Lecture No 27   June 10, 2022 (Fri)

4.2.2. $d \geq 2$: D is b'dd domain in $\mathbb{R}^d$ with smooth boundary

- Consider for $D \subset \mathbb{R}^d$, $d \geq 2$:

$$\partial_t u = \Delta u + \frac{1}{\varepsilon^2} f(u) + \dot{W}^\varepsilon(t,x), \quad x \in D, \tag{1}$$

  with Neumann boundary condition: $\partial u / \partial n = 0$ $(x \in \partial D)$.
- We assume $A(f) = 0$, but don't require that f is odd.
- When $d \geq 2$, SPDEs with space-time Gaussian white noise are ill-posed.
- Sharp interface limit and the derivation of SMMC are discussed only for limited noises, actually only for time-dependent noises (except Lions-Souganidis).
- F (1999) studied the case that the limit interfaces are convex in 2D setting, which is extended by Weber (2010), see below.

Smeared noise

- The noise $\dot{W}^\varepsilon(t,x) = \frac{1}{\varepsilon}\xi_t^\varepsilon$ depends only on t, and
  $\xi_t^\varepsilon = \varepsilon^{-\gamma}\xi(\varepsilon^{-2\gamma}t), 0 < \gamma < 2/3$.
- Here, $\xi(t) \in C^1(\mathbb{R}_+)$, a.s. is a stationary process with mean 0 and strong mixing property.
- We have that $\xi_t^\varepsilon \Rightarrow \alpha\dot{W}_t$ $(\varepsilon \downarrow 0)$ (more precisely, $\int_0^t \xi_s^\varepsilon ds \Rightarrow \alpha W_t$ in law by CLT), but cannot treat the case that $\xi_t^\varepsilon = \alpha\dot{W}_t$. Instead, we consider a mild noise converging to $\alpha\dot{W}_t$. Here, W_t is 1D Brownian motion and α is a constant given by

$$\alpha := \sqrt{2\int_0^\infty E[\xi(0)\xi(t)]\,dt}.$$

  This is a kind of Green-Kubo formula.
- Thus, the equation (1) is written with $\dot{W}_t^\varepsilon = \frac{1}{\alpha}\xi_t^\varepsilon\,(\sim \dot{W}_t)$,

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}f(u^\varepsilon) + \frac{1}{\varepsilon}\alpha\dot{W}_t^\varepsilon.$$

- The reason that we need $\frac{1}{\varepsilon}$ in front of the noise was explained heuristically in Lect-24 (Section 3.1.3).

Limit evolution

- Then, the evolution of Γ_t in the limit is given by

$$V = \kappa + \bar{c}\alpha \dot{W}_t,$$

where $\bar{c}$ is an inverse surface tension given by

$$\bar{c} = \frac{\sqrt{2}}{\int_{-1}^{1} \sqrt{V(u)}du} \left(= 2\|m'\|_{L^2(\mathbb{R})}^{-2} = 2\alpha_1^2 \right)$$

(with V normalized as $V(\pm 1) = 0$; α_1 is in Theorem 1).

- The above evolution of Γ_t is obtained by the heuristic argument in Section 3.1.3, just by replacing $\xi_t^\varepsilon \sim \alpha \dot{W}_t$.
- $\bar{c} = -c'(0)$ stated in Sect. 3.1.3 can be expressed as above.

[Theorem 3] (F, Acta Math Sinica 1999) Assume $d = 2$. As long as the limit curve Γ_t is strictly convex and stays inside D, we have as $\varepsilon \downarrow 0$

$$u^\varepsilon(t,x) \Longrightarrow \chi_{\Gamma_t}(x) := \begin{cases} +1, & \text{on one side of } \Gamma_t, \\ -1, & \text{on the other side of } \Gamma_t, \end{cases}$$

in law, where curve Γ_t moves according to the stochastic curvature motion:

$$V = \kappa + \bar{c}\alpha \dot{W}_t \tag{2}$$

where V denotes the inward normal velocity of Γ_t. □

- Meaning to SCM (2) was given by SPDE via Gauss map.
- H. Weber (2010) extended Theorem 3 to arbitrary dimensions $d \geq 2$ and established short time sharp interface limit under non-convex setting of interfaces. Convergence was shown in a.s.-sense due to the result by Dirr-Luckhaus-Novaga, who gave pathwise solution to $V = \kappa + \dot{W}_t$ via signed distance function.

