Extremal metrics on destabilising test configurations

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Extremal metrics on destabilising test configurations

In this lecture, we will discuss joint work with Cristiano Spotti. This is a construction of extremal metrics on the total space of certain destabilising test configurations. We will put the results in the wider context of constructing extremal metrics on fibrations, which is a well-studied area. This will allow us to see the main new challenges in the construction.

Main result

A simplified version of the main result is the following.

Theorem 1 (S.–Spotti '21).

Suppose (X, L) is analytically strictly K-semistable and has discrete automorphism group. On certain test configurations

$$\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1,$$

there exists an extremal metric

$$\Omega_k \in c_1(\mathcal{L} + \pi^* \mathcal{O}(k))$$

for all $k \gg 0$.

The actual result is more general, and we will discuss the more refined statement in due course.

The general setup

The general setup

The main result falls in the context of constructing extremal metrics on the total space of fibrations. This has a long history of study, with many different constructions. The general setup is that of a holomorphic submersion (fibration)

$$\pi : (Y, H) \to B,$$

where $H \to Y$ is an ample line bundle on Y. The goal is then to construct Kähler metrics on Y in H (or maybe another polarisation, somehow related to H).

Constructions

Two of the main avenues of producing extremal Kähler metrics on Y are:

Symmetry: These are explicit constructions using an ansatz in situations with high symmetry. E.g., Calabi's first examples of non-cscK extremal metrics on Hirzebruch surfaces $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(d)) \to \mathbb{P}^1$, generalisations of these by Hwang–Singer, Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman (which for example was used by Legendre and Jubert to show that $\mathbb{P}(L_1 \oplus L_2 \oplus L_3) \to C$ is Calabi dream, if C is a genus 0 or 1 curve), ...

Perturbation techniques: These involve working in "adiabatic classes". These are classes of the form $c_1(H + kD)$ for $k \gg 0$, where D is the pullback of an ample line bundle on B. This includes work of Fine, Hong, Brönnle, Dervan–S. and more.

Constructions in adiabatic classes

The new construction is of the latter type. The starting point for considering adiabatic classes is the observation that if

- $\omega_Y \in c_1(H)$ is relatively Kähler;
- $\omega_B \in c_1(D)$ is Kähler;

then the form

$$\omega_k = \omega_Y + k\omega_B \in c_1(H + kD)$$

is Kähler, and its scalar curvature expands as

$$S(\omega_k) = S(\omega_F) + O(k^{-1}),$$

where $S(\omega_F)$ is the scalar curvature of ω_Y restricted to the fibres, the fibrewise scalar curvature.

Constructions in adiabatic classes

$$S(\omega_k) = S(\omega_F) + O(k^{-1}),$$

The heurestic reason is that for a product, one would have $S(\omega_k) = S(\omega_F) + k^{-1}S(\omega_B)$. In general, one has

$$S(\omega_k) = S(\omega_F) + O(k^{-1}),$$

but the $O(k^{-1})$ term is more involved. In Lemma 2 below, we will prove why the expansion is this way.

Constructions in adiabatic classes

The expansion

$$S(\omega_k) = S(\omega_F) + O(k^{-1})$$

means that to leading order, the cscK equation is the cscK equation fibrewise. A natural assumption is therefore to start with a fibrewise cscK metric ω_Y (however, this will not be the case in the main result discussed today!). Note that there may be many such: if $Y = \mathbb{P}(E)$, any hermitian metric on E induces a relative Kähler metric on Y which is fibrewise Fubini-Study.

Constructions in adiabatic classes

The non-uniqueness of the fibrewise cscK metric comes from two sources:

- Pullback of functions from B;
- Fibrewise holomorphic vector fields (cscK metrics are only unique up to automorphisms).

The former is harmless and should be thought of as analogous to Kähler potentials only being unique up to a constant in the absolute setting. But as we will see later, the latter causes many complications.

Constructions in adiabatic classes

We want to construct extremal metrics on the total space of fibrations using perturbative techniques. Recall that the general strategy in such problems have the following key steps:

- Create approximate solutions to the extremal equation;
- Control the inverse of the linearised operator for the extremal equation;
- Apply a quantitative inverse/implicit function theorem, or contraction mapping theorem.

For the fibration setting, we will see that the control we get for the inverse of the linearised operator is not good enough that using just a fibrewise cscK metric together with some fixed pulled back metric from the base will allow us to use the implicit function theorem. We need a better approximate solution.

