Stochastic Analysis and its Applications

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Weeks of March 7 – June 10, 2022,
14 weeks, 28 lectures

Lecture No 1 March 8, 2022 (Tue)

Plan of the course "Stochastic Analysis and its Applications" (28 lectures)

Abstract: The first half of the course is devoted to explaining fundamental concepts, terms, facts and tools in probability theory and stochastic analysis. Then, in the second half, we pick up some topics in stochastic partial differential equations as applications of stochastic analysis.

Contents:

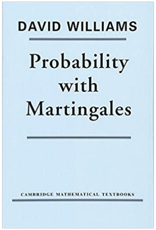
Part I. Foundations of Probability Theory (5∼6 lectures)
Probability space, Dynkin's π-λ theorem, Convergence of random variables, Independence, Conditional probability, Strong law of large numbers, Kolmogorov's inequality, Convergence in law, Central limit theorem

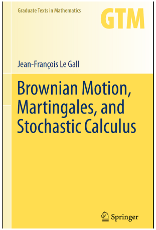
Part II. Foundations of Stochastic Analysis (9∼10 lectures)
Discrete and continuous time martingales, Brownian motion, Stochastic integrals, Ito's formula, Stochastic differential equations, Relation to PDEs

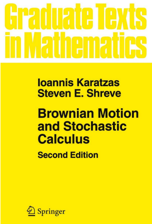
Part III. Applications of Stochastic Analysis (14∼12 lectures)
Stochastic partial differential equations, Random interfaces, (Stochastic) Motion by mean curvature, Stochastic Allen-Cahn equation, Time-dependent Ginzburg-Landau equation, Other topics.

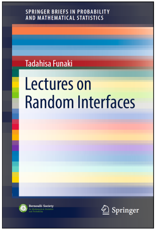
Textbooks:

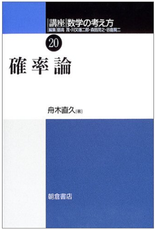
[1] D. Williams: Probability with Martingales, Cambridge, 1991. (→Part I, II)
[2] J-F. Le Gall, Brownian Motion, Martingales, and Stochastic Calculus, Springer, 2013. (→Part II)
[3] I. Karatzas and S.E. Shreve: Brownian Motion and Stochastic Calculus, Springer, 1991. (→Part II)
[4] T. Funaki, Lectures on Random Interfaces, SpringerBriefs, 2016. (→Part III)
[5] 舟木直久 (T. Funaki): 確率論, 朝倉書店, 2004. (→Part I, II)
[6] 舟木直久: 確率微分方程式, 岩波書店, 2005 (1997). (→Part II)
[7] 舟木直久, 乙部厳己 (Y. Otobe), 謝賓 (B. Xie): 確率偏微分方程式, 岩波書店, 2019. (→Part III)



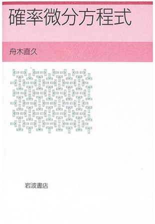




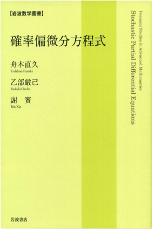




[5] **概率论**



[6] **随机微分方程**



[7] **随机偏微分方程**

Introduction

- Probability theory has applications to Natural Science (physics, biology, chemistry, $\cdots$), Engineering, Social Science (finance, economy, $\cdots$) and others.
- So, on the one hand, it is considered as an applied mathematics, but on the other hand, it has an axiomatic system and a field of pure mathematics.
- Probability theory recently attracts a lot of attention.

Gauss Prize   Fields Medalists in/related to probability

Itô
2006

Hairer
2014

Werner
2006

Smirnov
2010

Okounkov
2006

from web of Oberwolfach Math Institute

Modern probability theory:

- 1902: Establishment of Lebesgue integrals theory
- 1935: Formulation by Kolmogorov, for example, strong law of large numbers, in a framework of measure theory
- σ-additivity of probability plays a fundamental role to study several types of limits adjusting to the probability.