- The scaling in time and that of the noise are very different from 1D case

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2} f(u^\varepsilon) + \frac{1}{\varepsilon} \alpha \dot{W}_t^\varepsilon \underset{\varepsilon \downarrow 0}{\longrightarrow} V = \kappa + \bar{c} \alpha \dot{W}_t,$$

where $\dot{W}_t^\varepsilon = \frac{1}{\alpha} \xi_t^\varepsilon \sim \dot{W}_t$.

- Recall in 1D, we considered

$$\frac{\partial \bar{u}}{\partial t} = \varepsilon^{-2\gamma-1} \left\{ \Delta \bar{u} + \frac{1}{\varepsilon^2} f(\bar{u}) \right\} + \varepsilon^{-1/2} a(x) \dot{W}(t,x)$$

and showed that it converges to χ_{ξ_t} with ξ_t, a solution of SDE.

Proof of Theorem 3

- Since we assume the noise is mild, we can directly apply the PDE methods, in particular, we can construct super/sub solutions of (1) due to comparison theorem.
- Those are given as functions close to
  $\tilde{u}^\varepsilon(t,x) := m(d(x,\Gamma_t^\varepsilon)/\varepsilon; \varepsilon\xi_t^\varepsilon)$
  (assume this for $t=0$), with curve Γ_t^ε in D moving by
  $$V = \kappa - \frac{1}{\varepsilon} c(\varepsilon\xi_t^\varepsilon). \tag{3}$$
  Recall Lect-24 for $m(y;a)$ and $c(a)$.
- Indeed, to construct super/sub solutions, we slightly shift $d(x,\Gamma_t^\varepsilon) \to d(x,\Gamma_t^\varepsilon) \pm \varepsilon^a e^{ct}$ and $\varepsilon\xi_t^\varepsilon \to \varepsilon\xi_t^\varepsilon \pm \varepsilon^\beta$.
- However, if Γ_t^ε is convex, in terms of the Gauss map ($\theta \in S^1 \mapsto x(\theta) \in \Gamma_t^\varepsilon$), (3) can be rewritten into a PDE for the curvature function $\kappa = \kappa^\varepsilon(t,\theta)$:
  $$\frac{\partial \kappa}{\partial t} = \kappa^2 \left\{ \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa - \frac{1}{\varepsilon} c(\varepsilon\xi_t^\varepsilon) \right\}.$$

- (Wong-Zakai type theorem for SPDE) Then, one can study its limit as $\varepsilon \downarrow 0$ and obtain the following SPDE in the limit:

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \left\{ \frac{\partial^2 \kappa}{\partial \theta^2} + \kappa + \bar{c}\alpha \circ \dot{W}_t \right\} \tag{4}$$

  where $\circ$ denotes the Stratonovich's stochastic integral.
- (4) gives a precise mathematical meaning to SCM (2).
- This completes the proof of Theorem 3. □

- Numerical simulation (by K. Lee)
  2D case

Related topics

Generation of interfaces

- We assumed that interface is already created at initial time. We can show that starting from rather general initial condition an interface is generated in a short time.
  Alfaro-Antonopoulou-Karali-Matano (2018), Lee (2018).