The fibrewise cscK case

Constructions in adiabatic classes

We now want to discuss the method of proof in the case discussed above in more detail. Recall that the setup is that we have a holomorphic submersion

$$(Y, H) \to (B, D)$$

The polarisation H is *fibrewise cscK*, i.e. there exists an $\omega_Y \in c_1(H)$ whose restriction to any fibre is cscK. The material in this section was developed by Fine and further refined by Dervan–S. building also on the other works in the adiabatic setting mentioned above.

The expansion of the scalar curvature

We now prove exactly what the expansion of the scalar curvature is, to leading two orders. We will need to introduce some notation. Functions on Y split as

$$C^\infty(Y) = C_0^\infty(Y) \oplus \pi^* C^\infty(B),$$

where $C_0^\infty(Y)$ consists of fibrewise average 0 functions wrt ω_Y. Notation: ψ_0 is the $C_0^\infty(Y)$ component of a functions ψ on Y. Using ω_Y, we also have a splitting of the tangent bundle of Y as

$$TY = \mathcal{V} \oplus \mathcal{H}$$

where

- $\mathcal{V} = \ker \pi_*$ is the vertical tangent bundle;
- $\mathcal{H} \cong \pi^* TB$ is the horizontal tangent bundle.

We have similar splittings of any tensor bundle.

The expansion of the scalar curvature

We will let

- ρ be the curvature of $\Lambda^m \mathcal{V}$ induced by ω_Y^m;
- $\mathrm{Ric}(\omega_F)$ be the Ricci curvature of the metric induced on the fibres, which is simply the vertical component $\rho_{\mathcal{V}}$ of ρ;
- $\rho_{\mathcal{H}}$ be the horizontal component of ρ;
- $\Delta_{\mathcal{V}} = \Lambda_{\mathcal{V}} i\partial\bar{\partial}$ be the vertical Laplacian, where $\Lambda_{\mathcal{V}}$ is the contraction in the vertical direction;
- $F_{\mathcal{H}}$ be the curvature of the Ehresmann connection of the fibration $Y \to B$ given by the splitting $TY = \mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of Y.

The expansion of the scalar curvature

The term $F_{\mathcal{H}}$ is a two form with values in fibrewise hamiltonian vector fields. By using the co-moment map μ^* taking the vector field to the corresponding hamiltonian of average 0 on the each fibre, we can view $F_{\mathcal{H}}$ as a two form on B with values in $C_0^\infty(Y)$. We can then contract in the base direction to get an element

$$\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}) \in C_0^\infty(Y).$$

The expansion of the scalar curvature

The expansion of the scalar curvature is given by the following

Lemma 2.

$$S(\omega_k) = S(\omega_F) + k^{-1}(\psi_0 + \psi_B) + O(k^{-2}),$$

where

$$\psi_0 = (\Lambda_{\omega_B}\rho_{\mathcal{H}})_0 + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}})) \in C_0^\infty(Y)$$

and

$$\psi_B = \pi^*\left(S(\omega_B) - \Lambda_{\omega_B}(\alpha)\right) \in \pi^* C^\infty(B),$$

for some semipositive Weil–Petersson type 2-form α.

The expansion of the scalar curvature

The geometric interpretation of the form α is that it is the pullback of the Weil–Petersson Kähler form on the moduli space $\mathcal{M}$ of cscK manifolds, via the map $B \to \mathcal{M}$ sending b to the cscK manifold $\left(Y_b, (\omega_Y)\big|_b\right)$. Note that the map to the moduli space is not necessarily an embedding, so α is only semipositive, not necessarily positive. E.g. for isotrivial fibrations, such as $\mathbb{P}(E)$, α is actually 0.

Proof of Lemma 2

It suffices to prove that we have the expansion

$$\mathrm{Ric}(\omega_k) = \mathrm{Ric}(\omega_F) + \rho_{\mathcal{H}} + \mathrm{Ric}(\omega_B) + k^{-1} i\partial\bar{\partial}(\Lambda_{\omega_B}\omega_Y) + O(k^{-2}).$$

Note that the vertical component of ρ is $\mathrm{Ric}(\omega_F)$. This expansion in turn implies that

$$\begin{aligned} S(\omega_k) &= S(\omega_F) + k^{-1}\big(\Lambda_{\omega_B}\rho_{\mathcal{H}} + S(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}((\omega_Y)_{\mathcal{H}}))\big) + O(k^{-2}) \\ &= S(\omega_F) + k^{-1}\big(\Lambda_{\omega_B}\rho_{\mathcal{H}} + S(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}))\big) + O(k^{-2}), \end{aligned}$$

by the expansion

$$\Lambda_{\omega_k} = \Lambda_{\mathcal{V}} + k^{-1}\Lambda_{\mathcal{H}} + O(k^{-2}).$$

Proof of Lemma 2

$$S(\omega_k) = S(\omega_F) + k^{-1}\big(\Lambda_{\omega_B}\rho_{\mathcal{H}} + S(\omega_B) + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}))\big) + O(k^{-2}),$$

We also used

- $\mu^* F_{\mathcal{H}}$ and $(\omega_Y)_{\mathcal{H}}$ differ by a form pulled back from B;
- Pulled back functions are in the kernel of $\Delta_{\mathcal{V}}$;
- The horizontal component of $\Lambda_{\omega_B}\rho_{\mathcal{H}}$ is the negative of the contraction of the Weil–Petersson form α.

