there are two natural operations: multiplication and comultiplication, the operation $\Delta$.
We are going to use these operations for constructing the Khovanov homology theory for links. Meanwhile we (for evident reasons) restrict ourselves only to the case of commutative rings; moreover, we forget the operator $\varepsilon$ (this operator is used for defining invariants of cobordisms and proving functoriality). In other aspects we follow the paper [8] by Khovanov.

Khovanov construction for Frobenius extensions

As it was described earlier in this chapter the standard Khovanov theory is constructed over some arbitrary ring $\mathcal{R}$ (for example, the ring $\mathbb{Z}$ or the field $\mathbb{Q}$, or the field $\mathbb{Z}_p$); herewith the homology of the unknot is a graded two-dimensional module $\mathcal{A}$ over this ring, generated by vectors 1 ($v_+$) and X ($v_-$) having gradings $+1$ and $-1$, respectively.

Two maps are defined on these vectors: the multiplication m and comultiplication $\Delta$. If one shifts the gradings of vectors (this requires a slight change (renormalisation) in the construction of the homology theory), then one can set deg $1 = 0$, deg $X = 2$. Then the element 1 ($v_+$) can be considered as a unit (so us denote it by 1, and denote $v_-$ by X), and the multiplication and comultiplication defined earlier turn the module $\mathcal{A}$ into a Hopf algebra over $\mathcal{R}$, in which the multiplication is defined by rules $X^2 = 0$, and the comultiplication looks like $\Delta(1) = 1 \otimes X + X \otimes 1$, $\Delta(X) = X \otimes X$.
In [8] Khovanov solved the following problem: How can one find a condition for a couple of linear spaces $(\mathcal{A}, \mathcal{R})$ to get a link homology theory, where $\mathcal{R}$ is the basic coefficient ring and $\mathcal{A}$ (some Hopf algebra over $\mathcal{R}$) is the homology of the unknot (the main building bricks)? That means that we consider the state cube, with each vertex associated with a tensor power of $\mathcal{A}$ (over $\mathcal{R}$), with exponent equal to the number of circles in the given state, and define partial differentials by means of multiplication and comultiplication, and then add signs on edges and normalise the whole construction by grading shifts.

Khovanov showed that the invariance under the first Reidemeister move requires that $\mathcal{A}$ is two-dimensional as an $\mathcal{R}$-module and gave necessary and sufficient conditions for the existence of such a link homology theory.
In the same paper [8], it is shown that any such theory can be obtained by some operations (base change, twisting and duality) from the following solution:

1. $\mathcal{R} = \mathbb{Z}[\mathrm{h}, \mathrm{t}]$,
2. $\mathcal{A} = \mathcal{R}[\mathrm{X}]/(\mathrm{X}^2 - \mathrm{hX} - \mathrm{t})$,
3. $\deg \mathrm{X} = 2$, $\deg \mathrm{h} = 2$, $\deg \mathrm{t} = 4$,
4. $\Delta(1) = 1 \otimes \mathrm{X} + \mathrm{X} \otimes 1 - \mathrm{h}1 \otimes 1$,
5. $\Delta(\mathrm{X}) = \mathrm{X} \otimes \mathrm{X} + \mathrm{t}1 \otimes 1$.

As we see, the multiplication in the algebra $\mathcal{A}$ preserves the grading, and the comultiplication raises it by two.
We omit normalisations regulating these gradings.

We call this construction the universal $(\mathcal{R}, \mathcal{A})$-construction. The corresponding homology of a (classical oriented) link K will be denoted by $\mathrm{Kh}_{\mathrm{U}}(\mathrm{K})$.

M. Khovanov proved that all other cases followed from the universal $(\mathcal{R}, \mathcal{A})$-construction. First, he investigates Frobenius extensions for the invariance of the obtained homology theory under the first classical Reidemeister move $\Omega_1$. This leads it to two-dimensional $\mathcal{A}$ as an $\mathcal{R}$-module.

Later, M. Khovanov considers the universal topological construction by Bar-Natan [4], and constructs a functor from the topological category of Bar-Natan to the category of Frobenius extensions of rank two. The constructed functor is neither injective nor surjective, but it enjoys all nice properties needed for the invariance under the Reidemeister moves.

Thus M. Khovanov shows that any rank two Frobenius extension as above defines an extraordinary link homology theory. He shows also that any such theory without loss of information can be reduced to the universal theory described above by some algebraic operations.

We shall not go into the details of Khovanov's and Bar-Natan's constructions. We shall just consider the universal $(\mathcal{R}, \mathcal{A})$-construction.
Also, note that M. Khovanov also studied functoriality of his new homology theory, for example, its "good behaviour" under cobordisms (projective functoriality). To this end, besides multiplication and comultiplication operations, he also defined the unit and counit map and their transformations; we shall not touch on this subject.

# Contents

In the classification and tabulation of knots the important step is to describe diagrams having a minimal number of crossings. One of the main achievements in the development of knot theory is the Kauffman–Murasugi–Thistlethwaite theorem (Theorem 5.1) and the classification of alternating links by Menasco and Thistlethwaite [14] following from this theorem. (This theorem is introduced in Lecture 2.)