Lebesgue Kolmogorov
from Wikipedia

- We denote a thing which may happen by ω, and Ω is the set of all ω.
- Example (dice): $\Omega = \{1, 2, \ldots, 6\}$
  Probability: $P(\{\omega\}) = \frac{1}{6}$, ${}^{\forall}\omega \in \Omega$.
- In general Ω is an infinite set, for example, for dice thrown many times, we take $\Omega = \{1, 2, \ldots, 6\}^{\mathbb{N}}$ and a probability $P(A)$ is defined for $A \subset \Omega$. The set A for which the probability $P(A)$ is defined is called an event.
- The probability of the whole set is normalized as 1 so that $P(\Omega) = 1$.

- Modern probability theory assumes not only the finite additivity of P:

$$A, B : \text{disjoint events} \Longrightarrow P(A \cup B) = P(A) + P(B)$$

but also the σ-additivity:

$$A_1, A_2, \ldots : \text{disjoint events} \Longrightarrow P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

- Thus, the probability theory is built on the Lebesgue's measure theory. For instance for Brown motion discussed later, we take $\Omega = C([0, \infty), \mathbb{R}^d)$ that is the set of all possible continuous paths $x(t) \in \mathbb{R}^d, t \in [0, \infty)$.
- In particular, we need to consider measures on infinite-dimensional spaces.

Part I. Foundations of Probability Theory

§1 Probability space, Random variables, Probability distributions, Expectation and variance

1.1 Probability space

- Ω: a certain set
- $\mathcal{F}$: σ-field (σ-algebra) of Ω i.e., $\mathcal{F} \subset \mathcal{P}(\Omega)$ (i.e., $\mathcal{F}$ is a family of subsets of Ω) and satisfies
  (1) $\Omega \in \mathcal{F}$
  (2) $A \in \mathcal{F} \Longrightarrow A^c := \Omega \setminus A \in \mathcal{F}$
  (3) $A_n \in \mathcal{F}, n = 1, 2, \ldots \Longrightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$
- The pair of Ω and its σ-field $(\Omega, \mathcal{F})$ is called a measurable space. In probability theory, Ω is called a sample space, $\omega \in \Omega$ a sample and $A \in \mathcal{F}$ an event.

- A measure P on a measurable space $(\Omega, \mathcal{F})$ satisfying $P(\Omega) = 1$ is called a probability measure i.e.
  (1) $P : \mathcal{F} \to [0, 1]$ and $P(\Omega) = 1$
  (2) (σ-additivity) If $A_n \in \mathcal{F}, n = 1, 2, \ldots$ are disjoint (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$P\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} P(A_n)$$

[Definition 1.1] A triplet $(\Omega, \mathcal{F}, P)$ is called a probability space. $\square$

- For a σ-field $\mathcal{F}$, we have
  (1)' $\emptyset \in \mathcal{F}$
  (3)' $A_n \in \mathcal{F}, n = 1, 2, \ldots \Longrightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
- For a probability space $(\Omega, \mathcal{F}, P)$,
  - $P(\emptyset) = 0$
  - (Monotonicity) $A, B \in \mathcal{F}$: $A \subset B \Longrightarrow P(A) \leq P(B)$

- For any family $\mathcal{A}$ of subsets of Ω, that is $\mathcal{A} \subset \mathcal{P}(\Omega)$, there exists uniquely a smallest σ-field $\mathcal{F}$ which contains $\mathcal{A}$, i.e. a σ-field $\mathcal{F}$ satisfying

$$\mathcal{G} : \sigma\text{-field and } \mathcal{A} \subset \mathcal{G} \Longrightarrow \mathcal{F} \subset \mathcal{G}$$

  We denote such $\mathcal{F}$ by $\sigma(\mathcal{A})$, and call it the σ-field generated by $\mathcal{A}$. Indeed, we may take

$$\mathcal{F} = \bigcap_{\mathcal{G}:\sigma\text{-field}, \mathcal{A} \subset \mathcal{G}} \mathcal{G}$$

  P: Check that this $\mathcal{F}$ is a σ-field.

- A special character of probability theory, different from other areas, is to consider several σ-fields on a common space Ω at the same time ($\rightarrow$ Independence, Conditional probability, Martingale).

