3/8-10

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Reference

- [B68] Borel, "On the automorphisms of certain subgrps of semi-simple Lie grps", 1968-1969
- [G87] Gromov, "Hyperbolic groups", 1987.
- [FH78] Farrell, Hsiang, "on the rational homotopy grps of diffeomorphism grps of discs, spheres and aspherical mfds," 1978
- [FJ89] Farrell, Jones, "negatively curved mfds with exotic smooth structures." 1989

Recall we proved the following last time:

Fact: (M, h) = compact Riem. mfd.
$(\widetilde{M}, \widetilde{h})$ = universal cover with covering metric
$\Rightarrow \forall f \in A(\widetilde{M})$, $\exists$ geodesic $L_f \subset \widetilde{M}$
s.t. $f(L_f) = L_f$

sketch of pf for $f \neq id$: Take $\widetilde{p} \in \widetilde{M}$, $\pi(\widetilde{p}) = p \in M$ $\quad \pi: \widetilde{M} \to M$ covering map

$$A(\widetilde{M}) \xrightarrow[\Phi_{\widetilde{p}}^{-1}]{\cong} \pi_1(M, p)$$
$$f \longmapsto [\alpha]$$

Cartan'Th $\Rightarrow$ $\alpha \underset{\text{free}}{\sim}$ closed geodesic γ

Take $\widetilde{q} \in \gamma$, $\pi(\widetilde{q}) = q$
Let $\widetilde{\gamma}$ be the lifting geodesic of γ starting from $\widetilde{q}$ $\Rightarrow f(\widetilde{\gamma}) = \widetilde{\gamma}$

Q: How does the choice of $\widetilde{p}$ affect the

construction of γ?

A: No effect.

Reason: $A(\widetilde{M}) \xrightarrow{\Phi_{\widetilde{p}}^{-1}} \pi_1(M,p)$

$f \longmapsto [\alpha_f]$

α_f is given as follows:

Let $\widetilde{\alpha}_f : I \to \widehat{M}$ be a path s.t. $\widetilde{\alpha}_f(0) = \widetilde{p}$.
$\widetilde{\alpha}_f(1) = f(\widetilde{p})$

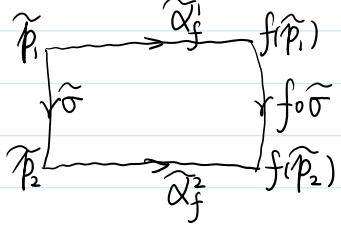
$\alpha_f \triangleq \pi \circ \widetilde{\alpha}_f$

Now take $\widetilde{p}_1, \widetilde{p}_2 \in \widehat{M}$ s.t. $\pi(\widetilde{p}_i) = p \in M$

Claim: $\exists\, [\sigma] \in \pi_1(M,p)$ s.t. $[\alpha_f^1] = [\sigma]^{-1}[\alpha_f^2][\sigma]$
$(\Rightarrow \alpha_f^1 \widetilde{\text{free}} \alpha_f^2)$

pf: Let $\widetilde{\sigma} : [0,1] \to \widehat{M}$ a path s.t.
$\widetilde{\sigma}(0) = \widetilde{p}_1$, $\widetilde{\sigma}(1) = \widetilde{p}_2$.

$\pi \circ \widetilde{\sigma} = \sigma$



$\widehat{M}$ simply-conn. $\Rightarrow \widetilde{\alpha}_f^1 \sim (f\circ\widetilde{\sigma})^{-1}\widetilde{\alpha}_f^2\widetilde{\sigma}$ (preserve endpts)

$\Rightarrow \widetilde{\alpha}_f^1 \sim \sigma^{-1}\alpha_f^2\sigma$ (preserve basept)

$\Rightarrow [\alpha_f^1] = [\sigma]^{-1}[\alpha_f^2][\sigma]$.

□

Claim: If M is neg. curved, then

for $i \geq 2$,

$$\pi_i BDiff_0(M) \xrightarrow{\cong} \pi_i BDiff(M)$$

Pf: Consider the fiber bundle

$$Diff(M)/Diff_0(M) \hookrightarrow BDiff_0(M) \to BDiff(M).$$

$Diff_0(M) \overset{\text{closed}}{\underset{\text{open}}{\subset}} Diff(M) \Rightarrow Diff(M)/Diff_0(M)$ with discrete top.

