Dispersive estimates for non-linear waves in mathematical General Relativity

1   Lecturer:

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2   Mode of the discussion, venue and time, audience:

- Offline, in English, at A513 in the Shuangqing Complex Building A, at Tsinghua University.

- During the Spring semester 2025, starting February 19, 2025. Two sessions per week: every Wednesday, at 11:00–12:30 and then at 14:00–15:30.

- Audience: Graduate, Postdoc, Researcher.

3   Prerequisite:

Basic knowledge from my previous courses on "The Cauchy problem in mathematical General Relativity" and on "Non-linear wave equations in General Relativity", graduate level knowledge in differential geometry and in Riemannian geometry, and basic knowledge in partial differential equations and analysis.

4   Introduction:

This course introduces the tools of analysis for proving dispersive estimates for solutions of non-linear wave equations arising in General Relativity from the point of view of analysis and partial differential equations. The course material builds on one hand, on a previous course that I gave in the Spring 2024 on the Cauchy problem in mathematical General Relativity, and on the other hand, on a previous course that I gave in the Fall 2024 on non-linear wave equations in General Relativity. The goal of this course is to show a method to obtain dispersive estimates for solutions of tensorial coupled non-linear hyperbolic partial differential equations, provided that one exploits the non-linear structure of the wave equations. We shall exhibit how this can be applied in General Relativity for perturbations of the Minkowski space-time governed by the Einstein-Yang-Mills system in the Lorenz gauge and in wave coordinates, by studying first the simpler case of higher dimensions.

5   Keywords:

Energy estimates, non-linear wave equations, hyperbolic partial differential equations, Minkowski vector fields, Klainerman-Sobolev inequality, weighted energy norms, bootstrap argument, decay, Hardy type inequality, commutator term, Grönwall type inequality, Einstein equations, Yang-Mills fields, Einstein-Yang-Mills system, gauge transformations, Minkowski metric, wave coordinates, Lorenz gauge, stability of Minkowski space-time.

6   Syllabus:

1. **Reminders of prerequisites:**
   The Einstein equations, the Yang-Mills equations, the Einstein-Yang-Mills system, the Lorenz gauge, wave coordinates, recasting the Einstein-Yang-Mills system as a coupled system of non-linear hyperbolic partial differential equations, the hyperbolic Cauchy problem, the constraint equations, the gauges invariance of the equations.

2. **Set-up of analysis for proving decay for solutions of non-linear wave equations:**

   - The Minkowski vector fields.
   - Weighted Klainerman-Sobolev inequality.
   - Definition of the norms.
   - The energy norm.
   - The bootstrap argument.
   - The bootstrap assumption.
   - The big O notation.

3. **À priori decay estimates:**

   - The spatial asymptotic behaviour of the fields on the initial hypersurface.
   - Estimates on the time evolution of the fields.

4. **Looking at the structure of the source terms of the coupled non-linear wave equations for the Einstein-Yang-Mills system in the Lorenz gauge and in wave coordinates.**

5. **Using the bootstrap assumption to exhibit the structure of the source terms of the Einstein-Yang-Mills system:**

   - Using the bootstrap assumption to exhibit the structure of the source terms for the Yang-Mills potential.
   - Using the bootstrap assumption to exhibit the structure of the source terms for the metric.
   - The source terms for $n \geq 5$.

6. **Energy estimates.**

7. **A Hardy type inequality.**

8. **The commutator term for $n \geq 4$:**

   - Using the Hardy type inequality to estimate the commutator term.

9. **The energy estimate for $n \geq 4$:**

   - The main energy estimate for the fields for $n \geq 4$.

10. **The proof of global stability of Minkowski space-time for $n \geq 5$:**

    - Using the Hardy type inequality for the space-time integrals of the source terms for $n \geq 5$.
    - Grönwall type inequality on the energy for $n \geq 5$.
    - The proof of the global stability for $n \geq 5$.

7 References:

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