[Example] (Generalization of dice throwing)

- Let a measurable space $(S, \mathcal{S})$ be given ($S = \{1, 2, \dots, 6\}$, $\mathcal{S} = \mathcal{P}(S)$ for dice).
- $\Omega = S^{\mathbb{N}}$: product space i.e., $\Omega = \{S$-valued trials repeated infinitely many times$\}$
- Kolmogorov's σ-field $\mathcal{F}_K$ is a natural σ-field of this Ω:

$$\mathcal{F}_K := \sigma\{C; \text{cylinder sets}\},$$

where a cylinder set means a subset of Ω of the following form

$$\begin{aligned} C &\equiv C(t_1, \dots, t_n; A_1, \dots, A_n), \quad t_i \in \mathbb{N}, A_i \in \mathcal{S}\,(1 \le i \le n) \\ &= \{\omega = (\omega(t))_{t \in \mathbb{N}} \in \Omega; \omega(t_1) \in A_1, \dots, \omega(t_n) \in A_n\} \end{aligned}$$

- We need to define probability at least for C of this form. □

- Equivalent definitions of σ-additivity
  Assume P is finitely additive. Then the following (1)–(3) are mutually equivalent:
  (1) P is σ–additive
  (2) If $A_n \in \mathcal{F}, n = 1, 2, \ldots$ is monotone increasing (that is, $A_1 \subset A_2 \subset \cdots$, we denote $A_n \nearrow$) , then
  $$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$
  (3) If $A_n \in \mathcal{F}, n = 1, 2, \ldots$ is monotone decreasing (that is, $A_1 \supset A_2 \supset \cdots$, we denote $A_n \searrow$), then
  $$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$
  (2), (3) are called continuity of a measure.
- Subadditivity: For $A_n \in \mathcal{F}, n = 1, 2, \ldots,$
  $$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

- If $A \in \mathcal{F}$ satisfies $P(A) = 1$, we say A happens almost surely, and write "A holds for P-$a.s.$ ω" or "A $a.s.$".

1.2 Random variables (denoted by r.v.'s)

$(\Omega, \mathcal{F}, P)$: Probability space

[Definition 1.2] (1) Let $(S, \mathcal{S})$ be a measurable space and let an $\mathcal{S}$-valued function $X = X(\omega)$ on Ω be given. If X is measurable as a map

$$X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$$

(i.e., For any ${}^{\forall}A \in \mathcal{S}$, $X^{-1}(A) \equiv \{\omega \in \Omega; X(\omega) \in A\} \in \mathcal{F}$), then X is called an S-valued random variable (r.v.).
(2) In particular, when $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, X is called a real-valued random variable, where $\mathcal{B}(\mathbb{R}) := \sigma\{$open sets of $\mathbb{R}\}$ is a Borel field of $\mathbb{R}$. $\square$

- The following five conditions are mutually equivalent:
  1. X is a real-valued r.v.
  2. For ${}^{\forall}a \in \mathbb{R}$, $\{X \leq a\} \in \mathcal{F}$
  3. For ${}^{\forall}a \in \mathbb{R}$, $\{X < a\} \in \mathcal{F}$
  4. For ${}^{\forall}a \in \mathbb{R}$, $\{X \geq a\} \in \mathcal{F}$
  5. For ${}^{\forall}a \in \mathbb{R}$, $\{X > a\} \in \mathcal{F}$

  [Remark] $\{X \leq a\} := \{\omega \in \Omega; X(\omega) \leq a\}$
  $\equiv X^{-1}((-\infty, a])$ (i.e., We often omit ω)
- composite function: $X_1, X_2, \ldots, X_n$: real-valued r.v.'s
  $g : (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$: measurable
  $\implies Y := g(X_1, X_2, \ldots, X_n)$ is also a r.v.
  [Remark] Also here, we omit ω. Precisely,
  $Y(\omega) = g\big(X_1(\omega), X_2(\omega), \ldots, X_n(\omega)\big)$.

  P: Show the above property for composite functions.

- Limits of r.v.'s: $X_n, n = 1, 2, \ldots$: real-valued r.v.'s, then

$$\inf_{n\geq 1} X_n, \quad \sup_{n\geq 1} X_n, \quad \liminf_{n\to\infty} X_n, \quad \limsup_{n\to\infty} X_n$$

  are all r.v.'s (if they take finite values). In particular, if $X = \lim_{n\to\infty} X_n$ exists, X is a r.v.
  [Note] To show this, it is essential that $\mathcal{F}$ is a σ-field.
- σ-field generated by a r.v.: For S-valued r.v. X, set

$$\mathcal{F}_X := \{X^{-1}(A)\,;\, A \in \mathcal{S}\}.$$

  Then, $\mathcal{F}_X$ is a σ-field of Ω. We call $\mathcal{F}_X$ a σ-field generated by X and denote also by $\sigma(X)$.

