Conformal Invariance in 2D Lattice Models

Part 4: Ising Model

Hao Wu (THU)

Ising Model

Curie temperature [Pierre Curie, 1895]

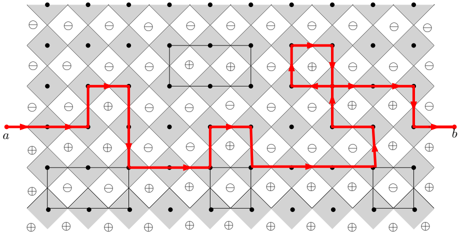
Ferromagnet exhibits a phase transition by losing its magnetization when heated above a critical temperature.

Ising Model [Lenz, 1920]

A model for ferromagnet, to understand the critical temperature

- $G = (V, E)$ is a finite graph
- $\sigma \in \{\oplus, \ominus\}^V$
- The Hamiltonian

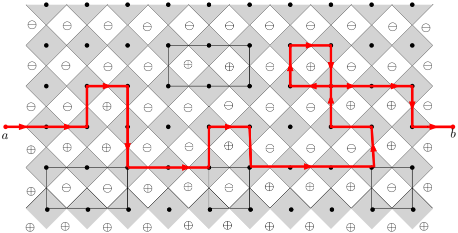
$$H(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y$$



Ising Model

Ising model is the probability measure of inverse temperature $\beta > 0$:

$$\mu_{\beta,G}[\sigma] \propto \exp(-\beta H(\sigma))$$



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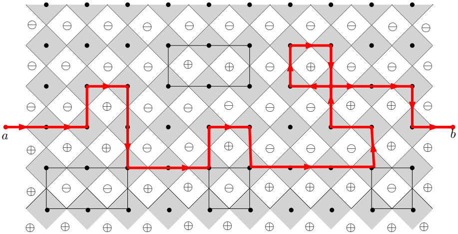

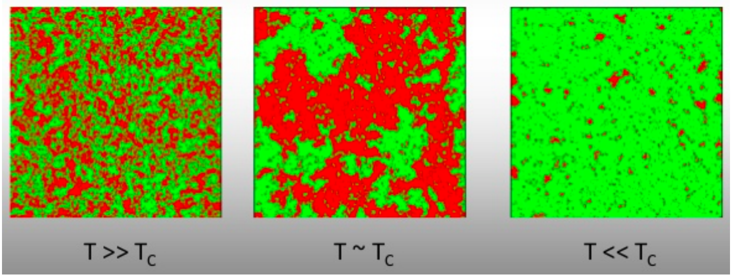

Ising Model

- FKG Inequality
- Phase Transition
- Critical Value
- Fermionic Observable
- Convergence of the Fermionic Observable

Ising Model—boundary conditions

Fix some boundary conditions (b.c.) $b \in \{\ominus, \oplus\}^{\partial G}$. The Ising model on G with b.c. b is the proba. measure :

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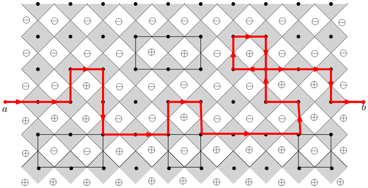
- free-b.c. $\mu^{f}_{\beta,G}$
- $\mu^{\oplus}_{\beta,G}$ and $\mu^{\ominus}_{\beta,G}$
- Dobrushin b.c. $\mu^{dobr}_{\beta,G}$
- b.c. induced by the config. outside G.



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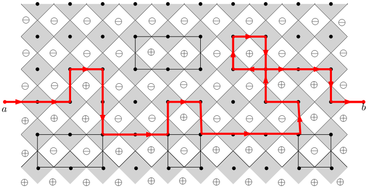

Domain Markov Property

Suppose $G' \subset G$, and for b.c. $b \in \{\ominus, \oplus\}^{\partial G}$ and $\psi \in \{\ominus, \oplus\}^{G \setminus G'}$ such that $\psi = b$ on ∂G,

$$\mu^{b}_{\beta,G}[X \,|\, \sigma_x = \psi_x, x \in G \setminus G'] = \mu^{\psi}_{\beta,G'}[X].$$

FKG Inequality

Theorem (FKG Inequality)

