YMSC Lectures Week 3

September 2024

1 Integral transforms

Fix $\mathsf{H} \in \mathfrak{D}_{al}$ (a Hamiltonian). Let $f : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(t)$, be a real piecewise-continuous function which decays faster than any power of t. For any quasi-local observable a we can define a new observable

$$\mathscr{I}_{\mathsf{H},f}(a) = \int_{-\infty}^{\infty} f(t)\alpha_{\mathsf{H}}(t)(a)dt. \tag{1}$$

It is easy to see that if a traceless, so is $\mathscr{I}_{\mathsf{H},f}(a)$, and that if a is anti-self-adjoint, so is $\mathscr{I}_{\mathsf{H},f}(a)$.

Lemma 1. If $a \in \mathscr{A}_{a\ell}$, then $\mathscr{I}_{\mathsf{H},f}(a) \in \mathscr{A}_{a\ell}$.

Sketch of a proof: pick an $r > 0$ and let $T = r/C$ for some constant C. Since $f(t)$ decays faster than any power, it is sufficient to check that

$$\int_{-T}^{T} f(t)\alpha_{\mathsf{H}}(t)(a)dt$$

can be approximated by a local observable with $\mathcal{O}(r^{-\infty})$ accuracy. By Lieb-Robinson bound, if we choose C sufficiently large, for any $t \in [-T, T]$ $\alpha_{\mathsf{H}}(t)(a)$ has tails outside a ball of radius r which are of order $\mathcal{O}(r^{-\infty})$. Therefore the integral over $[-T, T]$ also has tails of the same order.

Further, the map $\mathscr{I}_{\mathsf{H},f}$ continuous. More precisely, for any a-localized $\mathcal{A} \in \mathfrak{d}_{al}$ we have

$$\|\mathscr{I}_{\mathsf{H},f}(\mathcal{A})\|_{j,\alpha} \leq C_\alpha \|\mathcal{A}\|_{j,\alpha}, \quad \alpha \in \mathbb{N}_0 \tag{2}$$

where $C_\alpha > 0$ depends on H, f and a. This estimate implies that $\mathscr{I}_{\mathsf{H},f}$ is continuous but is stronger than continuity because C_α does not depend on j. In other words, the map $\mathscr{I}_{\mathsf{H},f}$ is equicontinuous w. r. to a family of metrics on $\mathfrak{d}_{al}$ labeled by j. This ensures that $\mathscr{I}_{\mathsf{H},f}$ extend to continuous chain maps $\mathscr{I}_{\mathsf{H},f} : C_\bullet \to C_\bullet$. On k-chains with $k \geq 0$ it is defined by

$$\mathscr{I}_{\mathsf{H},f}(\mathsf{a})_{j_0 \dots j_k} = \mathscr{I}_{\mathsf{H},f}(\mathsf{a}_{j_0 \dots j_k}), \tag{3}$$

while on derivations it is defined by

$$\mathscr{I}_{\mathsf{H},f}(\mathsf{A})^Y = \sum_Z \left(\mathscr{I}_{\mathsf{H},f}(\mathsf{A}^Z)\right)^Y. \tag{4}$$

If $\alpha_{\mathsf{H}}(t)$ preserves a state ψ, then $\mathscr{I}_{\mathsf{H},f}$ preserves the subspace $\mathfrak{d}_{al}^{\psi}$ of anti-self-adjoint traceless observables not exciting ψ and the subcomplex $C_\bullet^{\psi}$. Indeed, for any $a, b \in \mathfrak{d}_{al}$ we have

$$\langle[\mathscr{I}_{\mathsf{H},f}(a), b]\rangle_\psi = \int_{-\infty}^{+\infty} f(t)\langle[a, \alpha_{\mathsf{H}}(-t)(b)]\rangle_\psi dt. \tag{5}$$

Therefore if $a \in \mathfrak{d}_{al}^{\psi}$, then $\mathscr{I}_{\mathsf{H},f}(a) \in \mathfrak{d}_{al}^{\psi}$.

