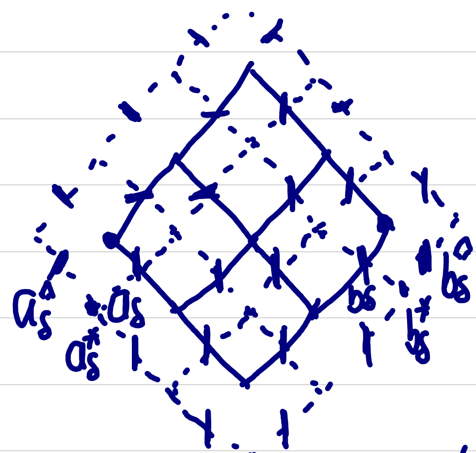
1.

- Convergence of FK-Ising model
- Ising model interface : tightness

2. Convergence of Ising model
   - positive - negative
   - positive - negative - free

$L_\delta := \sqrt{2}\delta e^{\frac{i\pi}{4}} \mathbb{Z}^2$



Ω_δ : discrete simply-connected domain on L_δ.

$\partial\Omega_\delta$: closed simple curve L_δ

Ω_δ^* : dual domain

$\Omega_\delta^\diamond$:

$V(\Omega_\delta^\diamond) := \{$midpoints of edges of $\Omega_\delta \cup \Omega_\delta^*\}$

$E(\Omega_\delta^\diamond) := \{$straight lines connecting nearest vertices$\}$

Fix two boundary points on medial graph $a_\delta^\diamond, b_\delta^\diamond$.

FK-Ising model on Ω_δ

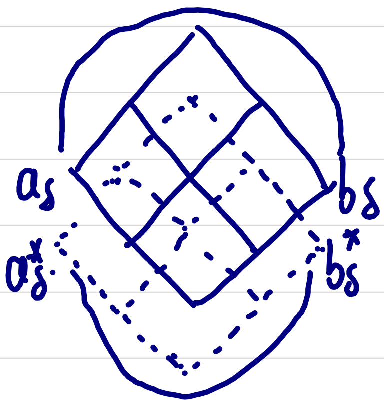
edge configuration w, $P(w) = \frac{1}{Z_{FK}} \left(\frac{p}{p-1}\right)^{|o(w)|} 2^{C(w)}$

para : p Z_{FK} : normalization

$p = p_c = \frac{\sqrt{2}}{\sqrt{2}+1}$ — dual edge configuration on Ω_δ^* is also critical FK-Ising.

$(a_\delta b_\delta)$ wired $(b_\delta a_\delta)$ free

$(a_\delta^* b_\delta^*)$ free $(b_\delta^* a_\delta^*)$ wired



$\exists!$ $\underline{\gamma_\delta}$ $\Omega_\delta^\diamond$ from $a_\delta^\diamond$ to $b_\delta^\diamond$ which does not cross open edges or dual open edges.

Convergence of discrete domains

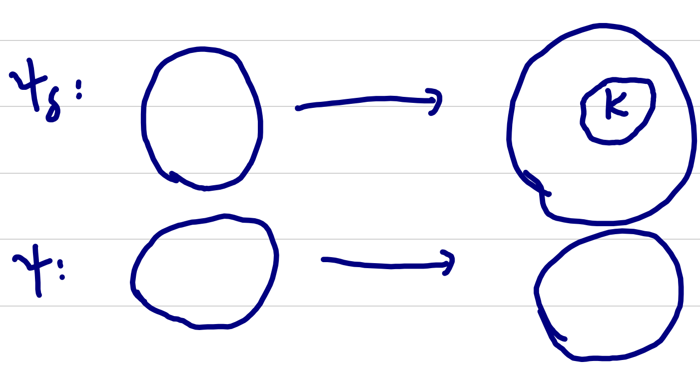
$(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond) \longrightarrow (\Omega; a, b)$ <u>in the Carathéodory sense</u>:

- $a_\delta^\diamond \to a$, $b_\delta^\diamond \to b$
- $\exists \varphi_\delta : (\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond) \longrightarrow (U; -1, 1)$

  $\varphi : (\Omega; a, b) \longrightarrow (U; -1, 1)$

$\varphi_\delta^{-1} \longrightarrow \varphi^{-1}$ locally uniformly

$\varphi_\delta^{-1}|_K \rightrightarrows \varphi^{-1}|_K$.