Fix $\beta > 0$, a finite graph G and some boundary conditions b. For any two increasing events A and B, we have

$$\mu^b_{\beta,G}[A \cap B] \geq \mu^b_{\beta,G}[A]\mu^b_{\beta,G}[B].$$

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Corollary (Comparison between boundary conditions)

Fix $\beta > 0$, a finite graph G. For boundary conditions $b_1 \leq b_2$ and any increasing event A, we have

$$\mu^{b_1}_{\beta,G}[A] \leq \mu^{b_2}_{\beta,G}[A].$$

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Phase Transition—Infinite-volume measure

Proposition

Fix $\beta > 0$. There exist two (possibly equal) infinite-volume measures $\mu_\beta^\oplus$ and $\mu_\beta^\ominus$ such that for any event A depending on a finite number of edges,

$$\lim_{n\to\infty} \mu_{\beta,\Lambda_n}^\oplus[A] = \mu_\beta^\oplus[A], \quad \lim_{n\to\infty} \mu_{\beta,\Lambda_n}^\ominus[A] = \mu_\beta^\ominus[A].$$

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Proposition

- $\mu_\beta^\oplus$ and $\mu_\beta^\ominus$ are translation invariant
- $\mu_\beta^\oplus$ and $\mu_\beta^\ominus$ are ergodic

For $\mu_\beta^\oplus$ or $\mu_\beta^\ominus$, there is no infinite cluster almost surely, or there exists a unique infinite cluster almost surely.

Phase Transition—the critical value

Theorem

There exists $\beta_c \in (0, \infty)$ *such that*

$$\mu_\beta^{\oplus}[\sigma_0] = 0, \quad \textit{if } \beta < \beta_c; \quad \mu_\beta^{\oplus}[\sigma_0] > 0, \quad \textit{if } \beta > \beta_c.$$

Moreover,

$$\beta_c = \frac{1}{2}\log(1+\sqrt{2}).$$

Critical Value : Edwards-Sokal coupling

Edwards-Sokal coupling

Fix a finite graph G and $p \in (0, 1)$.

- Sample $\omega \sim$ random-cluster model with $(p, 2)$ and free-b.c.
- Assign indep. to each cluster of ω a spin $\oplus$ or $\ominus$ with proba. $1/2$.

The obtained spin config. has the law of Ising model with free-b.c. and

$$\beta = \frac{1}{2} \log \frac{1}{1-p}.$$

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Consequence

$$\mu^f_{\beta,G}[\sigma_x\sigma_y] = \phi^0_{p,2,G}[x \leftrightarrow y], \quad \mu^{\oplus}_{\beta,G}[\sigma_x] = \phi^1_{p,2,G}[x \leftrightarrow \partial G].$$

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Moreover,

$$\beta_c = \frac{1}{2}\log(1 + \sqrt{2}).$$

In fact, we have $\mu_{\beta_c}^{\oplus}[\sigma_0] = 0$.

- For $\beta \le \beta_c$, we have $\mu_\beta^{\oplus} = \mu_\beta^{\ominus}$ and this is the unique infinite-volume measure.
- For $\beta > \beta_c$, the set of infinite-volume measures is given by

$$\{\lambda\mu_\beta^{\oplus} + (1-\lambda)\mu_\beta^{\ominus} : \lambda \in [0, 1]\}.$$

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High temperature expansion

Fix a finite graph $G = (V, E)$

$\mathcal{E}_G$: the collection of even subgraph of G, i.e. the set of subgraphs ω of G such that every vertex in V is the end-point of an even number of edges of ω.

Generally, for $A \subset V$, let $\mathcal{E}_G(A)$ be the set of subgraphs ω of G s.t.

- every vertex of $V \setminus A$ is the end-point of an even number of edges of ω,
- every vertex of A is the end-point of an odd number of edges of ω.

Note that if $\#A$ is odd, then $\mathcal{E}_G(A)$ is empty.