Finally, also have the following easy result:

$$\mathscr{I}_{\mathsf{H},f}([\mathsf{H}, a]) = -\mathscr{I}_{\mathsf{H},\frac{df}{dt}}(a). \tag{6}$$

2 Integral transforms in the presence of an energy gap

Now suppose ψ is a gapped ground state of H with gap $\geq \Delta > 0$. Then one can choose f so that $\mathscr{I}_{\mathsf{H},f}(a)$ does not excite ψ. Namely, we let $f(t) = w_\Delta(t)$ where w_Δ is an even continuous function whose Fourier transform is supported on some interval contained in $(-\Delta, \Delta)$. It will be convenient to normalize w_Δ so that $\int w_\Delta(t)dt = 1$.

Lemma 2. For any $a \in \mathfrak{d}_{al}$ and $b \in \mathfrak{d}_{al}$ we have

$$\langle \mathscr{I}_{\mathsf{H},w_\Delta}(a)b\rangle_\psi = \langle \mathscr{I}_{\mathsf{H},w_\Delta}(a)\rangle_\psi \langle b\rangle_\psi, \tag{7}$$

and thus $\mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A}) \in \mathfrak{d}_{al}^{\psi}$ and $\mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{N}_\bullet) \subseteq \mathcal{N}_\bullet^{\psi}$.

Proof. Let π be the GNS representation corresponding to ψ and dP_ω, $\omega \in \mathbb{R}$, be the projection-valued measure on $\mathbb{R}$ corresponding to the self-adjoint operator $\hat{H}$. Then

$$\begin{aligned}\langle \mathscr{I}_{\mathsf{H},w_\Delta}(a)b\rangle_\psi &= \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dt\, w_\Delta(t)e^{-i\omega t}\langle 0|\pi(a)dP_\omega \pi(b)|0\rangle = \\ &= \int_{-\Delta'}^{+\Delta'} d\omega \int_{-\infty}^{+\infty} dt\, w_\Delta(t)e^{-i\omega t}\langle 0|\pi(a)dP_\omega \pi(b)|0\rangle = \\ &= \int_{-\infty}^{+\infty} dt\, w_\Delta(t)\langle a\rangle_\psi\langle b\rangle_\psi = \langle \mathscr{I}_{\mathsf{H},w_\Delta}(a)\rangle_\psi\langle b\rangle_\psi. \end{aligned} \tag{8}$$

□

The proof is just a formalization of the observation that $\mathscr{I}_{\mathsf{H},w_\Delta}$ projects out all matrix elements of a which separate states with energy $\geq \Delta$ and in particular project out matrix elements of a between the ground state and excited states.

Next, let us show that if ψ is a gapped ground state for H, then one can choose the "energy density" h, $\partial\mathsf{h} = \mathsf{H}$, in such a way that h does not excite ψ.

Lemma 3. For any $\mathsf{H} \in \mathfrak{D}_{al}$ with a gapped ground state ψ there exists $\mathsf{h}^\psi \in C_0(\mathfrak{d}^\psi_{al})$ such that $\mathsf{H} = \partial\mathsf{h}^\psi$.

Proof. Suppose $\mathsf{H} = \partial\mathsf{h}$ for some $\mathsf{h} \in C_0(\mathfrak{d}_{al})$. Let $\mathsf{h}^\psi = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathsf{h})$. Then for any $a \in \mathscr{A}_{a\ell}$ we have

$$\partial\mathsf{h}^\psi(a) = \int_{-\infty}^{+\infty} w_\Delta(t)\alpha_{\mathsf{H}}(t)\mathsf{H}(\alpha_{\mathsf{H}}(-t)(a))dt = \\ = \int_{-\infty}^{+\infty} w_\Delta(t)\mathsf{H}(a)dt = \mathsf{H}(a). \quad (9)$$

□

Remark 1. The construction of h^ψ by means of the map $\mathscr{I}_{\mathsf{H},w_\Delta}$ is due to A. Kitaev ("Anyons in an exactly solved model").