Proof of Lemma 2

The expansion

$$\mathrm{Ric}(\omega_k) = \mathrm{Ric}(\omega_F) + \rho_{\mathcal{H}} + \mathrm{Ric}(\omega_B) + k^{-1} i\partial\bar{\partial}(\Lambda_{\omega_B}\omega_Y) + O(k^{-2}).$$

for the Ricci curvature follows by using that

$$TY^{n+m} \cong \mathcal{V}^m \otimes \mathcal{H}^n,$$

coming from the splitting of the tangent bundle into vertical and horizontal components. Note that $\mathcal{H}^n \cong \pi^* TB^n$.

Proof of Lemma 2

$$\mathrm{Ric}(\omega_k) = \mathrm{Ric}(\omega_F) + \rho_{\mathcal{H}} + \mathrm{Ric}(\omega_B) + k^{-1} i\partial\bar{\partial}(\Lambda_{\omega_B}\omega_Y) + O(k^{-2}).$$
$$TY^{n+m} \cong \mathcal{V}^m \otimes \mathcal{H}^n.$$

The $\mathcal{V}^m$ part in this decomposition comes from taking the vertical component of ω_k. This is the induced fibrewise metric that remains unchanged with k. This gives the term ρ, whose

- vertical component is $\mathrm{Ric}(\omega_F)$;
- horizontal component is $\rho_{\mathcal{H}}$.

Proof of Lemma 2

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$$TY^{n+m} \cong \mathcal{V}^m \otimes \mathcal{H}^n.$$

For the $\mathcal{H}^n$ part, note that we have another metric we can use on $\mathcal{H}^n \cong \pi^* TB^n$, namely the one induced by $\pi^*\omega_B$. We then have that the horizontal part can be divided into

- the curvature induced by $\pi^*\omega_B$, i.e. $\pi^* \mathrm{Ric}(\omega_B)$;
- $i\partial\bar{\partial}$ of the ratio of the volumes, computed in $\mathcal{H}^n$.

This last term is

$$i\partial\bar{\partial}\left(\log \frac{(\omega_k)^n_{\mathcal{H}}}{\omega_B^n}\right).$$

Proof of Lemma 2

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Proof of Lemma 2

Now,

$$\begin{aligned} i\partial\bar\partial\left(\log\frac{(\omega_k)^n_{\mathcal{H}}}{\omega_B^n}\right) =& i\partial\bar\partial\left(\log\frac{k^n\omega_B^n + k^{n-1}n(\omega_Y)_{\mathcal{H}}\wedge\omega_B^{n-1}+\dots}{\omega_B^n}\right)\\ &= i\partial\bar\partial\left(\log(1+k^{-1}\Lambda_{\omega_B}(\omega_Y)_{\mathcal{H}}+\dots)\right)\\ &= k^{-1}i\partial\bar\partial\Lambda_{\omega_B}(\omega_Y)_{\mathcal{H}} + O(k^{-2}). \end{aligned}$$

This completes the proof of the expansion of the Ricci curvature and hence of the scalar curvature, too.

The linearised operator

The expansion of the linearised operator

At this stage we see that we have a sequence of metrics ω_k on Y in Ω_k, which by the fibrewise cscK condition satisfies

$$S(\omega_k) = c + O(k^{-1}),$$

for a constant c. As remarked above, fibrewise cscK metrics are not necessarily unique, and at this stage we have used any fixed metric ω_B on the base. We need to determine these, and the previous expansion, as well as the linearisation of the equation, is the key to understanding this.

The expansion of the linearised operator

In the product case, it would only be when we pick a cscK metric on the base that we would solve the cscK equation on the product. In particular, if we in the product case pick a metric ω_B that is far away from cscK, we do not stand a chance in using perturbative techniques to produce a cscK metric on the product fibration. In general, therefore, we expect that there is at least a choice of ω_B that will have to come into play. Also, this suggests that one should expect that it is not enough to perturb from just the initial sequence of metrics ω_k.