High temperature expansion

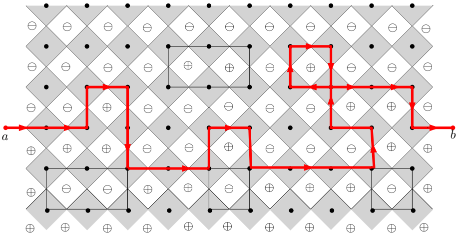
Let G be a finite graph and $\beta > 0$. We have

$$\mathcal{Z}^f_{\beta,G} = 2^{\#V(G)} \cosh(\beta)^{\#E(G)} \sum_{\omega \in \mathcal{E}_G} \tanh(\beta)^{o(\omega)}.$$

Low temperature expansion

- For each $\sigma \in \{\oplus, \ominus\}^{V(G)}$,
- define $\omega[\sigma] \in \{0,1\}^{E(G^*)}$: $\forall e = (x,y) \in E(G)$,

$$\omega[\sigma](e^*) = \begin{cases} 1, & \text{if } \sigma_x \neq \sigma_y, \\ 0, & \text{otherwise.} \end{cases}$$



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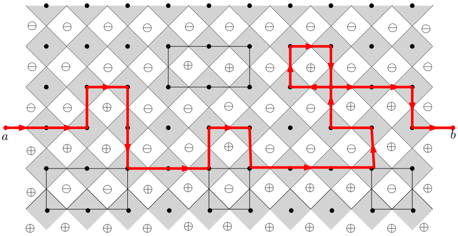

- Two-to-one for free b.c.

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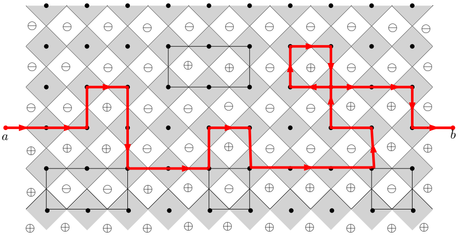

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$$\text{If } \mathbb{P}[\sigma] \propto e^{-\beta H(\sigma)}, \quad \text{then } \mathbb{P}[\omega] \propto e^{-2\beta o(\omega)}.$$

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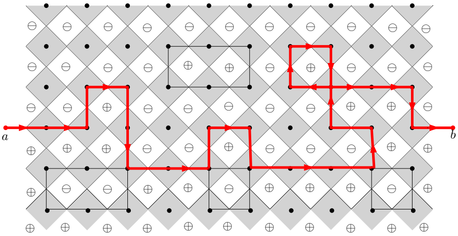

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Proposition

$$\mathcal{Z}^{\oplus}_{\beta,G} = e^{\beta \# E(G^*)} \sum_{\omega \in \mathcal{E}_{G^*}} e^{-2\beta o(\omega)}, \quad \mathcal{Z}^{dobr}_{\beta,G} = e^{\beta \# E(G^*)} \sum_{\omega \in \mathcal{E}_{G^*}(\{a,b\})} e^{-2\beta o(\omega)}.$$

Krammers-Wannier duality

Let $\beta > 0$ and define $\beta^* \in (0, \infty)$ such that

$$\tanh(\beta^*) = e^{-2\beta}.$$

Then for every graph G, we have

$$2^{\#V(G^*)} \cosh(\beta^*)^{\#E(G^*)} \mathcal{Z}^{\oplus}_{\beta,G} = e^{\beta \#E(G^*)} \mathcal{Z}^{f}_{\beta^*,G^*}.$$

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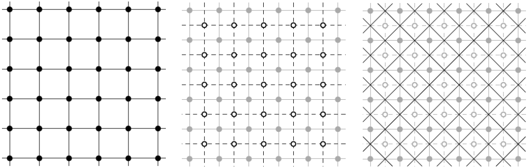
Fermionic observable



- $\mathbb{L}$: the square lattice
- $\mathbb{L}^*$: the dual lattice
- $\mathbb{L}^\diamond$: the medial lattice.

$V(\mathbb{L}^\diamond) = \{$centers of edges in $E(\mathbb{L})\}$ and $E(\mathbb{L}^\diamond) \sim$ nearest neighbours.