Another useful choice of f is as follows. Let $w_\Delta(t)$ be as above. Let u_Δ be an odd piecewise-continuous function which decays faster than any power of t and satisfies

$$\frac{du_\Delta(t)}{dt} = w_\Delta(t) - \delta(t). \quad (10)$$

The Dirac δ on the r.h.s. is needed because $\int w_\Delta(t)dt = 1$, while the integral of the l.h.s. vanishes. Note that the Fourier transform of $u(t)$ satisfies

$$\tilde{u}_\Delta(E) = \frac{1}{iE} \quad (11)$$

for $|E| \geq \Delta$. So the matrix elements of $\mathscr{I}_{\mathsf{H},u_\Delta}(a)$ between states separated by energy $E \geq \Delta$ are simply the matrix elements of a times $(iE)^{-1}$.

Theorem 1. Let ψ be a gapped state. For any $\mathsf{Q} \in \mathfrak{D}^\psi_{al}$ there exists $\mathsf{q}^\psi \in C^\psi_1$ such that $\partial\mathsf{q}^\psi = \mathsf{Q}$.

Proof. Suppose $\mathsf{Q} = \partial\mathsf{q}$. Let $\mathsf{h}^\psi \in C^\psi_1$ be a 0-chain such that $\partial\mathsf{h}^\psi = \mathsf{H}$ (such a 0-chain exists by the preceding lemma). Let

$$\mathsf{q}^\psi = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathsf{q}) + \mathscr{I}_{\mathsf{H},u_\Delta}([h^\psi, \mathsf{Q}]).$$

The first term is in C^ψ_1 by Lemma 2. The second term is in C^ψ_1 because $[h^\psi, \mathsf{Q}] \in C^1_\psi$. Finally:

$$\partial\mathsf{q}^\psi = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathsf{Q}) + \mathscr{I}_{\mathsf{H},u_\Delta}([\mathsf{H},\mathsf{Q}]) = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathsf{Q}) - \mathscr{I}_{\mathsf{H},\frac{du_\Delta}{dt}}(\mathsf{Q}) = \mathsf{Q}.$$

Here we used the differential equation (11). □

Similarly, one can prove the following more general result:

Theorem 2. Let ψ be a gapped state. If $\mathsf{q} \in C_p^\psi$ satisfies $\partial \mathsf{q} = 0$, then there exists $\mathsf{p} \in C_{p+1}^\psi$ such that $\partial \mathsf{p} = \mathsf{q}$. In other words, the homology of the chain complex $(C_\bullet^\psi, \partial)$ vanishes.

3 The Hall conductance

Let A be a finite set and U_a, $a \in A$, be cones with apex 0 such that $\cup_a U_a = \mathbb{R}^2$ and $U_a \cap U_b = \{0\}$ unless $a = b$, $a = b+1$ or $b = a+1$. Let ψ be a gapped $U(1)$-invariant state. Let $\mathsf{Q} \in \mathfrak{D}_{al}^\psi$ be the electric charge (generator of $U(1)$ transformations). By the result of the previous section, there exists $\mathsf{q} \in C_1^\psi$ such that $\partial \mathsf{q} = \mathsf{Q}$.

Lemma 4. One can choose q so that $[\mathsf{Q}, \mathsf{q}] = 0$.