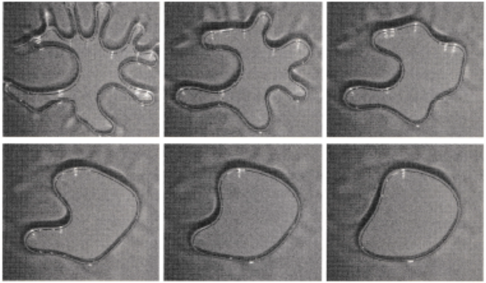
Sharp interface limit in ill-posed setting (Open Problem)

- Even in an ill-posed setting, it might be possible to discuss under the formulations of Hairer or Gubinelli noting the theory of Glimm-Jaffe-Spencer for the phase transition in $P(\phi)_2$-model, though the field is realized as a generalized function (a.s.)
- Note that $P(\phi)_2$-model provides the invariant measure of the singular SPDE with space-time Gaussian white noise.

Cahn-Hilliard case ($d \geq 2$)

- Stochastic Cahn-Hilliard equation:
  4th order SPDE, mass-conservation law
  - 1D: Antonopoulou-Blömker-Karali 2012 (smooth noise), Bertini-Brassesco-Buttà 2015 (space-time white noise, infinitesimal in time, fractional BM, including phase field model)
  - Higher dimensions: Antonopoulou-Karali-Kossioris 2011, Cahn-Hilliard eq with deterministic noise (under white noise scaling) and gave formal expansion of the solutions. For the deterministic Cahn-Hilliard eq, it is known that Hele-Shaw free boundary problem appears in the sharp interface limit (instead of mean curvature motion in Allen-Cahn case).
  - ($d = 2$) Stochastic Cahn-Hilliard equation to (noise-vanishing) Hele-Shaw problem; cf. Yang's thesis, Röckner-Yang-Zhu, 2021

Hele-Shaw free boundary problem



(taken from S. Yazaki's book)

Pour viscous fluid into narrow gap of two glass plates settled in parallel and observe its change in time. Volume of fluid is preserved and the (free) boundary of fluid moves.

Role of stochasticity

- Dirr-Luckhaus-Novaga (2001) and Souganidis-Yip (2004) used SMMC with $\dot{W} = \varepsilon \dot{W}_t$ (only time-dependent noise) to pick up a right (unique, non-fattening) solution from non-unique deterministic solutions as $\varepsilon \downarrow 0$.

4.2.3. Stochastic mass conserving Allen-Cahn eq

(1) Equation

- $u = u^\varepsilon(t,x)$: solution of the stochastic PDE with non-local term in a smooth bounded domain D of $\mathbb{R}^n$:

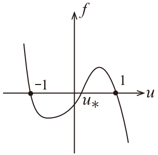
$$\begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}\left(f(u^\varepsilon) - \fint_D f(u^\varepsilon)\right) + \alpha \dot{W}_t^\varepsilon, & x \in D \\ u^\varepsilon(0,\cdot) = g^\varepsilon(\cdot), & x \in D \end{cases} \tag{5}$$

  with Neumann boundary condition $\partial_\nu u^\varepsilon = 0$ at ∂D, where $\alpha > 0$ and

$$\fint_D f(u^\varepsilon) = \frac{1}{|D|}\int_D f(u^\varepsilon(t,x))dx.$$

- We write n instead of d for dimension.

- The reaction term $f \in C^\infty(\mathbb{R})$ is bistable and balanced ($\Rightarrow$ standing wave):



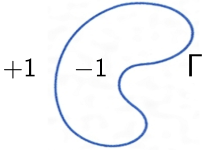
$$f(\pm 1) = 0,\ f'(\pm 1) < 0,\ A(f) = 0.$$

- $\dot{W}_t^\varepsilon$: mild noise, i.e., a time derivative of $W_t^\varepsilon = W_t^\varepsilon(\omega) \in C^\infty([0,\infty))$ (in t a.s.-ω) defined on $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$ such that $W_t^\varepsilon \to W_t$, 1D Brownian motion, a.s. as $\varepsilon \downarrow 0$.
- $\dot{W}_t^\varepsilon$ is like $\frac{1}{\alpha}\xi_t^\varepsilon$ taken before, but should be carefully scaled in time. We discuss later.
- Especially, note that we don't have $\frac{1}{\varepsilon}$ in front of noise.