The expansion of the linearised operator

We will now investigate how good of an approximate solution we will actually need in order to perturb to a genuine extremal metric on the fibration. This means establishing the mapping properties of the linearised operator. We will later see that these mapping properties will be important also to be able to create a good enough approximate solution, too.

The expansion of the linearised operator

The key lies in an asymptotic expansion of the linearised operator.

Proposition 1 (Dervan–S.).

The linearisation of the scalar curvature operator at ω_k admits an expansion

$$dS(\omega_k)_0 = -L_F + k^{-1} D_1 + k^{-2} D_2 + O(k^{-3}),$$

where

- *L_F is the fibrewise Lichnerowicz operator of ω_Y;*
- *D_1 vanishes on base functions;*
- *the base component of the operator D_2, acting on base functions, is the linearisation of the twisted scalar curvature operator.*

The expansion of the linearised operator

$$dS(\omega_k)_0 = -L_F + k^{-1}D_1 + k^{-2}D_2 + O(k^{-3}),$$

The interpretation of this is that we can remove any decaying term orthogonal to fibrewise holomorphy potentials, and we can remove any base term that decays to order greater than k^{-2}. In fact, one can even do better in the base direction:If ϕ is pulled back from B,

$$\omega_k + i\partial\bar{\partial}(\phi) = \omega_Y + k(\omega_B + k^{-1}i\partial\bar{\partial}\phi).$$

So, we are changing ω_B to $\omega_B + k^{-1}i\partial\bar{\partial}\phi$ for a function pulled back from B, and this changes the scalar curvature by the linearisation of the twisted scalar curvature at order k^{-1} and hence gives this contribution to the order k^{-2}-term of the scalar curvature of ω_k. Thus we can actually deal with any horizontal term decaying at order strictly greater than k^{-1}.

The expansion of the linearised operator

That the leading order term in the expansion is $-L_F$ means that we can remove any vertical term orthogonal to fibrewise holomorphy potentials using the linearisation. If the fibres have holomorphic vector fields, however, this is not the full space $C^\infty_0(Y)$. We will let $C^\infty_E(Y)$ denote the space of fibrewise average 0 holomorphy potentials with respect to ω_Y. Then the image of L_F is precisely the orthogonal complement to $C^\infty_E(Y)$.

The expansion of the linearised operator

To ensure that we can hit anything apart from global holomorphy potentials, we need to understand when we hit the whole of $C^\infty_E(Y)$. This is the content of the next result.

Proposition 2 (Dervan–S.).

The operator $q \circ D_1$, acting on $C^\infty_E(Y)$, can be identified with a self-adjoint elliptic operator on a vector bundle $E \to B$ with kernel precisely the fibrewise holomorphy potentials that are globally holomorphy potentials with respect to ω_k for any/all k sufficiently large.

In particular, this means that we can deal with any vertical term orthogonal to global holomorphy potentials at order k^{-1}.

The expansion of the linearised operator

The above can be used to give a bound on the (right) inverse of the linearised operator of the metrics ω_k and perturbations thereof. For simplicity, we will assume that both B and Y have trivial reduced automorphism group, which in particular means that we are solving the cscK equation on the total space. Adjustments can be made in the case when this does not hold, and one is then solving the extremal equation instead, as Y has global holomorphic vector fields.

The expansion of the linearised operator

Proposition 3 (Fine/Dervan–S.).

The linearisation of the scalar curvature operator at ω_k (or a suitable perturbation of ω_k) has an inverse Q_k such that there for any j, α there exists a $C > 0$ such that

$$\|Q_k\| \leq Ck^3$$

in operator norm, as an operator $C^{j,\alpha} \to C^{j+4,\alpha}$.

From the above proposition, we see that we need a better approximate solution in order to apply our perturbative techniques.

The expansion of the linearised operator

We wish to use the following Quantitative Inverse Function theorem.

Theorem 3.

Let $F : V \to W$ be a differentiable map of Banach spaces, whose derivative at 0 is an isomorphism with inverse η. Let

- *r' be the radius of the ball in V centered at 0 in which $F - dF$ is Lipschitz of constant $\frac{1}{2\|\eta\|}$;*
- *$r = \frac{r'}{2\|\eta\|}$.*

Then for all $w \in W$ such that $\|F(0) - w\| \leq r$, there exists a $v \in V$ with $\|v\| \leq r'$ such that $F(v) = w$.

The expansion of the linearised operator

A quick calculation then shows that with the bound established in Proposition 3 we need an approximate solution that is extremal to order at least

$$k^{-7}$$

to apply the above theorem to produce a solution to the cscK/extremal equation.