Proof. We take any q and average over $U(1)$, i.e. let

$$\mathsf{q}' = \frac{1}{2\pi} \int_0^{2\pi} \alpha_{\mathsf{Q}}(\phi)(\mathsf{q}) d\phi.$$

It is easy to check that q' satisfies all the conditions. □

Now let's form a 2-chain $[\mathsf{q}, \mathsf{q}]$ with components $[\mathsf{q}_i, \mathsf{q}_j]$. This 2-chain is actually a cycle thanks to $[\mathsf{Q}, \mathsf{q}] = 0$. Let's define the following derivations:

$$\mathsf{F}_{ab} = \sum_{i \in U_a, j \in U_b} [\mathsf{q}_i, \mathsf{q}_j], \quad a, b \in I.$$

Clearly, $\mathsf{F}_{ab} \in \mathfrak{D}_{al}^\psi$ and satisfies $[\mathsf{Q}, \mathsf{F}_{ab}] = 0$. Also, each F_{ab} decays rapidly away from $U_a \cap U_b$.

Lemma 5. F_{ab} is an inner derivation, i.e. there is $f_{ab} \in \mathfrak{d}_{al}{}^\psi$ such that $\mathsf{F}_{ab}(c) = [f_{ab}, c]$ for any $c \in \mathscr{A}_{a\ell}$.

Proof. If $b \neq a \pm 1$, this is clear. If $b = a+1$, we note that one hand $\mathsf{F}_{a,a+1}$ is localized near the cone $U_a \cap U_{a+1}$ and on the other hand $\mathsf{F}_{a,a+1} = -\sum_{b \neq a} \mathsf{F}_{b,a+1}$ and thus it is localized near $\cup_{b \neq a} U_b \cap U_{a+1}$. Therefore it is localized near 0. □

Thus $\psi(f_{ab}) \in i\mathbb{R}$ is well-defined.

Lemma 6. If $b \neq a \pm 1$, then $\psi(f_{ab}) = 0$.

Proof. For $b \neq a \pm 1$ we have an explicit formula for f_{ab}:

$$f_{ab} = \sum_{j \in U_a, k \in U_b} [\mathsf{q}_j, \mathsf{q}_k].$$

This sum is convergent, so we can first do the average over ψ and then sum. Since q_j does not excite ψ, the average of each summand vanishes, and thus $\psi(f_{ab}) = 0$. □

Thus we are left with the numbers $\psi(f_{a,a+1})$. There is no simple formula for $f_{a,a+1}$. Let us pick some numbers β_a and form a linear combination

$$\langle \mathsf{F}, \beta \rangle = \sum_a \beta_a \psi(f_{a,a+1}).$$

Lemma 7. This number is unchanged under $\beta_a \mapsto \beta_a + \gamma_a - \gamma_{a+1}$.

Proof. One can easily see that the change is

$$\sum_a \gamma_a \psi(f_{a,a+1} + f_{a,a-1}).$$

But since $\sum_a \mathsf{F}_{ab} = 0$, we have $f_{a,a+1} + f_{a,a-1} = -\sum_{b \neq a-1,a+1} f_{ab}$, and the average of the latter vanishes. □

Note that the set of numbers β_a modulo the above equivalence is isomorpic to $\mathbb{R}$. In fact, this quotient is the the degree-1 Cech cohomology of S^1 at infinity. We can normalize by requiring $\sum_a \beta_a = 1$.

Conclusion: we get a single well-defined number $\sigma = \langle \mathsf{F}, \beta \rangle$.

Remark 2. We assumed a rather special choice of cones which induce a cover of S^1 which does not have any triple overlaps. As a result, the cocycle condition on the 1-cochain β was vacuous. One can generalize the argument to an arbitrary conical cover. Then $\langle \mathsf{F}, \beta \rangle$ is inner iff β is a 1-cocycle. So one gets a single numerical invariant of a state for any choice of the conical cover.

Lemma 8. The number σ is independent of the choice of q.

Proof. The ambiguity in the choice of q is $\mathsf{q} \mapsto \mathsf{q} + \partial \mathsf{m}$ where $\mathsf{m} \in C_2^{\psi}$. Easy to check that under such a change the derivation $\mathsf{F}_{a,a+1}$ changes by an inner derivation whose ψ-average vanishes. □

Finally, we need to check the choice of the cones U_a does not matter. It is sufficient to show that refining the conical "cover" does not change σ.