- Mass-conservation law in stochastic sense:

$$\fint_D u^\varepsilon(t) = \fint_D u^\varepsilon(0) + \alpha W_t^\varepsilon$$

- Initial value: $g^\varepsilon \to \chi_{\Gamma_0}$ with some hypersurface $\Gamma_0 \subset D$, where $\chi_\Gamma = 1$ (outside of Γ), -1 (inside of Γ).
(More precisely, we assume g^ε is close to a function determined by the asymptotic expansion up to Kth order with $K > \max\{n+2, 6\}$.)



- Goal is to study the limit $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x)$.

(2) Sharp interface limit

- Evolution of limit hypersurfaces $\Gamma_t \subset D$:

$$V = \kappa - \fint_{\Gamma_t} \kappa + \frac{\alpha|D|}{2|\Gamma_t|} \circ \dot{W}_t, \quad t \in [0, \sigma], \tag{6}$$

  up to a certain stopping time $\sigma > 0$ (a.s.), where
  - V = inward normal velocity of Γ_t,
  - κ = mean curvature of Γ_t (multiplied by $n-1$),
  - $\dot{W}_t$ = white noise in time,
  - $\circ$ means Stratonovich stochastic integral,
  - $\fint_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma$.

- (6) can be formulated as an SPDE for signed distance function $d(t, x)$ from Γ_t:

$$\partial_t d(t,x) = \Delta d(t,x) - \fint_{\Gamma_t} \Delta d(t,y)dy + \frac{\alpha|D|}{2|\Gamma_t|} \circ \dot{W}_t$$

- Evolution of approximating herpersurfaces $\Gamma_t^\varepsilon \subset D$:

$$V^\varepsilon = \kappa - \fint_{\Gamma_t^\varepsilon} \kappa + \frac{\alpha |D|}{2|\Gamma_t^\varepsilon|} \dot{W}_t^\varepsilon, \quad t \in [0, \sigma^\varepsilon] \tag{7}$$

  We assume $\Gamma_t^\varepsilon \to \Gamma_t$ (and $\sigma^\varepsilon \to \sigma$) a.s. (in smooth class).
- This is shown in 2D as long as Γ_t is convex in law sense, which is Wong-Zakai type theorem.
- Another assumption is that the diverging speeds of derivatives of W_t^ε as $\varepsilon \downarrow 0$ are extremely slow:

$$\sup_{\omega \in \Omega} \left| \frac{d^k}{dt^k} W_t^\varepsilon(\omega) \right| \le H_\varepsilon, \ t \in [0, T], 1 \le k \le n(K),$$

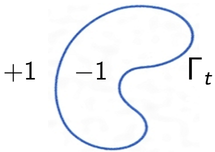
$$\lim_{\varepsilon \downarrow 0} H_\varepsilon = \infty, \quad \lim_{\varepsilon \downarrow 0} \frac{H_\varepsilon^{2n(K)}}{\log \log |\log \varepsilon|} = 0,$$

  where $n(K) \in \mathbb{N}$ is a certain number determined from K (=order of expansion). (We count how many products of derivatives of $\dot{W}_t^\varepsilon$ appear in the expansion up to Kth order.)