Space of Curves

path: $[0,1] \longrightarrow \mathbb{C}$ continuous map

C unparameterized paths in $\mathbb{C}$

$\forall \gamma_1, \gamma_2 \in C$ $\quad \boxed{d}(\gamma_1,\gamma_2) := \inf \sup_{t\in[0,1]} |\hat{\gamma}_1(t) - \hat{\gamma}_2(t)|$

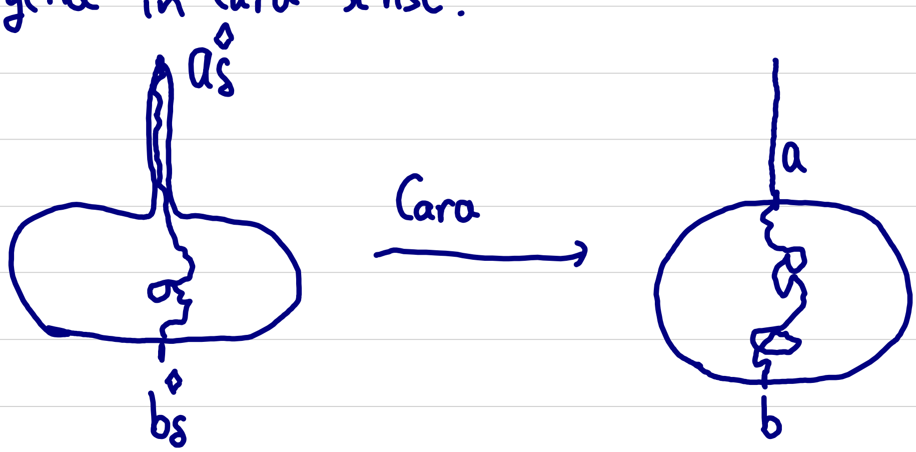
$\inf \begin{cases} \hat{\gamma}_1 \longleftrightarrow \gamma_1 \\ \hat{\gamma}_2 \longleftrightarrow \gamma_2 \end{cases}$

Theorem 1.1: Suppose $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ converges to $(\Omega; a, b)$ in the Carathéodory sense. Then, $\underline{\underline{\varphi_\delta(\gamma_\delta)}}$ converges to $SLE_{16/3}$ from -1 to 1 <u>as curves</u> in law.

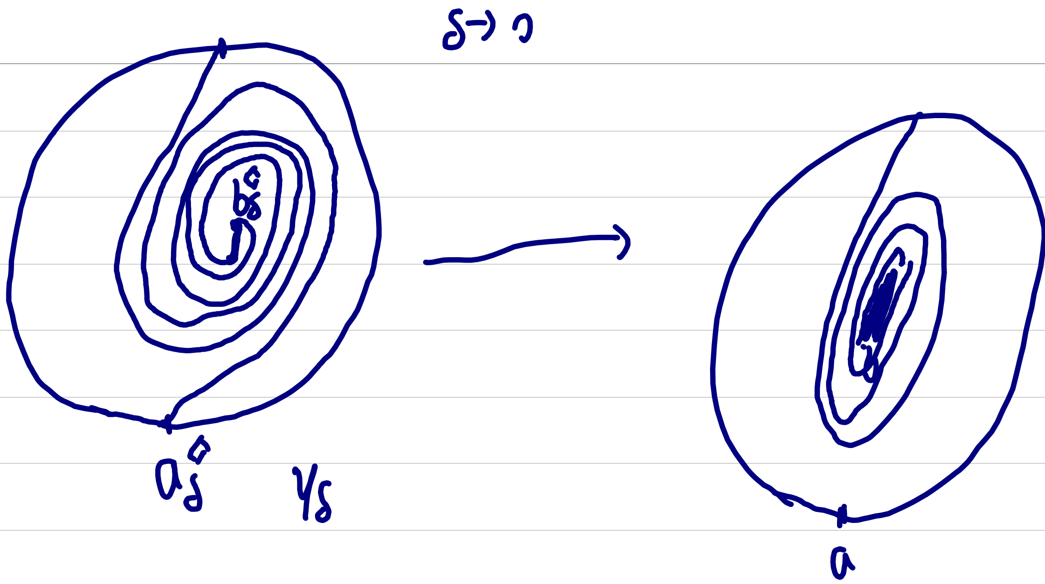
$\gamma_\delta \ \times$

We consider the convergence of $\varphi_\delta(\gamma_\delta)$:

- Convergence in Cara sense.