Lemma 9. σ is invariant under a refinement of cover.

Proof. If a cover refines another cover, a choice of q with respect to the finer cover obviously gives a choice of q for the coarser cover. Since the choice of $\partial \mathsf{q}$ does not affect σ, the lemma is proved. □

To summarize, to every gapped $U(1)$-invariant state ψ we can assign a number σ. There are two more things to check.

- σ does not depend on the choice of the origin in $\mathbb{R}^2$.
- σ is unaffected by the action of a $U(1)$-invariant LGA.

The latter is rather obvious, as all constructions are invariant under such LGAs. To prove the former, one should generalize to covers which are only conical sufficiently far from the origin. This is also fairly straightforward.

Remark 3. One can show that the invariant σ is proportional to the Hall conductance at zero temperature. This requires some manipulations with the Kubo formula which computes the conductance tensor. See pp. 17-18 of https://arxiv.org/abs/2006.14151 for details. One of the steps is noting that the Kubo formula involves a current-current correlator of the form

$$\langle J_{jk}(1-P)\frac{1}{H^2}(1-P)J_{kl}\rangle,$$

where P is the projector to the ground state, H is the Hamiltonian, and $J_{jk} = [H_j, Q_k] - [H_k, Q_j]$ is the current 2-chain. This can be re-written in terms of $\mathscr{I}_{\mathsf{H},u_\Delta}(J_{jk})$ as

$$\langle \mathscr{I}_{\mathsf{H},u_\Delta}(J_{jk})\mathscr{I}_{\mathsf{H},u_\Delta}(J_{kl})\rangle - \langle \mathscr{I}_{\mathsf{H},u_\Delta}(J_{jk})\rangle\langle \mathscr{I}_{\mathsf{H},u_\Delta}(J_{kl})\rangle.$$

On the other hand, if we choose H_j which do not excite ψ (i.e. write $\mathsf{H} = \partial\mathsf{h}^\psi$ with $\mathsf{h}^\psi \in C_1^\psi$), then $\partial\mathscr{I}_{\mathsf{H},u_\Delta}(J) = \mathsf{q}^\psi - \mathsf{q}$, where $\mathsf{q}^\psi \in C_1^\psi$ and $\mathsf{q} \in C_1^\psi$ satisfy $\partial\mathsf{q}^\psi = \partial\mathsf{q} = \mathsf{Q}$.

4 Systems living on subsets of the lattice

It is interesting to ask what happens if the lattice system occupies only some part of $\mathbb{Z}^2$. Let's call this part Γ. Nothing in the above arguments depended on whether Γ fills the whole $\mathbb{Z}^2$ or not. So we still can define an invariant σ for such state. However, this invariant may vanish identically for some choices of Γ.

Indeed, suppose there is a conical region $X \subset \mathbb{R}^2$ such that every $x \in \Gamma$ is within distance r from X. Let X_∞ be the base of this cone. It is a closed subset of the circle "at infinity." One can use X instead of $\mathbb{R}^2$ to define an invariant σ_X. That is, one can pick some cover of X by cones and proceed as before. The resulting number σ_X will depend on a simplicial 1-cocycle β_X on X_∞. (Each conical cover of X gives a triangulation of X_∞, and β_X is a 1-cocycle with respect to this triangulation.) The number σ_X actually depends on the cohomology class of β_X. In particular, if β_X is exact, then $\sigma_X = 0$.

On the other hand, we can always complete the conical cover of X to a conical cover of $\mathbb{R}^2$ and reinterpret the derivations F_{ab} as associated to pairs of regions of the enlarged cover. Of course, $\mathsf{F}_{ab} = 0$ when either a or b label the elements of the cover that we added to get at cover of $\mathbb{R}^2$. We can define σ using a 1-cocycle β on S^1. It is easy to see that $\sigma = \sigma_X$ provided $\beta_X \in H^1(X_\infty)$ is taken to be a restriction of $\beta \in H^1(S^1)$. Thus if $H^1(X_\infty) = 0$, we must have $\sigma = 0$.