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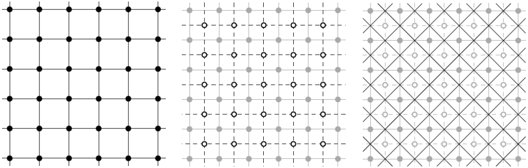

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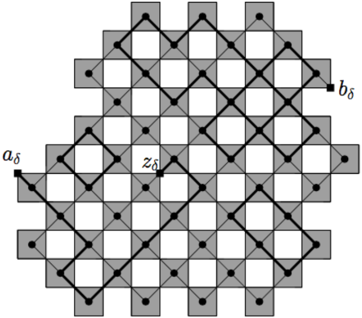
- $(\Omega_\delta^\diamond; a_\delta, b_\delta)$: a spin-Dobrushin domain
- For $z_\delta \in \Omega_\delta^\diamond$, let $\mathcal{E}(a_\delta, z_\delta) =$ collections of contours on Ω_δ composed of loops and an interface from a_δ to z_δ.

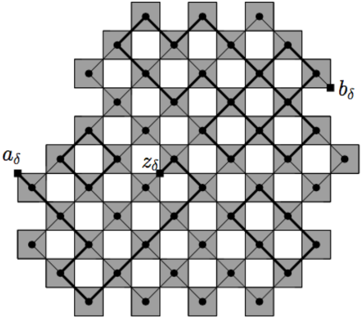


Fermionic observable

The spin-Ising fermionic observable :

$$F_\delta(z_\delta) = \frac{\sum_{\omega \in \mathcal{E}(a_\delta, z_\delta)} e^{-\frac{1}{2} i W_{\gamma(\omega)}(a_\delta, z_\delta)} (\sqrt{2}-1)^{o(\omega)}}{\sum_{\omega \in \mathcal{E}(a_\delta, b_\delta)} e^{-\frac{1}{2} i W_{\gamma(\omega)}(a_\delta, b_\delta)} (\sqrt{2}-1)^{o(\omega)}}$$

- for $\omega \in \mathcal{E}(a_\delta, z_\delta)$, denote by $\gamma(\omega)$ the interface from a_δ to z_δ
- $W_\gamma(a_\delta, z_\delta)$ is the total winding of the curve γ between a_δ and z_δ



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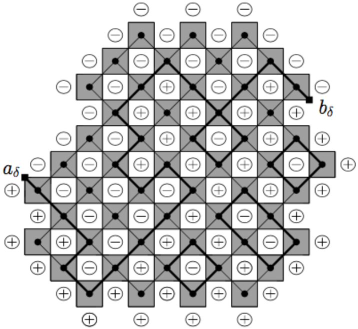
Theorem

$$F_\delta(\cdot) \to \sqrt{\varphi'(\cdot)/\varphi'(b)}, \quad as\ \delta \to 0,$$

where φ is any conformal map from Ω onto $\mathbb{H}$ that sends $a \to \infty$ and $b \to 0$. The convergence is local uniform.



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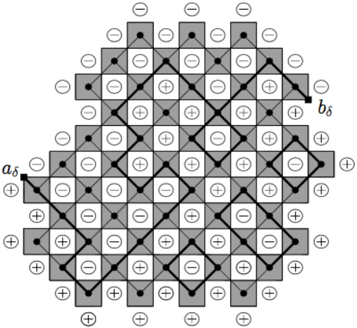


Strategy

- F_δ is s-holomorphic
- the associated function H_δ converges to $\Im\varphi$
- F_δ converges to $\sqrt{\varphi'(\cdot)/\varphi'(b)}$

Fermionic Observable

For $\omega \in \mathcal{E}(a_\delta, x_\delta)$, define

$$x_\omega = \frac{1}{\mathcal{Z}}(\sqrt{2}-1)^{o(\omega)} \exp\left(-\frac{i}{2}(W_{\gamma(\omega)}(a_\delta, x_\delta) - W_{\gamma_0}(a_\delta, b_\delta))\right).$$