  P: Show that $\mathcal{F}_X$ is a σ-field.

[Example] (Generalization of dice throwing) When $(S, \mathcal{S})$ is given, we defined $\Omega = S^{\mathbb{N}}$ and $\mathcal{F}$ as its Kolmogorov's σ-field $\mathcal{F}_K$. For $\omega \in \Omega$ written as $\omega = (\omega(1), \omega(2), \cdots)$,

$$X_n(\omega) := \omega(n)$$

is an S-valued r.v. (that is, $\mathcal{F}_K/\mathcal{S}$-measurable function). X_n represents the number of the dice on its nth throw. □

1.3 Probability distribution

- Distribution For S-valued r.v. X,

  $$P_X(A) := P(X^{-1}(A)), \qquad A \in \mathcal{S}$$

  determines a probability measure (image measure) on $(S, \mathcal{S})$. It is called a distribution of X.
- Distribution function For a real-valued r.v. X,

  $$F_X(x) = P(X \le x), \quad x \in \mathbb{R}$$

  is called a distribution function of X.

- Properties of distribution function $F = F_X$
  (1) Monotone increasing (non-decreasing):
  $x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$
  (2) $\lim_{x\to\infty} F(x) = 1, \quad \lim_{x\to-\infty} F(x) = 0$
  (3) $F(x)$ is right-continuous

  The converse is known and true: If a function $F : \mathbb{R} \to [0, 1]$ satisfies three conditions (1)–(3), then there exists uniquely a probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

  $$F(x) = \mu((-\infty, x]), \quad {}^{\forall}x \in \mathbb{R}.$$

  μ is called the Lebesgue–Stieltjes measure of F.

  [Remark] Lebesgue–Stieltjes measure corresponding to $F(x) = 0\,(x \leq 0), x\,(0 \leq x \leq 1), 1\,(x \geq 1)$ is the Lebesgue measure on $[0, 1]$.

1.4 Expectation and variance

Since a real-valued r.v. X defined on a probability space $(\Omega, \mathcal{F}, P)$ is measurable as a map $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, one can define its integral with respect to the measure P in Lebesgue's sense:

$$\int_\Omega X(\omega)\, P(d\omega)$$

We call this Expectation (or Mean) of X and denote $E[X]$. We first briefly recall the definition of integrals in Lebesgue's sense.

Definition of expectation

(1) If a r.v. X takes only finitely many values, i.e.,

$$X(\omega) = \sum_{i=1}^{k} a_i 1_{C_i}(\omega), \quad a_i \in \mathbb{R},\ C_i \in \mathcal{F},$$

X is called simple and its expectation $E[X]$ is defined by

$$E[X] := \sum_{i=1}^{k} a_i P(C_i).$$

(2) For a non-negative r.v. $X \geq 0$, one can take an increasing sequence of non-negative simple r.v.'s $(X_n)_{n=1,2,\ldots}$ such that $X(\omega) = \lim_{n\to\infty} X_n(\omega)$, ${}^\forall\omega$ holds. Then the expectation $E[X]$ is defined by

$$E[X] := \lim_{n\to\infty} E[X_n] \in \bar{\mathbb{R}}_+ (\equiv [0, \infty])$$

Note that this value is determined independently of the choice of an approximating sequence X_n.

(3) General r.v. X is decomposed as $X = X^+ - X^-$. If $E[X^+] < \infty$ or $E[X^-] < \infty$ hold, the expectation of X is defined by

$$E[X] = E[X^+] - E[X^-] \in [-\infty, \infty],$$

where $X^\pm (= X^\pm(\omega))$ denote positive and negative parts of X:

$$X^+(\omega) = \max\{X(\omega), 0\}, \quad X^-(\omega) = \max\{-X(\omega), 0\}.$$

When both $E[X^\pm]$ are finite (i.e., $|E[X]| < \infty$), X is called integrable. $\square$