Completing the construction

The case of no automorphisms

We will now produce a solution to the extremal equation. From the above, we see that the initial sequence ω_k is not good enough to guarantee that we can perturb to an actual solution. We need to we improve the approximate solution so that it becomes approximately cscK or extremal to higher order. That is, we need to find new metrics $\omega_{k,l} \in [\omega_k]$ that are extremal to order k^{-l}. When ω_Y is fibrewise cscK, this gives us the case $l = 1$. As remarked above, we need to get to $l = 7$, but the process works to produce approximate solutions to arbitrary high order.

The case of no automorphisms

To improve this approximate solution, we go back to the analysis of the $O(k^{-1})$ term. Recall that we have an expansion

$$\begin{aligned} S(\omega_k) =& S(\omega_F) + k^{-1}(\psi_0 + \psi_B) + O(k^{-2}), \\ \psi_0 =& (\Lambda_{\omega_B}\rho_{\mathcal{H}})_0 + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}})), \\ \psi_B =& S(\omega_B) - \Lambda_{\omega_B}(\alpha). \end{aligned}$$

The case of no automorphisms

Looking at the term

$$\psi_B = S(\omega_B) - \Lambda_{\omega_B}(\alpha)$$

we see that, to improve the solution to $O(k^{-2})$, one therefore needs ω_B to solve a *twisted* cscK equation

$$S(\omega_B) - \Lambda_{\omega_B}(\alpha) = c_B$$

or more generally a twisted extremal equation

$$S(\omega_B) - \Lambda_{\omega_B}(\alpha) \in \overline{\mathfrak{h}}_B,$$

where $\overline{\mathfrak{h}}_B$ denotes the holomorphy potentials on B with respect to ω_B.

The case of no automorphisms

Recall that we can only remove horizontal terms at order strictly greater than k^{-1}, so to deal with the $O(k^{-1})$ we need to make a good choice of metric to pull back from B, and the expansion tells us what this good choice is. Assuming one can find such an ω_B, we have now determined ω_B: but we still have some freedom as we can perturb!

The case of no automorphisms

We therefore need to understand the contribution to the above if we perturb to $\omega_{k,2} = \omega_k + k^{-1} i\partial\bar{\partial}\phi$, for some ϕ. This boils down to understanding the linearisation P_k of the scalar curvature operator, at ω_k. This was the content of Proposition 1. Recall that the linearisation has an expansion

$$P_k = -L_F + O(k^{-1}),$$

where L_F is a fibrewise Lichnerowicz operator, if ω_Y is fibrewise cscK.

The case of no automorphisms

Now, if the fibres have no automorphisms, L_F is surjective on $C_0^\infty(Y)$. We can therefore remove the ψ_0 term, and continue to improve the solution to arbitrary order, also using that any horizontal term at order strictly greater than k^{-1} can be removed by perturbing, using the expansion of the linearisation. Eventually, the approximate solutions are sufficiently good so that the Inverse Function Theorem 3 can be applied to find a solution to the cscK equation for $k \gg 0$.

What if the fibres have automorphisms?

If the fibres have automorphisms, we cannot remove the ψ_0 term completely using perturbations. However, in this case, there is also some choice of ω_Y, since the metric is not uniquely determined on each fibre. Indeed, a theorem of Berman–Berndtsson shows that extremal metrics are only unique up to the action of the automorphism group. So while it is not automatic that we can remove the ψ_0 term, we also have many fibrewise cscK metrics to consider initially.

What if the fibres have automorphisms?

Dervan–S. found an equation for a good choice of fibrewise cscK metric. The equation is

$$q\left(\Lambda_{\omega_B}\rho_{\mathcal{H}} + \Delta_{\mathcal{V}}(\Lambda_{\omega_B}(\mu^* F_{\mathcal{H}}))\right) = 0,$$

where q is the fibrewise L^2-orthogonal projection to average 0 holomorphy potentials. This says that precisely the term that cannot be removed at the crucial stage using the linearisation vanishes. This is an equation on fibrewise cscK metrics (an important fact in the theory is that one can view these as smooth sections of a finite dimensional vector bundle over X). In the case when $Y = \mathbb{P}(E)$, this reduces to the Hermite–Einstein equation for a hermitian metric on E.

What if the fibres have automorphisms?

Under the assumption that such a metric ω_Y exists, the construction still goes through, following the same type of steps as in the case of trivial automorphism groups of the fibres. This requires one to use the subleading order terms in the asymptotic expansion of the vertical part of the linearised operator, see Proposition 1 and Proposition 2. From this, one sees that at order k^{-1}, one can hit anything orthogonal to global holomorphy potentials. This means that any vertical error after the crucial term at order k^{-1} can be removed, which is the essential step in constructing a good approximate solution. This completes the outline of the proof in the fibrewise cscK case.

