# 2020-9-27 Kähler geometry

$[c_p(g)] \overset{\text{def}}{=} c_p(M)$ is called the $p$-th Chern class.

Look at the case $p=1$.

$$c_1(g) = \frac{\sqrt{-1}}{2\pi}\,\operatorname{tr}(H) = \frac{\sqrt{-1}}{2\pi}\,\underbrace{R^{i}{}_{ik\bar{l}}}_{R_{k\bar{l}}{}^{i}{}_{i} \,=\, R_{k\bar{l}}}\, dz^k \wedge d\bar{z}^l$$

$$= -\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g_{i\bar{j}})$$

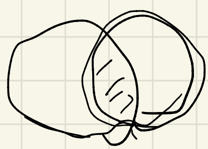
If $g'$ is another Kähler form, then

$$-\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g_{i\bar{j}}) - \left(-\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\det(g'_{i\bar{j}})\right)$$

$$= -\frac{\sqrt{-1}}{2\pi}\,\partial\bar{\partial}\log\frac{\det(g_{i\bar{j}})}{\det(g'_{i\bar{j}})}$$

Lemma

$\dfrac{\det(g_{i\bar{j}})}{\det(g'_{i\bar{j}})} \in C^\infty(M)$, i.e. indep of local coordinate.

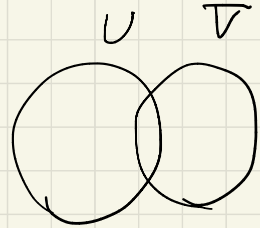


proof

If $z^1, \dots, z^m$ is holo coord on $U$.

$w^1, \dots, w^m$ is holo coord on $V$

$U \cap V \neq \emptyset$




$$g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right) = g\left(\frac{\partial z^k}{\partial w^i}\frac{\partial}{\partial z^k}, \overline{\frac{\partial z^l}{\partial w^j}\frac{\partial}{\partial z^l}}\right)$$

$$\therefore \det\left(g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right) = \cancel{\left|\det\left(\frac{\partial z^i}{\partial w^j}\right)\right|^2} \det g\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)$$

In the same way

$$\det(g'(\quad , \quad)) = \cancel{\left|\det\left(\frac{\partial z^i}{\partial w^j}\right)\right|^2} \det(g'(\ ,\ ))$$

$$\therefore \frac{\det\left(g\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right)}{\det\left(g'\left(\frac{\partial}{\partial w^i}, \overline{\frac{\partial}{\partial w^j}}\right)\right)} = \frac{\det\left(g\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)}{\det\left(g'\left(\frac{\partial}{\partial z^i}, \overline{\frac{\partial}{\partial z^j}}\right)\right)}$$

$\therefore$

Thus

$$c_1(g) - c_1(g') = \overset{\partial+\bar\partial}{\widehat{d}}\left(-\frac{\sqrt{-1}}{2\pi}\,\bar\partial \log \frac{\det g}{\det g'}\right)$$

$$\therefore \left[c_1(g)\right] = \left[c_1(g')\right]$$

<u>Lemma</u> : If $\alpha$ is a real exact $(p,p)$-form then $\exists$ real $(p-1,p-1)$-form $\beta$ s.t.

$$\alpha = i\partial\bar\partial\beta.$$

In particular for $p=1$, i.e. for exact $(1,1)$ $\alpha$, we have $(0,0)$-form i.e. $\exists$ a smooth function $\varphi \in C^\infty(M)$ such that $\alpha = i\partial\bar\partial\varphi$.

( $\partial\bar\partial$-Lemma).

$\because$ This is an application of <u>Hodge theory</u>

see. A. Futaki Lecture Notes springer.

S. Kobayashi : Transformation groups in differential geometry springer, <u>Appendix</u>.

Recall

$c_p(g)$ is called the $p$-th Chern form

$c_1(g)$ the <u>first chern form</u>

the <u>Ricci form</u>

<u>Def</u> $(M, g)$ Riemannian mfd.

(1) $S(g) (= \mathrm{Scal}(g)) = g^{ij} R_{ij} = R^j{}_j$ $\quad R_i{}^i$

Scalar curvature.

(2) $g$ is called an Einstein metric

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists k \in \mathbb{R}$ s.t. $R_{ij} = k\, g_{ij}$

<u>Lemma</u> If $g$ is an Einstein metric then $S(g)$ is constant.

$(\because)$ $S(g) = g^{ij} R_{ij} = g^{ij}(k\, g_{ij}) = kn$

$n = \dim M$. $(\because)$

Consider the case of compact Kähler mfd.