[Theorem 4] (F-Yokoyama, Ann Probab 2019) Suppose that a smooth local solution $\{\Gamma_t \subset D\}_{0\le t\le\sigma}$ of (6) exists uniquely. Then, with a suitable choice of initial values $\{g^\varepsilon(\cdot)\}_{\varepsilon\in(0,1)}$ satisfying $\lim_{\varepsilon\downarrow 0} g^\varepsilon(x) = \chi_{\Gamma_0}$, $u^\varepsilon(t\wedge\sigma^\varepsilon,\cdot)$ converges as $\varepsilon\downarrow 0$ to $\chi_{\Gamma_{t\wedge\sigma}}(\cdot)$ on $C([0,T], L^2(D))$ a.s. □

$$(5)\qquad \begin{cases} \partial_t u^\varepsilon = \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2}\left(f(u^\varepsilon) - \displaystyle\fint_D f(u^\varepsilon)\right) + \alpha \dot{W}_t^\varepsilon, & x\in D \\ \partial_\nu u^\varepsilon = 0, & x\in\partial D \\ u^\varepsilon(0,\cdot) = g^\varepsilon(\cdot), & x\in D \end{cases}$$

$$(6)\qquad V = \kappa - \fint_{\Gamma_t}\kappa + \frac{\alpha|D|}{2|\Gamma_t|}\circ \dot{W}_t, \quad t\in[0,\sigma]$$



(6) has non-local multiplicative noise (due to conservation law)

Heuristic reason for the difference in scaling of noises

- Stoch. AC (1): $\frac{1}{\varepsilon}\dot{W}_t^\varepsilon \qquad (\dot{W}_t^\varepsilon \sim \dot{W}_t)$
- Stoch. mass-conserving AC (5): $\dot{W}_t^\varepsilon$
  (More sensitive to noise in the present case)
- With noise $\dot{W}^\varepsilon(t,x)$ s.t. $\int_D W^\varepsilon(t,x)dx = 0$, we expect $\frac{1}{\varepsilon}\dot{W}^\varepsilon(t,x)$.

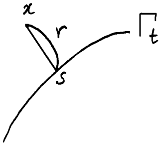
- From $g^\varepsilon \approx \pm 1$, one can expect $u^\varepsilon(t,x) \approx \pm 1$ or $f(u^\varepsilon) \approx 0$, so that $⨍_D f(u^\varepsilon) \approx 0$.
- In fact, one can show $⨍_D f(u^\varepsilon) = O(\varepsilon)$, i.e., $⨍_D f(u^\varepsilon) = \varepsilon\lambda_0(t) + O(\varepsilon^2)$, so that $\varepsilon^{-2} ⨍_D f(u^\varepsilon) = \varepsilon^{-1}\lambda_0(t) + \cdots$. This is the same order as $\varepsilon^{-1}\alpha\dot{W}^\varepsilon$ in the stochastic Allen-Cahn equation.
- The evolution of $\lambda_0(t)$ is governed by the noise term in the present case.
- We will see how $\lambda_0(t)$ and the limit dynamics (6) are determined by the asymptotic expansion method.

(3) Asymptotic expansion method

- Stoch. mass-conserving AC eq (5) is rewritten as

$$0 = f(u^\varepsilon) + \varepsilon^2(-\partial_t u^\varepsilon + \Delta u^\varepsilon + \alpha \dot{W}_t^\varepsilon) - \varepsilon\lambda_\varepsilon(t),$$

where $\lambda_\varepsilon(t) := \varepsilon^{-1} \fint_D f(u^\varepsilon)$.



- Near Γ_t, we introduce a coordinate $x = (r, s)$ and stretched (micro) variable $\rho = r/\varepsilon$ ($\rho > 0$ outside

- Comparison argument doesn't work. Instead, we introduce an asymptotic expansion in stretched variable (ρ, s):

$$\begin{aligned}
\tilde{\Gamma}_t^\varepsilon &= \{x; u^\varepsilon(t,x) = 0\} = \{r = \varepsilon h_\varepsilon\}, \\
h_\varepsilon &= h_1 + \varepsilon h_2 + \varepsilon^2 h_3 + \cdots, \\
u^\varepsilon &= u_{-1} + \varepsilon u_0 + \varepsilon^2 u_1 + \cdots, \\
\lambda_\varepsilon &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \\
u^{\pm,\varepsilon} &= \pm 1 + \varepsilon u_0^\pm + \varepsilon^2 u_1^\pm + \cdots \ (\text{outer solutions} : \lim_{\rho\to\pm\infty} u^\varepsilon)
\end{aligned}$$