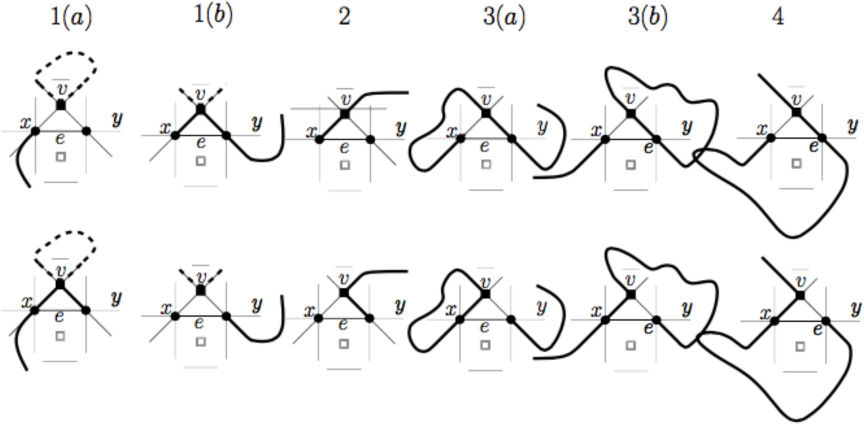
For $\omega' \in \mathcal{E}(a_\delta, y_\delta)$, define

$$y_{\omega'} = \frac{1}{\mathcal{Z}}(\sqrt{2}-1)^{o(\omega')} \exp\left(-\frac{i}{2}(W_{\gamma(\omega')}(a_\delta, y_\delta) - W_{\gamma_0}(a_\delta, b_\delta))\right).$$

Suppose $e = (x_\delta, y_\delta)$ is horizontal, we wish to show

$$\sum_{\omega \in \mathcal{E}(a_\delta, x_\delta)} \Re(x_\omega) = \sum_{\omega' \in \mathcal{E}(a_\delta, y_\delta)} \Re(y_{\omega'}).$$

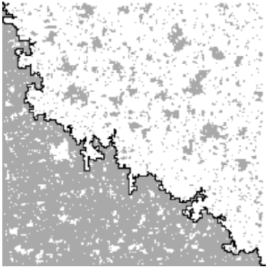
Fermionic Observable



configuration	Case 1(a)	Case 1(b)	Case 2	Case 3(a)	Case 3(b)	Case 4
x_ω	x_ω	x_ω	x_ω	x_ω	x_ω	x_ω
$y_{\omega'}$	$(\sqrt{2}-1)e^{i\pi/4}x_\omega$	$\frac{e^{i\pi/4}}{\sqrt{2}-1}x_\omega$	$e^{-i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{3i\pi/4}x_\omega$	$e^{-5i\pi/4}x_\omega$
arg. x_ω mod π	$5\pi/8$	$\pi/8$	$\pi/8$	$5\pi/8$	$5\pi/8$	$5\pi/8$

Summary : Ising Model

- Phase transition
- Critical Value : $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$
- Conformal Invariance of the critical phase



Thm [Chelkak-Smirnov Invent.Math.'10]

The interface of critical Ising model on $\mathbb{Z}^2$ with Dobrushin b.c. converges to SLE(3). (Nov. 2nd)

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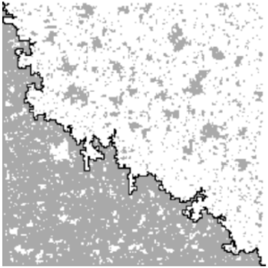

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Consequence : the critical arm exponent

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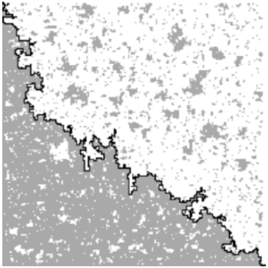

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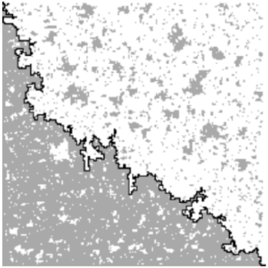

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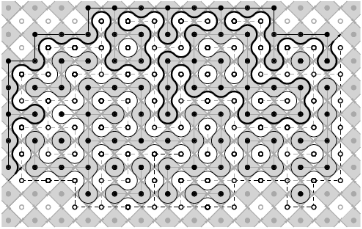
- critical $\beta = \beta_c$
- supercritical $\beta > \beta_c$

- $\mu^{\oplus}_{\beta,\Lambda_n}[\sigma_0] \le e^{-cn}$
- $\mu^{\oplus}_{\beta,\Lambda_n}[\sigma_0] = n^{-1/8+o(1)}$
- $\mu^{\oplus}_{\beta,\Lambda_n}[\sigma_0] \to \mu^{\oplus}_{\beta}[\sigma_0] > 0.$

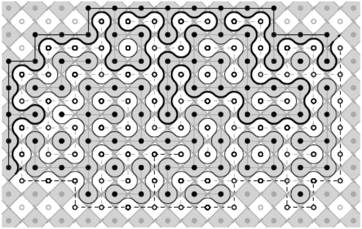
FK-Ising model, RCM with $q = 2$



[Chelkak-Smirnov, Invent.Math.'10]

Critical FK-Ising on $\mathbb{Z}^2$ with Dobrushin b.c. The interface converges to $SLE_{16/3}$.