For example, if the 2d system actually lives on a strip of some finite width in $\mathbb{R}^2$, one can take X to be a straight line contained in this strip, and then X_∞ consists of two points. Since the latter has nonzero cohomology only in degree 0, $\sigma = 0$ for such systems. This is as it should be: a quasi-1d system with a nonzero Hall conductance cannot be gapped because of gapless chiral modes on the boundaries.

Similarly, suppose we cut out a sector out of $\mathbb{R}^2$ and consider a system which lives on the complement of the sector. X_∞ has trivial degree-1 cohomology in this case, so $\sigma = 0$.

Similarly, a system living on any bounded subset of $\mathbb{R}^2$ must have $\sigma = 0$ (because $X_\infty = \emptyset$ in this case).

On the other hand, suppose our system lives on $\mathbb{R}^d$ with $d > 2$ and occupies some region Γ which is contained in an r-thickening of a conical region $X \subset \mathbb{R}^d$. Then the number of invariants depends on the degree-1 cohomology of $X_\infty \subset S^{d-1}$ (the base of the cone X). This cohomology group can be quite large. For example, let $d = 3$, and take X_∞ to be an arbitrary finite graph on S^2 whose edges are large circles. Let X be a cone with base X_∞. Then the number of "Hall conductances" is determined by the number of independent loops in X_∞. Say, a quasi-2d system living on a cone over the Mercedes-Benz logo has three independent invariants.

5 Higher-dimensional systems

If we consider systems on $\mathbb{R}^d$, then σ cannot be constructed (or rather, it vanishes because $H^1(S^{d-1}) = 0$. However, one can look for more complicated invariants "iterating" the construction of σ. Turns out one gets non-trivial invariants only for d even. I will call them higher Hall conductances, but I do not know their interpretation in terms of transport theory.

Let me explain the construction of "higher Hall conductance" for $d = 4$. As before, we choose $\mathsf{q} \in C_1^\psi$ such that $\partial \mathsf{q} = \mathsf{Q}$. The 2-cochain $[\mathsf{q}, \mathsf{q}]$ is still a cocycle, so by the acyclicity of our DGLA we must have $\frac{1}{2}[\mathsf{q}, \mathsf{q}] = \partial \mathsf{q}^{(3)}$, where $\mathsf{q}^{(3)} \in C_3^\psi$. We can also choose $\mathsf{q}^{(3)}$ so that $[\mathsf{q}^{(3)}, \mathsf{Q}] = 0$ Then $\mathsf{p}^{(4)} = [\mathsf{q}^{(3)}, \mathsf{q}]$ is a 4-cocycle. Indeed:

$$\partial[\mathsf{q}^{(3)}, \mathsf{q}] = \frac{1}{2}[[\mathsf{q}, \mathsf{q}], \mathsf{q}] - [\mathsf{q}^{(3)}, \mathsf{Q}] = \frac{1}{2}[[\mathsf{q}, \mathsf{q}], \mathsf{q}].$$

This vanishes by the Jacobi identity.

Now we pick a conical cover $\{U_a\}_{a \in A}$ of $\mathbb{R}^4$ which induces a triangulation of S^3 "at infinity", and pick a simplicial 3-cocycle $\beta^{(3)}$ with respect to this triangulation. Consider a derivation

$$\mathsf{F}_{abcd} = \sum_{i \in U_a, j \in U_b, \ldots} \mathsf{p}^{(4)}_{ijkl} \in \mathfrak{D}^\psi_{al}.$$

I claim that

$$\sum_{a,b,c,d} \beta_{abcd} \mathsf{F}_{abcd}$$

is an inner derivation (thanks to the cocycle condition on β. Thus we can define a numerical invariant by averaging the corresponding local observable over ψ. One can show that the invariant depends only on the cohomology class of β in $H^3(S^3)$.