- We have the conservation law:

$$\int_D \left\{\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\big(f(u^\varepsilon) - \varepsilon\lambda_0(t)\big)\right\}dx = 0,$$

  or

$$\int_D \left\{\Delta u^\varepsilon + \frac{1}{\varepsilon^2}\big(f(u^\varepsilon) - \varepsilon\lambda_0(t) + \varepsilon^2\alpha\dot{W}_t^\varepsilon\big)\right\}dx = |D|\alpha\dot{W}_t^\varepsilon.$$

- The first order approximation u_{-1} of u^ε would be

$$u_{-1} = m\big(\tfrac{d(x,\Gamma_t^\varepsilon)}{\varepsilon}; \varepsilon\lambda_0(t) - \varepsilon^2\alpha\dot{W}_t^\varepsilon\big)$$

- Recall perturbed traveling wave sol. $m = m(\rho; a)$ and its speed $c = c(a)$ are defined by

$$m'' + cm' + \{f(m) - a\} = 0,\ m(\pm\infty) = m_\pm^*,\ m(0) = 0,$$

  for $a \sim 0$, where $m_\pm^* = \pm 1 + O(a)\,(a \to 0)$ are solutions of $f(m_\pm^*) - a = 0$.

- As before, $\Delta u^\varepsilon \sim \frac{1}{\varepsilon^2} m''|\nabla d|^2 + \frac{1}{\varepsilon} m' \Delta d$ and note that

$$m'' + cm' + (f - \varepsilon\lambda_0 + \varepsilon^2 \alpha \dot{W}_t^\varepsilon) = 0$$

  with $c = c(\varepsilon\lambda_0 - \varepsilon^2 \alpha \dot{W}_t^\varepsilon) \sim \varepsilon \bar{c} \lambda_0(t)$
- Since $\frac{1}{\varepsilon} \int_D m' = 2|\Gamma_t|$ $(D \sim \Gamma_t \times \mathbb{R})$, we see $\lambda_0(t)$ is determined by

$$2\bar{c}\lambda_0(t)|\Gamma_t| = 2\int_{\Gamma_t} \kappa ds - |D|\alpha \dot{W}_t^\varepsilon, \qquad (8)$$

  where $\bar{c} := 2(\int_{\mathbb{R}} m'(\rho)^2 d\rho)^{-1} \equiv 2\alpha_1^2$ is the inverse surface tension, also expressed as before: $\bar{c} = \sqrt{2}/\int_{-1}^{1} \sqrt{V(u)} du$. We treat $\dot{W}_t^\varepsilon$ as if $\dot{W}_t^\varepsilon \approx \dot{W}_t$, so that independent of ε.

- Looking at the term of $O(\varepsilon)$, u_0 should satisfy

$$\mathcal{L}u_0 \left(\equiv -\partial_\rho^2 u_0 - f'(m)u_0\right) = (-V+\kappa)m' - \lambda_0(t)$$

- Solvability condition for u_0: $\int \mathcal{L}u_0 m' d\rho = 0$ implies

$$V = \kappa - \bar{c}\lambda_0(t) \quad \text{on } \Gamma_t.$$

  (Recall $\mathcal{L}m' = 0$ i.e., m' is 0-eigenfunction)
- (Derivation of the dynamics of limit hypersurface)
  This combined with the equation (8) for $\lambda_0(t)$ leads to the limit dynamics (6) of Γ_t.