**Theorem 5.1 (Kauffman–Murasugi–Thistlethwaite theorem)**

The length of the Jones polynomial for a link with connected shadow is less than n or equal to n. The equality holds only for alternating diagrams without splitting points and connected sums of them.

In this section we shall prove theorems establishing the minimality of virtual and classical diagrams, see also [15, 16]. The inequality span $\langle K\rangle \leqslant 4n + 2(\chi - 2)$ for a virtual diagram K with n classical crossings and the atom with the Euler characteristic $\chi$ allowed one to prove the minimality in those cases, when the Euler characteristic could not be increased. If the inequality turns into the equality, then to decrease the number of crossings we have to increase the Euler characteristic of the atom or, the same, to decrease its genus. It turns out that by using Khovanov homology one can get estimates on the atom genus, at the same time in some cases one can see that this genus cannot be decreased. In this case the previous arguments together with non-reducibility of the genus lead to the minimality of the diagram.

We shall first mention the spanning tree theorem for Khovanov homology, proved independently by S.Wehrli [12] and A.Champanerkar and J.Kofman [17].
More precisely, in [12] it is shown that the Khovanov homology is isomorphic to the homology of a certain complex. Let $\mathcal{V}_1(\mathrm{K})$ be the set of single-circle states of the virtual diagram K. From this a generalisation of Theorem 3.2 follows.

**Lemma 5.2**

The non-zero Khovanov homology $\mathrm{Kh}(\mathrm{K})$ can have the bigrading only of the form $(\mathrm{C}_1 + \beta - \mathrm{w}, \mathrm{C}_2 + \beta - 2\mathrm{w} \pm 1)$, where w belongs to some finite set of integers, $\beta$ belongs to the set of values $\beta(\mathrm{s})$ over all states $\mathrm{s} \in \mathcal{V}_1(\mathrm{K})$, and $\mathrm{C}_1$, $\mathrm{C}_2$ are constants.

An important particular case of this lemma is the statement of the Khovanov homology thickness (thickness was first introduced by Shumakovitch [Shu2, Shu3]).
Consider a link diagram K and its Khovanov homology over a certain non-graded ring R. Denote by $t_{max}$ and $t_{min}$ the maximal and minimal values of $2x - y$ over all pairs x, y such that the homology group of K with the bigrading $(x, y)$ is non-trivial.

**Definition 5.3**

The thickness (width) $T_R(K)$ of the Khovanov complex is $(t_{max} - t_{min})/2 + 1$.

**Remark 5.4**

This quantity is an integer for all links.

Later on, by a diagonal we call the set of pairs of integer numbers $(x, y)$ for which the number $2x - y$ is constant. Among diagonals there are the extreme left and the extreme right, at which the number $2x - y$ is minimal and maximal, respectively. Thus, the thickness measures the number of diagonals between two extreme diagonals.

**Definition 5.5**

By thickness (width) $T(K)$ of the link diagram $K$ we mean the maximum of all $T_R(K)$ over all rings $R$ without additional grading.

From Lemma 5.2 and the definition of atom, we get the following lemma.

**Lemma 5.6**

For any diagram $K$ (with a connected atom) of a link we have: $T(K) \leqslant g(K) + 2$, where $g(K)$ is the genus of the atom corresponding to $K$.

**Definition 5.7**

Let us call a link diagram K 2-complete, if $T(K) = g(K) + 2$.

Indeed, for an estimate of the number of diagonals of the Wehrli complex (see Theorem 3.2) it is necessary for us to estimate the range of numbers $\beta(s)$ over all states $s \in \mathcal{V}_1(L)$. It is easy to see that in the case of alternating link diagrams all these numbers equal each other (this leads to the presence of two diagonals $t_{max}$ and $t_{min}$ such that $t_{max} = t_{min} + 2$), in the case of atoms with genus one the numbers $\beta(s)$ can equal $x$, $x+1$, $x+2$ for some $x$; in the case of atoms with the Euler characteristic $\chi$ they can take values in an interval from some number $x$ to $x + (2 - \chi)$.

Now we have the following

**Theorem 5.8**

Let $\mathrm{T(K)} = \mathrm{g} + 2$, $\mathrm{span}\,\langle \mathrm{K} \rangle = \mathrm{s}$. Then the number of crossings of any connected diagram equivalent to K cannot be smaller than $\mathrm{s}/4 + \mathrm{g}$. In particular, if a diagram with n crossings and the atom with genus g is 1-complete and 2-complete, then it is minimal.

The last assertion means that all diagrams for which two properties of "natural non-reducibility" hold (in the decomposition of the Kauffman bracket polynomial the leading and lowest terms are not equal to zero and in the Wehrli complex each of the two extreme diagonals has at least one non-trivial element of the Khovanov homology) are minimal.

# Contents

# Exercises

1. Calculate explicitly the Khovanov homology over $\mathbb{Z}$ for the Hopf link.
2. Calculate the integral Khovanov homology for trefoil knots. Is there any torsion?
3. Calculate the integral Khovanov homology for the figure eight knot.

# Contents

# Research problems:

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