If $g$ is a Kähler-Einstein metric

then $R_{i\bar{j}} = k\, g_{i\bar{j}}$.

If $k>0 \Rightarrow c_1(M)>0$ i.e.

$c_1(M)$ is represented by a positive (1,1)-form.

$\sqrt{-1}\,\alpha_{i\bar{j}}\,dz^i\wedge d\bar{z}^j$

$(\alpha_{i\bar{j}})$ positive def Hermitian

( ☺ $R_{i\bar{j}} = k\, g_{i\bar{j}}$ $k>0$ $(g_{i\bar{j}})$ positive def Hermitian )

$K_M = \overset{m}{\wedge}\, T'M^*$ canonical line bdl

$K_M^{-1} = \overset{m}{\wedge}\, T'M$

$c_1(K_M^{-1}) = c_1(M) > 0$

$K_M^{-1}$ ample. $\Longleftrightarrow$ $M$ : Fano mfd.

If $k=0 \Longrightarrow c_1(M)=0$ in $H^2_{DR}(M)$

If $k<0 \Longrightarrow c_1(M)<0 \Longleftrightarrow K_M$ is ample.

Is converse True ?

Yes for $k<0$. Yau, Aubin 1976

$\boxed{\text{Yes for } k=0 \qquad \text{Yau}}$

No in general for $k>0$. Matsushima

Calabi-Yau mfd
the metric is called the Calabi-Yau metric

K-stability is necessary and sufficient.

How is the proof for $k<0$ and $k=0$.

$\omega$ fixed Kähler m. $[\omega] \in H^2_{DR}(M:\mathbb{R})$

$\sqrt{-1}\, g_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ Kähler class.

$\Omega$ = the set of all Kähler forms cohomologous to $\omega$.

$\forall \omega' \in \Omega$ $\quad \omega' - \omega = \sqrt{-1}\, \partial\bar{\partial}\varphi$, $\exists \varphi \in C^\infty(M)$.

Suppose $c_1(M) = 0$.

$$c_1(M) \ni c_1(g) = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j = -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\det(g_{i\bar{j}}) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial F \quad \text{—— } (*)$$

for $\exists F \in C^\infty(M)$.

If $\exists g'$ s.t. $c_1(g') = 0$ with $\omega' \in \Omega$.

then $\omega' = \omega + \sqrt{-1}\,\partial\bar\partial\varphi$.

i.e. $g'_{i\bar{j}} = g_{i\bar{j}} + \dfrac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$ $= \varphi_{i\bar{j}}$

$$= g_{i\bar{j}} + \varphi_{i\bar{j}}$$

$$\therefore\ -\frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\det(g'_{i\bar{j}}) = 0. \quad \text{—— } (**)$$

$(*) + (**)$

$$\partial\bar\partial \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = \partial\bar\partial F$$

$$\therefore\ \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = \overbrace{F + \text{const}}^{\to F}$$

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{\sqrt{t}F} \quad\text{———}\quad (1).$$

② Given $g$ and $F$ on a compact Kähler mfd $M$, can we solve this equation (1) ?

Monge-Ampère equation.

= A PDE involving the determinant of a function.

Theorem (Yau 1976):

(1) is solvable on any compact Kähler manifold.

Theorem (Aubin, Yau)

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{\varphi + \sqrt{t}F}$$ is solvable

on any compact Kähler mfd.

$$\frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} = e^{-t\varphi + F} \qquad (*_t)$$

$t=1$

$t=0$

$S = \{ t \in [0,1] \mid (*_t) \text{ has a solution} \}$.

We need to show $S = [0,1]$

Difficulty is showing closedness of $S$ in $[0,1]$.

If $\varphi_t$ is a solution for $(*_t)$ and if we can show

$\{\varphi_t\}$ is bounded in $C^3(M)$

$(|\varphi|, |\nabla\varphi|, |\nabla\nabla\varphi|, |\nabla\nabla\nabla\varphi|$

$\leq C$ indep of $t$.

Yau's work shows it is sufficient to show

$|\varphi_t| \leq C$ indep $t$ is enough.

Chen-Donaldson-Sun, Tian

For Fano mfd $M$.

$\exists$ KE $\Longleftrightarrow$ K-stability

twisted K-E metric

$Ric = \omega + \alpha$  $\alpha$ any given.

Many works do this for $\alpha \geq 0$.

$Ric > 0 \Longrightarrow M$ : Fano

Without this when can we solve twisted KE.