- However, to establish the error estimate of the asymptotic expansion from u^ε, we need to continue it up to Kth order with $K > \max\{n+2, 6\}$ (Sobolev imbedding type estimate is needed)..
- This requires additional works, in particular, because
- Powers of $\dot{W}_t^\varepsilon, \ddot{W}_t^\varepsilon, \ldots$ like $(\dot{W}_t^\varepsilon)^2$, $(\dot{W}_t^\varepsilon)^3, \ldots$, appear repeatedly and these are diverging.
    - In the 0 th order term, only $\dot{W}_t^\varepsilon$ appears, so that the limit can be established in stochastic sense.
    - In the k th order terms with $k \geq 1$, diverging terms appear. If the diverging speed H_ε of derivatives of W_t^ε is sufficiently slow as we stated before Theorem 4, these terms multiplied by the prefactor ε^k can be controlled.

- Indeed, to give an error estimate, we use the Schauder estimate: For the solution u of

$$\begin{cases} (\partial_t - L + \mathcal{L})u(t,s) = F(t,s), & (t,s) \in (0,T] \times \mathcal{S}, \\ u(0,s) = 0, & s \in \mathcal{S}, \end{cases}$$

we have

$$|u|_{2+\alpha} \leq M(|u|_0 + |F|_\alpha), \tag{9}$$

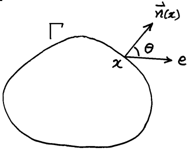
holds with $M = C(1 + |\dot{W}^\varepsilon|_{n(K)})^{n(K)}$, where $\mathcal{L} = \mathcal{L}^{\dot{W}^\varepsilon}$ is a certain integral operator, $\mathcal{S}$ is a fixed reference manifold which parametrizes the hypersurface Γ_t^ε, $n(K) \in \mathbb{N}$, and

$$L = L^{\dot{W}^\varepsilon} = \sum_{i,j=1}^{n-1} \alpha_{ij}^{\dot{W}^\varepsilon}(t,s)\partial^2_{s^i s^j} + \sum_{i=1}^{n-1} \beta_i^{\dot{W}^\varepsilon}(t,s)\partial_{s^i} + b^{\dot{W}^\varepsilon}(t,s),$$

- To show this, estimates on the fundamental solutions of the linear diffusion operator $\partial_t - L (= \partial_t - L^{\dot{W}^\varepsilon})$ on $\mathcal{S}$ (parameterizing Γ_t^ε) are required.
- But, the coefficients are determined from the noise and diverging. We examine the diverging speed carefully. We also need the lower bounds on $(\alpha_{ij}^{\dot{W}^\varepsilon})_{ij}$ and $|\Gamma_t^\varepsilon|$.
- This gives the error estimate for the asymptotic expansion and one can complete the proof of Theorem 4.

(4) Limit SPDE — 2D, convex curve

- Γ: strictly convex closed plane curve
- Gauss map: $\theta \in S := [0, 2\pi) \mapsto x = x(\theta) \in \Gamma$ if the angle between one fixed direction $\mathbf{e} := (1, 0)$ in the plane $\mathbb{R}^2$ and the outward normal $\vec{n}(x)$ at x to Γ is θ.



- Denote by $\kappa = \kappa(\theta) > 0$ the curvature of Γ at $x = x(\theta)$.
- In this setting., we show $\Gamma_t^\varepsilon \to \Gamma_t$, which we assumed.

Extremely Slowly diverging noise

- Choose the noise as

$$\dot{W}_t^\varepsilon = \frac{\beta_\varepsilon \xi\left(\beta_\varepsilon^2 t\right)}{\sqrt{2\int_0^\infty E[\xi(0)\xi(t)]dt}},$$

where $\xi(t)$ is a stationary, strong mixing, mean 0, bounded, smooth process, and we take $\beta_\varepsilon \nearrow \infty$ (slow enough, satisfying the assumption of Theorem 4):

$$\beta_\varepsilon = (\log\log\log|\log\varepsilon|)^\lambda, \lambda > 0.$$

Note that $W_t^\varepsilon (:= \int_0^t \dot{W}_s^\varepsilon ds) \to W_t$ in law.