6 Non-abelian symmetry groups

Let G be a compact Lie group. I will only do the case $d = 2$. Now we have a homomorphism $\mathsf{Q} : \mathfrak{g} \to \mathfrak{D}^{\psi}_{al}$ where $\mathfrak{g}$ is the Lie algebra of G. That is, upon choosing a basis e_a in $\mathfrak{g}$, we have charges $\mathsf{Q}_a = \mathsf{Q}(e_a) \in \mathfrak{D}^{\psi}_{al}$ such that

$$[\mathsf{Q}_a, \mathsf{Q}_b] = f^c_{ab}\mathsf{Q}_c,$$

where f^c_{ab} are structure constants of $\mathfrak{g}$ with respect to the basis e_a.

Note that Q_a is no longer G-invariant. But we can deal with this as follows. Let's introduce a formal variable $t = \{t^a\}$ which lives in $\mathfrak{g}$ and form $\mathsf{Q}(t) = \mathsf{Q}_a t^a$. Then $\mathsf{Q}(t)$ is G-invariant provided G also acts on t^a (via its adjoint representation). So we introduce a DGLA $C^{\psi}_{\bullet} \otimes \mathrm{Sym}^{\bullet}(\mathfrak{g}^*)$ and its G-invariant part $C^{\psi,G}_{\bullet}$. Its convenient to assign the element t degree -2, so that $\mathsf{Q}(t)$ is a degree -2 element in $C^{\psi,G}_{\bullet}$.

Lemma 10. $\mathsf{Q}(t)$ is a central element of $C^{\psi,G}_{\bullet}$.

Proof. An element of $C^{\psi}_{\bullet} \otimes \mathrm{Sym}^{\bullet}(\mathfrak{g}^*)$ can be regarded as a polynomial function $f : \mathfrak{g} \to C^{\psi}_{\bullet}$, $f : t \mapsto f(t)$. The G-invariance condition implies

$$\frac{\partial f}{\partial t^a}[s,t]^a + [\mathsf{Q}(s), f(t)] = 0, \quad \forall s,t \in \mathfrak{g}.$$

Setting $s = t$ we get the desired result. ☐

Theorem 3. The DGLA $C^{\psi,G}_{\bullet}$ is acyclic (i.e; its homology is trivial).

Proof. Use the averaging over G trick as before. ☐

Then there exists $\mathsf{q}(t) \in C^{\psi,G}_{1} = t^a\mathsf{q}_a$ such that $\partial\mathsf{q}(t) = \mathsf{Q}(t)$. We form a 2-cochain of $C^{\psi,G}_{\bullet}$[1]:

$$\mathsf{p}(t) = \mathsf{p}_{ab}t^a t^b = \frac{1}{2}[\mathsf{q}(t), \mathsf{q}(t)].$$

This is a 2-cocycle because $\mathsf{Q}(t)$ is a central element in the DGLA $C^{\psi,G}_{\bullet}$. We contract it with a simplicial 1-cocycle β of S^1 at infinity and average over ψ. The average is a quadratic polynomial in t. Since p is G-invariant and so is ψ, this quadratic polynomial is also G-invariant.

We conclude that for a general compact G the invariant σ takes values in G-invariant polynomials on $\mathfrak{g}$ of degree 2. This is precisely the same datum which allows one to write down a Chern-Simons action for a G-connection (on the classical level).

Similarly, in any even spatial dimension $d = 2k$ we get an invariant which takes values in G-invariant polynomials on $\mathfrak{g}$ of degree $k+1$. This datum classifies Chern-Simons actions in space-time dimension $d+1$.

[1]When I call it a 2-cochain, I only count the degrees of the coefficients of its Taylor expansion. If I also counted the t-degree, I would say that p has degree -2.