- $\partial\Omega$



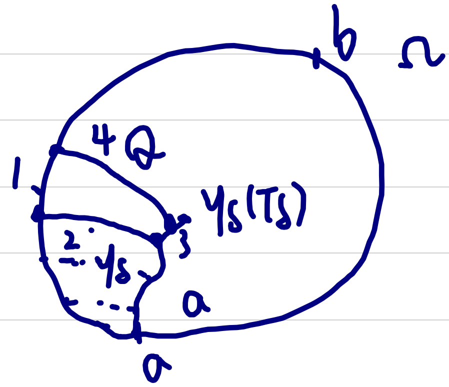
Three steps:

1. Tightness of $\varphi_\delta(\gamma_\delta)$
2. Construct observable and prove the convergence
3. derive the law of any sublimit by using observable.

1. Check that $\{\varphi_\delta(\gamma_\delta)\}$ satisfies C_2 condition:

$\exists M>0$, such that $\forall \delta>0$, $\forall 0\le T_\delta \le 1$ for γ_δ, $\forall$ $\boxed{\text{avoidable}}$ quadrilateral Q of $\Omega_\delta \setminus \gamma_\delta[0,T_\delta]$, $m(Q)$ is larger than M.

$$P(\gamma_\delta[T_\delta,1] \text{ crosses } Q \mid \gamma_\delta[0,T_\delta]) < \tfrac{1}{2}.$$



Q does not separate $\gamma_\delta(T_\delta)$ and b in $\Omega_\delta \setminus \gamma_\delta[0,T_\delta]$

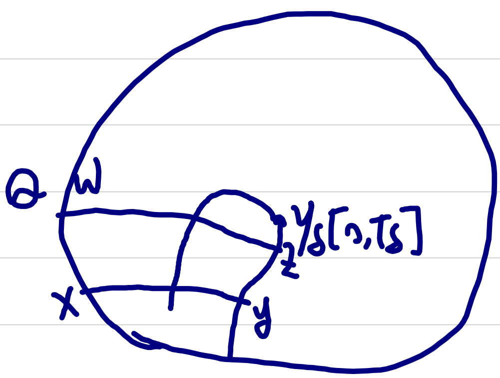
· γ_δ satisfies C_2 condition $\Longleftrightarrow$ $\varphi_\delta(\gamma_\delta)$ satisfies C_2 condition.

· $\frac{1}{2}$ is not important

Theorem: $\forall L>0$, there exists $\eta=\eta(L)>0$ such that:

$\forall\, (Q;x,y,z,w)$ $m(Q)>L$, $\forall$ boundary conditions ξ

$\underline{P}^{\xi}[\,(xy)$ connects to (zw) by open edges$\,] \leq \underline{1-\eta}$.



By [KS, Theorem 1.5], $\{\gamma_\delta\}$ satisfies C_2 condition

$\Longrightarrow \{\varphi_\delta(\gamma_\delta)\}_{\delta>0}$ is tight.

For the second step:

edge FK-fermionic observable:

$$F_\delta(e) \doteq E\left[\mathbb{1}_{\{e\in\gamma_\delta\}} e^{\frac{i}{2}\underline{W_{\gamma_\delta}(e,b_\delta^\diamond)}}\right]$$

we always assume the edge connecting to $b_\delta^\diamond$ is horizontal.

vertex FK-fermionic observable: $\quad F_\delta(v) = \frac{1}{2}\sum_{v\sim e} F_\delta(e)$.

Theorem 1.4: Suppose $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ converges to $(\Omega; a, b)$ in the Cara sense. Then, for the vertex FK-fermionic observable F_δ,

$$\frac{1}{\sqrt{2\delta}} F_\delta \longrightarrow \underline{\underline{\sqrt{\phi'}}} \text{ locally uniformly.}$$

ϕ is any conformal map from $(\Omega; a, b) \to (\mathbb{R}\times(0,1); -\infty, +\infty)$.