- Under these notation, the limit dynamics (6) for Γ_t is rewritten into the stochastic integro-differential equation for $\kappa = \kappa(t, \theta)$:

$$\partial_t \kappa = \kappa^2 \partial_\theta^2 \kappa + \kappa^3 - \kappa^2 \cdot \bar{\kappa} + \frac{c\alpha\kappa^2}{|\Gamma_t|} \circ \dot{W}_t, \tag{10}$$

  where $\bar{\kappa}$ denotes the average of κ over the curve Γ_t and $|\Gamma_t|$ stands for the length of Γ_t.
- Similarly, the dynamics (7) for Γ_t^ε is rewritten into the equation for $\kappa = \kappa^\varepsilon$:

$$\partial_t \kappa = \kappa^2 \partial_\theta^2 \kappa + \kappa^3 - \kappa^2 \cdot \bar{\kappa} + \frac{c\alpha\kappa^2}{|\Gamma_t^\varepsilon|} \dot{W}_t^\varepsilon, \tag{11}$$

  where $\bar{\kappa}$ denotes the average of κ over the curve Γ_t^ε.

- Since $x(\theta) \in \mathbb{R}^2 \cong \mathbb{C}$ is written as

$$x(\theta) = x(0) - \sqrt{-1} \int_0^\theta \frac{e^{\sqrt{-1}\theta'}}{\kappa(\theta')} d\theta',$$

  we see that $|x'(\theta)| = 1/\kappa(\theta)$.
- Therefore, $\bar{\kappa}$ and $|\Gamma|$ are given by

$$\bar{\kappa} := \frac{1}{|\Gamma|} \int_S \kappa(\theta)|x'(\theta)|d\theta = \frac{2\pi}{|\Gamma|},$$
$$|\Gamma| := \int_S |x'(\theta)|d\theta = \int_S \frac{d\theta}{\kappa(\theta)},$$

  respectively, which are functionals of $\kappa = \{\kappa(\theta); \theta \in S\}$.

$$\sigma_N^\varepsilon := \inf\{t > 0; \kappa^\varepsilon(t,\theta), \kappa^\varepsilon(t,\theta)^{-1}, |\kappa^\varepsilon(t,\theta)'| \geq N \\ \text{for some } \theta \text{ or } \operatorname{dist}(\Gamma_t^\varepsilon, \partial D) \leq 1/N\}, \quad N \geq 1$$

σ_N is defined similarly for $\kappa(t,\theta)$.

[Theorem 5] For each $m \in \mathbb{N}$ and $T > 0$, the distribution of the solution $\kappa^\varepsilon(t \wedge \sigma_N^\varepsilon, \cdot)$ of SPDE (11) converges weakly to that of the solution $\kappa(t \wedge \sigma_N, \cdot)$ of SPDE (10) on $C([0,T], C^m(S))$. □

- This is Wong-Zakai type theorem.
- We prove tightness (moments estimates under cutoff) and pathwise uniqueness for the SPDE (10).
- Thus, in the present setting, the assumption "$\Gamma_t^\varepsilon \to \Gamma_t$" for Theorem 4 holds (locally in time, in law sense).

(5) Summary of stochastic mass-conserving AC eq

- We studied the sharp interface limit for stochastic mass-conserving Allen-Cahn equation by applying the asymptotic expansion method.
- We need to control up to Kth order terms in the expansion, so that diverging terms like $(\dot{W}_t^\varepsilon)^2$, $(\dot{W}_t^\varepsilon)^3$ appear repeatedly.
- Investigating the Schauder estimate with diverging coefficients, one can show that, if the diverging speed of derivatives of W_t^ε is sufficiently slow, these diverging terms can be controlled (by ε in the expansion).

Thank you for attending the course.

I am happy if it was useful for you and if you could find any interest in it.

I hope to see you soon in Beijing.