Final remark

In the statement of the main result, we will make some assumptions on the automorphism groups. This is essentially to avoid running into these issues, which significantly complicate the construction.

The new construction

Statement of the result

The new construction produces extremal metrics on a special type of fibration. We have

- $(Y, H) = (\mathcal{X}, \mathcal{L})$, the total space of a test configuration for some (X, L) over $\mathbb{P}^1$, satisfying certain assumptions;
- $(B, D) = (\mathbb{P}^1, \mathcal{O}(1))$.

Statement of the result

If $(\mathcal{X}, \mathcal{L})$ is a product test configuration and (X, L) admits a cscK metric, it is already known that one can produce extremal metrics in the adiabatic classes $\mathcal{L} + \mathcal{O}(k)$. This follows e.g. by the construction of Dervan–S. detailed above.

In the construction under discussion now, however, we will have that *almost no fibre* actually admits a cscK metric. This is in stark contrast to the previous constructions. For us, (X, L) will be a strictly K-semistable manifold, and the degeneration above is to a cscK central fibre $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$.

Statement of the result

More precisely, we consider a Kähler manifold (X, L) which is *analytically K-semistable*. This means that there is a $\mathrm{Aut}_0(X)$-equivariant degeneration of (X, L) to a cscK Kähler manifold (X_0, L_0). Below we will be more precise on the test configurations we consider. The condition implies K-semistability of (X, L). It can also be seen as asking that (X, L) is a small equivariant deformation of a cscK manifold.

Statement of the result

Theorem 4 (S.–Spotti '21).

Suppose (X, L) is analytically strictly semistable with discrete automorphism group and let $\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$ be a test configuration degenerating (X, L) to a cscK central fibre $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$, arising from the Kuranishi family of $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$,. Assume further that the reduced automorphism group of the central fibre is $\mathbb{C}^*$. Then for all large k, there exists an extremal metric in $c_1(\mathcal{L} + \pi^*\mathcal{O}(k))$.

Statement of the result

The main importance of the result is two-fold:

- It is the first construction where one is able to remove the condition that all the fibres admit cscK/extremal metrics. The construction therefore provides an important technique that allows dealing with the semistable case, where no canonical choice of *fixed* fibrewise metric will exist.
- The construction actually provides a myriad of new extremal metrics. We will discuss this at the end of the lecture.

Statement of the result

Remark.

The assumption regarding the automorphism group can be thought of as analogous to the assumption that the automorphism group is discrete in similar perturbation problems. The central fibre has to have a $\mathbb{C}^*$ *action, and so the assumption says that the automorphism group is minimal. In particular, we see no obstructions, unlike the cases related to the OSC equation. The freedom we have in the choice of cscK metric on the central fibre, precisely equals the freedom we have on* $\mathcal{X}$*, which also has a* $\mathbb{C}^*$ *as its automorphism group.*

Statement of the result

Remark.
We actually relax the condition on the automorphism group somewhat. We will not go into details of this, but the condition is really that the discrepancy between the reduced automorphism group of the central fibre and the original semistable manifold is given by the additional $\mathbb{C}^*$ *action on the central fibre.*

Kuranishi theory

We give slightly more details on how exactly we assume the test configuration arises. This uses the Kuranishi theory of the cscK central fibre.

Suppose (X_0, L_0) is a polarised manifold with cscK metric $\omega \in c_1(L)$, and whose underlying smooth manifold is M. Let $\mathcal{J}(M, \omega)$ denote the space of ω-compatible almost complex structures on M. Let T be a maximal compact torus in the reduced automorphism group of X_0. Then there exists a complex space V with a T-action and a holomorphic embedding $V \to \mathcal{J}(M, \omega)$, equivariant with respect to the T-action, which induces a versal deformation space for X_0. One can moreover ensure that the scalar curvature of the complex structures in the family (with respect to the form ω) takes values in the holomorphy potentials of (X_0, ω) (Székelyhidi/Brönnle).

Kuranishi theory

It is certain test configurations produced from this family that we will consider in our construction. The key fact we are using is that the versality of the construction implies that if we have a test configuration taking some (X, L) to a cscK (X_0, L_0), then there exists a (potentially different) test configuration $(\mathcal{X}, \mathcal{L})$ in the Kuranishi family, taking (X, L) to (X_0, L_0). It is on this family that our construction takes place. That is, the test configurations $\mathcal{X}$ we consider are produced from the Kuranishi family near the polystable $(X_0, L_0) = (\mathcal{X}_0, \mathcal{L}_{|\mathcal{X}_0})$.