FK-Ising model, RCM with $q = 2$



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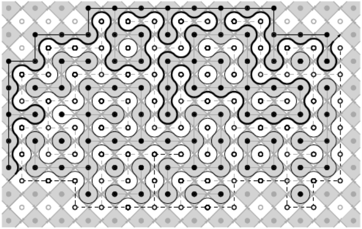

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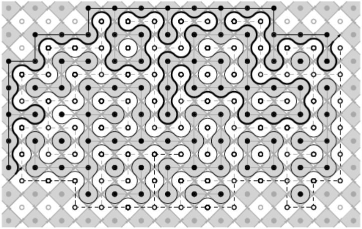

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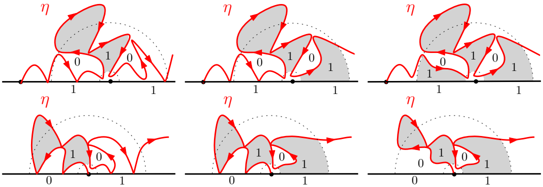
Conclusion

Arm exponents for Critical FK-Ising.

FK-Ising model

Boundary arm exponents : 6 patterns



boundary conditions (11).
(010), (0101), (10101)

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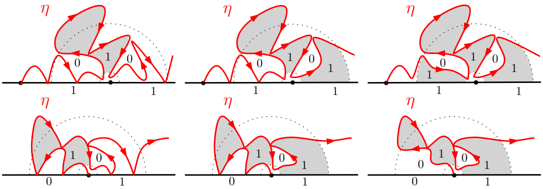

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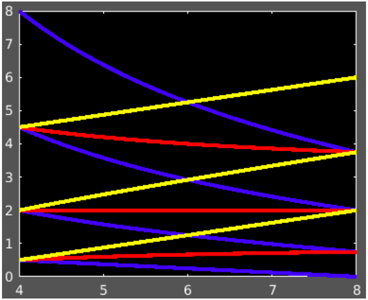

Interior arm exponents : 3 patterns

blue : (10), (1010), (101010)

red : (101), (10101), (1010101)

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FK-Ising model

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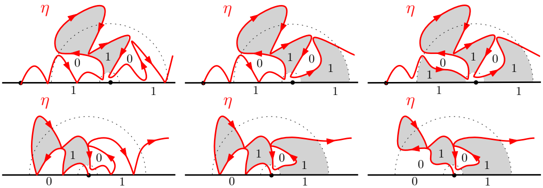

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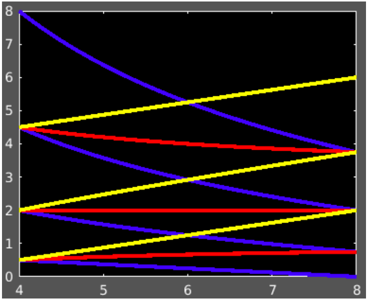

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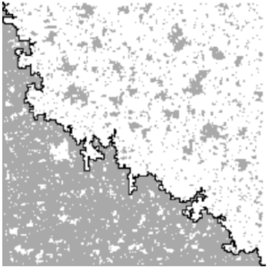
Universal arm exponents for RCM

$$\alpha_5 = 2, \quad \kappa \in (4, 8).$$

Critical Ising model, Dobrushin boundary condition



courtesy to Smirnov.