$\phi_1 \quad \phi_2 \quad \exists C$ s.t. $\phi_1 = \phi_2 + C$

for the third step:

For any stopping time T_δ for γ_δ, denote by $\underline{\underline{F_\delta^{T_\delta}}}$ the vertex obs

$\boxed{(\Omega_\delta^\diamond \setminus \gamma_\delta[0,T_\delta]; \gamma_\delta(T_\delta), b_\delta^\diamond).}$

Lemma 1.5 For every $v \in V(\Omega_\delta^\diamond)$, $\{F_\delta^n\}$ is mart before the hitting time of v.

Pf: The def of $F_\delta^{T_\delta}$ is in conditional sense.

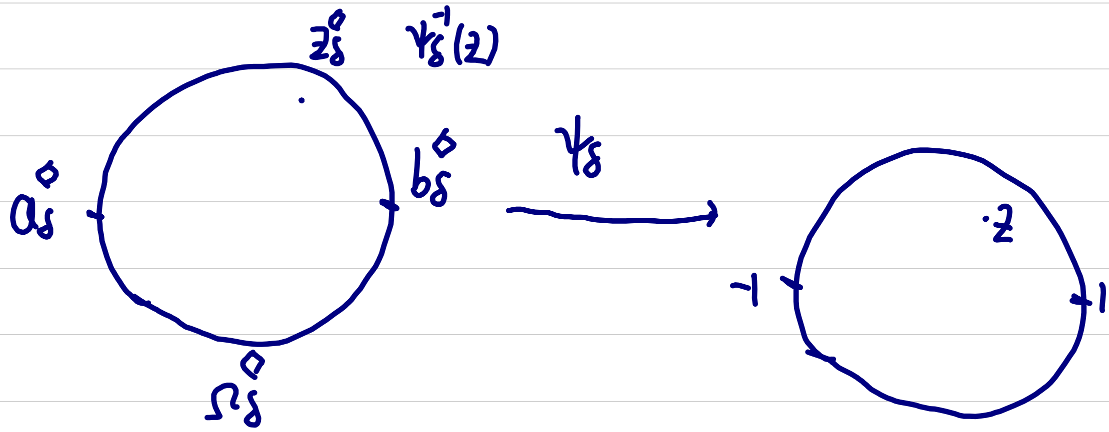
Thm 1.4 + lem 1.5 $\Longrightarrow$ Thm 1.1.

$\{\gamma_\delta(\gamma_\delta)\}$ is tight $\Longrightarrow$ We can choose sublimit γ.

We only need to prove that the law of γ is SLE16/3 from -1 to 1.

We may assume $\gamma_\delta(\gamma_\delta) \longrightarrow \gamma$ in law.

Lem 1.5 : $F_\delta^{n}(z_\delta^\diamond)$ is a mart before hitting $z_\delta^\diamond$



$$E[F_\delta^{n\wedge T_\delta}(z_\delta^\diamond) f(\psi_\delta(\gamma_\delta[0,n\wedge T_\delta]))] = E(F_\delta^{m\wedge T_\delta}(z_\delta^\diamond) f(\psi_\delta(\gamma_\delta[0,m\wedge T_\delta])))$$

f is a continuous function on $\mathbb{C}$.

$\delta\to 0$:

We can prove: $\exists\{T_i\}$ for γ, $\{T_i\}$ is dense in $[0,1]$.

$\exists T_i^\delta$ for γ_δ, $T_i^\delta \to T_i$ in law.

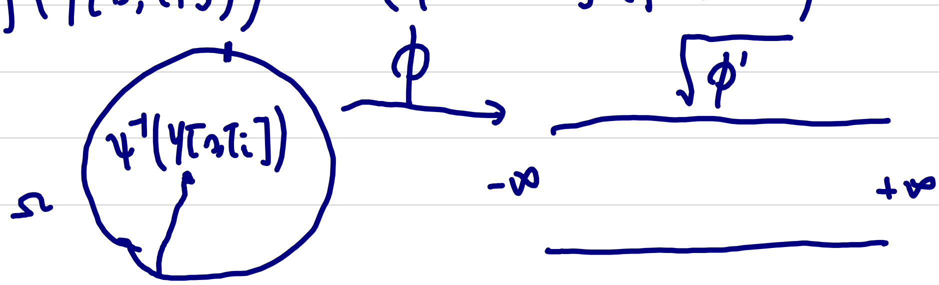
$\gamma_\delta(\underline{n_\delta\wedge T_\delta})$ $\quad n_\delta\wedge T_\delta \xrightarrow{?}$ stopping time for γ. (hard).