Kuranishi theory

Upshot: we can assume that we have a fixed smooth manifold M with a symplectic form ω and family J_t with t some disk Δ about 0 in $\mathbb{C}$, such that (M, J_0, ω) is cscK, and $(M, J_t) \cong X$ for $t \neq 0$. Moreover, $J_t = J_0 + O(|t|)$, and the $O(|t|)$ term is non-vanishing. The relationship between $\mathcal{X} \to \Delta$ and J_t is that as a smooth manifold, $\mathcal{X} = \Delta \times M$ and J_t is the complex structure of the fibre of $\mathcal{X}$ over t in Δ.

The key new issue

Recall the old approach for a cscK fibration $(Y, H) \to (B, D)$: one starts with a metric $\omega_Y \in c_1(H)$ and metric $\omega_B \in c_1(D)$ and considers the one parameter family of metrics $\omega_k \in c_1(H + kD)$ given by

$$\omega_k = \omega_Y + k\omega_B.$$

If ω_Y is fibrewise cscK, then we obtain

$$S(\omega_k) = c + O(k^{-1}).$$

The key new issue

The problem in our case $(Y, H) = (\mathcal{X}, \mathcal{L})$, is that there are no fibrewise cscK metrics in $c_1(\mathcal{L})$: all but one fibre does *not* admit a cscK metric! Thus, if we work with any fixed metric $\omega_{\mathcal{X}} \in c_1(\mathcal{L})$, we can never hope to obtain good approximate solutions.

Upshot: we need to work with a *sequence* of relative Kähler metrics $\widetilde{\omega}_k \in c_1(\mathcal{L})$.

A conjecture of Donaldson

To explain why there is some hope to get a good sequence of relative Kähler metrics, we recall the following conjecture of Donaldson.

Conjecture (Donaldson).

Let $(V, \mathcal{D})$ *be a polarised projective manifold. Then*

$$\inf_{\omega \in c_1(\mathcal{D})} \|S(\omega) - \hat{S}\| = \sup_{\mathcal{V} \text{ t.c.}} \frac{-DF(\mathcal{V})}{\|\mathcal{V}\|}.$$

In particular, if $(V, \mathcal{D})$ is strictly semistable, we should be able to find metrics that are *arbitrarily close to a cscK metric*, even though no actual cscK metric exists.

A good family near the central fibre

Under our assumptions, the fact that we can get a family of metrics $\alpha_\epsilon \in c_1(L)$ satisfying

$$\|S(\alpha_\epsilon) - \hat{S}\| \leq \epsilon$$

on the non-cscK general fibre X is automatic. This comes from the Kuranishi theory detailed above: since $J_\epsilon = J_0 + O(\epsilon)$,

$$S(\omega, J_\epsilon) = S(\omega, J_0) + O(\epsilon).$$

If we then use a diffeomorphism to identify the complex structure of the non-zero fibres with a fixed fibre and pullback the symplectic form, we get the symplectic forms α_ϵ.

Extending the metric

Next, we want to take our local family over some disk Δ and extend it to a relative Kähler metric on the corresponding compactified test configuration over $\mathbb{P}^1$, which we still call $\mathcal{X}$. As the metrics are *not* necessarily identical on the S^1-fibres, this is not automatic. We then use a rather crude interpolation so that

$$\omega_\varepsilon|_{\mathcal{X}_t} = \begin{cases} \alpha_t & \text{if } t \leq \frac{\varepsilon}{2} \\ \alpha_\epsilon & \text{if } t \geq \varepsilon \end{cases}$$

In the trivialisation identifying the test configuration with $\mathbb{C}^* \times X$ over $\mathbb{C}^*$, we therefore have a constant metric on every fibre, outside the disk of radius ε. We can then trivially extend over $\infty \in \mathbb{P}^1$, to obtain a relative Kähler metric on the full family. This metric satisfies that

$$\|S(\omega_\varepsilon|_{\mathcal{X}_t}) - \hat{S}\| \leq C\epsilon$$

for *every* fibre $\mathcal{X}_t$.

Relating the parameters

We now want to relate the parameter ε in our family of relative Kähler metrics on $\mathcal{X}$, to the parameter k of the polarisation. If we write

$$\Omega_\epsilon = \omega_\epsilon + \lambda\varepsilon^{-\delta}\pi^*\omega_{\mathbb{P}^1},$$

for some Kähler form $\omega_{\mathbb{P}^1}$ on $\mathbb{P}^1$, so $k = \lambda\varepsilon^{-\delta}$, then this is Kähler for all sufficiently small ε if either

- $\delta = 1$ and λ is sufficiently large;
- $\delta > 1$.