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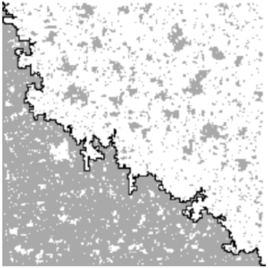

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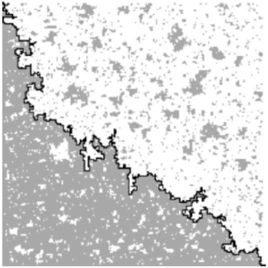

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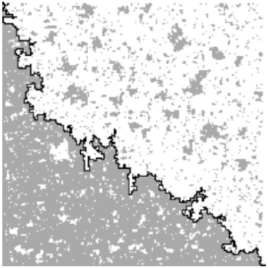

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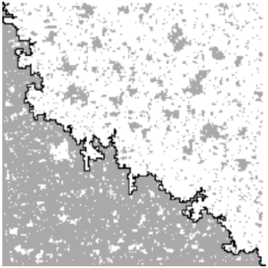
Conclusion

Arm exponents for critical Ising with Dobrushin boundary condition.

Critical Ising model, free boundary condition



courtesy to Smirnov.

[Hongler-Kytölä, JAMS'13]

Critical Ising model on $\mathbb{Z}^2$ with free b.c.
The interface converges to $\mathrm{SLE}_3(-3/2)$.

[W. AOP'18]

Arm exponents for $\mathrm{SLE}_3(-3/2)$.

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Conclusion

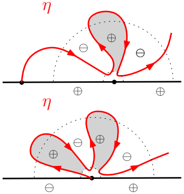
Arm exponents for critical Ising with free boundary condition

Critical Ising model, Arm exponents

Interior arm exponents : alternating $\alpha_{2j} = (16j^2 - 1)/24$.

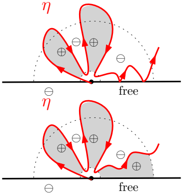
Boundary arm exponents : 6 patterns

b.c. (⊖⊕)



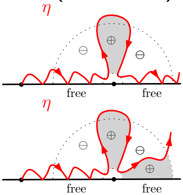
$$\begin{cases} \alpha^+_{2j-1} = j(4j+1)/3, \\ \alpha^+_{2j} = j(4j+5)/3. \end{cases}$$

b.c. (⊖free)



$$\beta^+_j = j(j+1)/3.$$

b.c.(freefree)



$$\gamma^+_j = j(2j-1)/6.$$

Critical Ising model, Arm exponents

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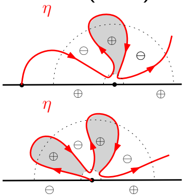
b.c. (⊖⊕)



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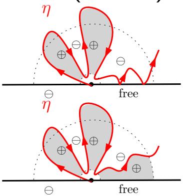

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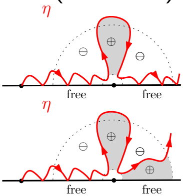
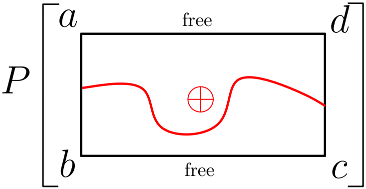


$$\gamma^+_j = j(2j-1)/6.$$

The asymptotic of the arm exponents is uniform over b.c. :

$$\alpha^+_j, \beta^+_j, \gamma^+_j \approx j^2/\kappa, \quad \forall \kappa.$$

Critical Ising model, Cardy's formula



[Benoist-Duminil-Copin-Hongler'16]

- It converges to $f(\Omega; a, b, c, d)$.
- It is conformally invariant.
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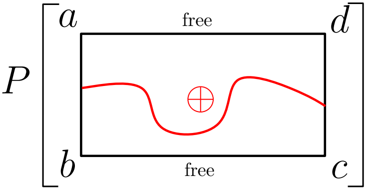

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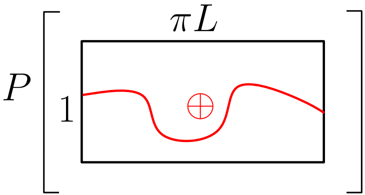

$$\approx \exp(-L/6)$$

Thanks !

Conformal Invariance of 2D Lattice Model

- Percolation
- RCM
- GFF

- SLE(6)
- SLE(16/3)
- SLE(4)

- Ising model
- Potts $q = 3$
- Potts $q = 4$

- SLE(3)
- SLE(10/3) ?
- SLE(4) ?