$T_1 < T_2$

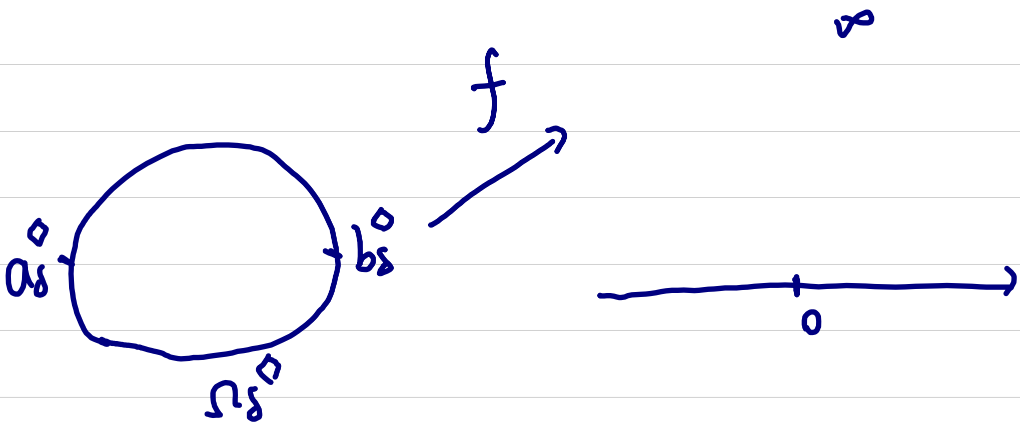
$n\to T_1^\delta$, $m\to T_2^\delta$ $\quad \delta\to 0 \Rightarrow$

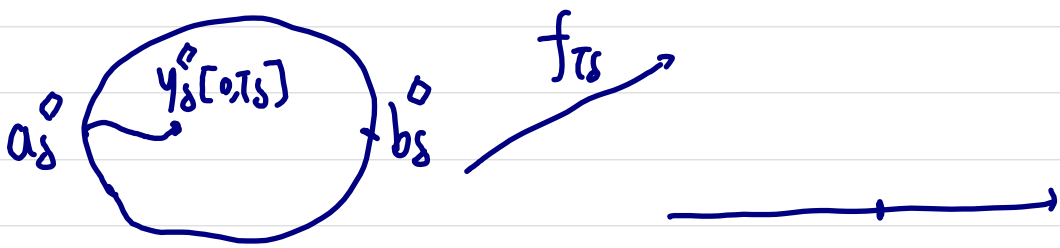
$$E(F^{T_1}(z) f(\gamma[0,T_1])) = E(F^{T_2}(z) f(\gamma[0,T_2]))$$

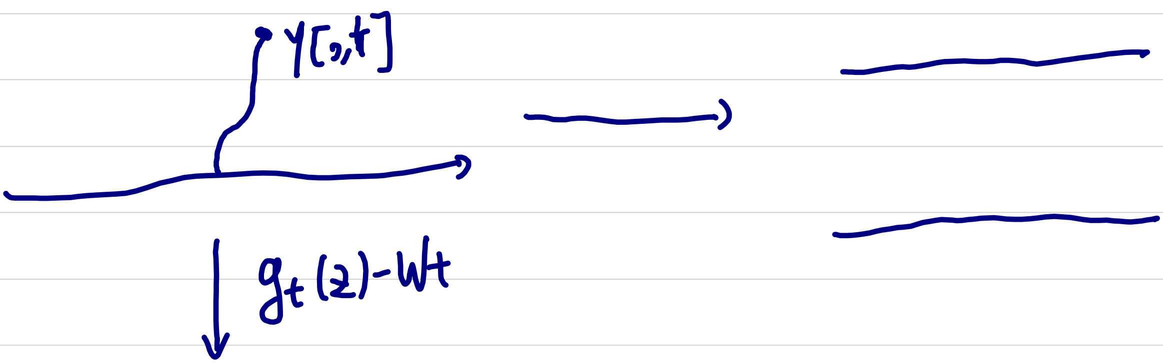
$F^{T_i}(z)$:

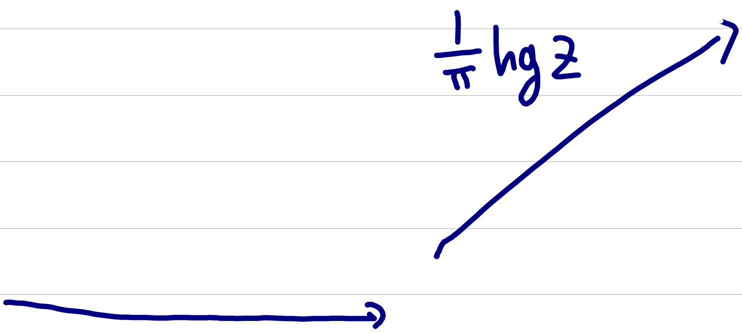


This implies $F^t(z)$ is mart (for filt generated by γ).









Thus, we have

$$F^t(z) = \sqrt{\left(\frac{1}{\pi}\log(g_t(z)-W_t)\right)'}$$

$$= \underline{\frac{1}{\sqrt{\pi}}\sqrt{\frac{g_t'(z)}{g_t(z)-W_t}}} \quad := M_t(z)$$

$(\Omega; a, b) = (H; 0, \infty)$

$W_t = g_t(z) - \frac{1}{\pi}\frac{g_t'(z)}{M_t^2} \Rightarrow W_t$ is a semimartingale.

$W_t = N_t + L_t$, $\{N_t\}$ is a martingale, $\{L_t\}$ is a bounded vari process.

$$\underline{d\,g_t'(z) = \partial_t g_t'(z)\,dt = \frac{-2g_t'(z)}{(g_t(z)-W_t)^2}\,dt}$$

$$\left(\partial_t g_t(z) = \frac{2}{g_t(z)-W_t}\right)$$

$$dM_t = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\frac{d\,g_t'(z)}{\sqrt{g_t'(z)}}\cdot\frac{1}{g_t(z)-W_t} - \frac{1}{2}\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^3}\,d(g_t(z)-W_t)\right.$$
$$\left. + \frac{3}{8}\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^5}\,d\langle W\rangle_t\right.$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{1}{2}\,\frac{-2g_t'(z)}{(g_t(z)-W_t)^2}\cdot\frac{1}{\sqrt{g_t'(z)}}\cdot\frac{1}{\sqrt{g_t(z)-W_t}}\,dt - \frac{1}{2}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,3}}\left(\frac{2dt}{g_t(z)-W_t} - dN_t - dL_t\right)\right.$$

$$\left. + \frac{3}{8}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,5}}\,d\langle W\rangle_t\right)$$

$g_t(z)-W_t = z + o(1) \quad z\to\infty$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{3}{8}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,5}}\left(d\langle W\rangle_t - \frac{16}{3}dt\right) + \frac{1}{2}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,3}}\left(dN_t + dL_t\right)\right)$$

drift term $=0$, $\quad \frac{3}{8}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,5}}\left(d\langle W\rangle_t - \frac{16}{3}dt\right) + \frac{1}{2}\,\frac{\sqrt{g_t'(z)}}{\sqrt{g_t(z)-W_t}^{\,3}}\,dL_t = 0$

$$\frac{3}{8}\,\frac{1}{g_t(z)-W_t}\left(d\langle W\rangle_t - \frac{16}{3}dt\right) + \frac{1}{2}dL_t = 0.$$

$z\to\infty \Rightarrow dL_t = 0 \Rightarrow d\langle W\rangle_t = \frac{16}{3}dt$

This implies that $W_t \overset{law}{=} \sqrt{\frac{16}{3}}\,B_t$.

$\Rightarrow \gamma$ is $SLE_{16/3}$.