For us, it suffices to let $\delta > 1$, and we can even take $\delta = 2$. The fact that we can get away with not have to pick a specific value of δ is related to the our assumptions on the automorphism group.

Completing the argument

The point of picking ω_ε the way we did is that now

$$S(\Omega_\varepsilon) = S(\omega, J_0) + O(\varepsilon).$$

I.e., we are seeing the scalar curvature of the *central fibre* as the leading order term in the expansion of the scalar curvature. We have lower order vertical terms we have to deal with, but these cause no problem. The expansion of the Lichnerowicz operator L_ε of Ω_ε is

$$L_\varepsilon = L_0 + O(\varepsilon),$$

where L_0 is the Lichnerowicz operator of the central fibre. This means that we can remove any error orthogonally to the generator of the $\mathbb{C}^*$ action on central fibre – but this comes from a global vector field on $\mathcal{X}$! So to solve the extremal equation on $\mathcal{X}$, this poses no obstruction.

Extensions of the theory

Remark.
When the discrepancy between the automorphism group of the central fibre and the general fibre is larger, more care is needed in the construction. This is analogous to the setting of the OSC equation, where we need to use the extra freedom of the choice of cscK metric on the central fibre, to make up for the terms we cannot remove using the linearisation. The tools to extend to this setting are on the way. A general theory extending the OSC equation to fibrations with semistable fibres is currently being developed by Annamaria Ortu.

Examples

Examples

The construction allows us to produce many new examples of extremal metrics. In order to produce a test configuration to which the construction applies we need to verify two things:

- We have strictly K-semistable manifold degenerating to a *smooth* cscK central fibre;
- The discrepancy between the reduced automorphism groups of the general fibre and central fibre is correct.

The former means a degeneration of (X, L) to a cscK central fibre exists in the Kuranishi family, by versality, and this produces the test configuration $\mathcal{X}$ we actually work on. Establishing the latter part relies on the work of Cheltsov–Przyjalkowski–Shramov on automorphism groups of Fano threefolds.

Examples

In general such examples can be difficult to produce. In the Fano setting remarkable progress has been made through work on the valuative criterion for stability, which has allowed one to actually check K-stability and thus verify whether or not there exists a Kähler–Einstein metric. This has been particularly fruitful on Fano threefolds, through the work of a wide group of researchers. These developments are described a new book by Araujo et al, and we refer to it and its references for details on what we will mention.

Examples

There are 105 deformation families of Fano threefolds, and the collection of these is referred to as the Mori–Mukai list of Fano threefolds. In most of these families, either all members are K-polystable or are strictly K-unstable. However, there exists some families with both K-polystable and K-unstable members, including strictly K-semistable ones, which is the most important for our purposes.

Examples

As an illustration of the method, we present one example here. All the explicit families we consider are described in the new book on K-stability of Fano threefolds.

Lemma 5.

Let X be the Fano threefold in the family 4.13 of the Mori–Mukai list given as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, with homogenous coordinates $([x, y], [u, v], [p, q])$, in the curve given by the two equations

$$xv - yu = x^3 p + y^3 q + xy^2 p = 0.$$

Then there exists a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ for which the construction of Theorem 4 applies.

Proof of Lemma 5

X admits a test configuration degenerating X to the Fano threefold X_0 given as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in the curve

$$xv - yu = x^3 p + y^3 q = 0$$

simply by scaling the latter equation. That is, the test configuration is given by the family obtained by blowing up in

$$xv - yu = x^3 p + y^3 q + sxy^2 p = 0$$

for $s \in \mathbb{C}$. The central fibre is K-polystable by the Fano threefolds project, and has automorphism group $\mathbb{C}^*$, while X has trivial automorphism group. It follows that the construction applies to a test configuration for $(X, -K_X)$.

Examples

Similar techniques show that the following families produce at least one test configuration to which the construction applies. In some of these cases, we produce infinitely many examples, and in others, only one.

Theorem 6 (S.–Spotti '21).

The following families from the Mori–Mukai list of Fano threefolds produce at least one test configuration for a strictly K-semistable manifold to which the construction applies: 1.10, 2.20, 2.21, 2.22, 2.24, 3.5, 3.8, 3.10, 3.12, 4.13. Together, they give infinitely many projective manifolds that admit extremal Kähler metrics in some classes.

Examples

Remark.

The above result says that in each of the cases mentioned, we have found at least one member, to which we can apply our construction. We have not checked every possible member of the family in order to get a complete classification. Also, not all the families will give test configurations to which we can apply the construction, even if there are strictly K-semistable members of these families. This can come both from there being no smooth K-polystable member in the family, or that the discrepancy condition on the reduced automorphism groups does not hold.

Thank you!