$\Rightarrow BDiff_0(M) \xrightarrow{p} BDiff(M)$ is a covering map,

claim follows. □

$H < G$

$EG/G = BG.$

$\Downarrow E \simeq *$

$EG/H = BH$

$G/H \hookrightarrow E/H \to E/G$

Rk: $Diff(M)/Diff_0(M)$ is finite when M is neg. curved.

Pf: $Diff(M) \xrightarrow{\Psi} \pi_0 G(M) \cong Out(\pi_1 M)$

$\ker \Psi = Diff_0(M)$

$\Rightarrow Diff(M)/Diff_0(M) \to Out(\pi_1 M)$

is monic.

$Out(\pi_1 M)$ finite if M is neg. curved and $\dim M \geq 3$ (cf. [B68], [G87, Th5.4A])

2.2 $T^{<0}(M)$: classifying space for neg. curved M-bundles (with FHT)

Let M be a neg. curved mfd. For any M-bundles $E \xrightarrow{\pi} B$, let $M_x \triangleq \pi^{-1}(x)$

Def: A smooth M-bundle $E \xrightarrow{\pi} B$ is neg. curved if $\forall x \in B$, $\exists h_x \in R^{<0}(M_x)$ s.t.

"h_x vary continuously over $x \in B$".

if $\forall x \in B$, $\exists h_x \in R^{<0}(M_x)$ s.t.
= "h_x vary continuously over $x \in B$".

Eg. $B \times M \longrightarrow B$ trivial M-bundle
$\forall x \in B$, $\exists h_x \in R^{<0}(M_x) = R^{<0}(M)$
= "h_x vary cont. over $x \in B$"

$\Updownarrow$

$M \longrightarrow R^{<0}(M)$ is cont.
$x \longmapsto h_x$

= $\forall x \in B$, $\exists x \in U \underset{\text{open}}{\subseteq} B$ and $\varphi : U \times M \xrightarrow{\cong} \pi^{-1}[U]$ (fiberwise diffeom.)

notation: $\varphi_x : M \longrightarrow M_x$

s.t.
$U \longrightarrow R^{<0}(M)$ is cont. //
$x \longmapsto \varphi_x^* h_x$

Def: Two neg. curved M-bundles $E_i \longrightarrow B$, $i = 0, 1$ are $\equiv$ if $\exists$ neg. curved M-bundle

$$E \longrightarrow B \times [0,1]$$

s.t. $E|_{B \times i} \cong E_i$ fiberwise isometric.

Th M : neg. curved mfd.
B : paracompact top. space
Then $\exists$ bijection
$[B, T^{<0}(M)] \xrightarrow{1:1}$ {$\equiv$ classes of neg. curved M-bundles over B with FHT}.

Cor M : neg. curved mfd
B : paracompact, simply-conn. path conn. locally path conn.
Then $\exists$ bijection
$[B, T^{<0}(M)] \xrightarrow{1:1}$ {$\equiv$ classes of neg. curved M-bundles over B}

Eg (The universal neg. curved M-bundles)

Let M be neg. curved mfd

$\mathrm{Diff}_0(M) \curvearrowright R^{<0}(M)$ free ($\Leftarrow$ $\mathrm{Diff}_0(M) \curvearrowright R(M)$ free)

$R^{<0}(M) \longrightarrow T^{<0}(M) = R^{<0}(M)/\mathrm{Diff}_0(M)$

has local sections (cf. [Ebin 70]).

$R^{<0}(M) \longrightarrow T^{<0}(M)$ is principal $\mathrm{Diff}_0(M)$-bundle

Claim: $R^{<0}(M) \times_{\mathrm{Diff}_0(M)} M \longrightarrow T^{<0}(M)$ is a neg. curved M-bundle.

pf: First note $R^{<0}(M) \times M \downarrow \bar{\eta}$ $R^{<0}(M)$ is neg. curved M-bundle

(For $h \in R^{<0}(M)$, equipped $M_h = h \times M = M$ with metric h)

This induces a neg. curved bundle structure for η by the diagram:

$$\begin{array}{ccc} R^{<0}(M)\times M & \xrightarrow{\pi} & R^{<0}(M) \times_{\mathrm{Diff}_0(M)} M \\ \bar{\eta}\downarrow & \hookrightarrow & \downarrow \\ R^{<0}(M) & \longrightarrow & T^{<0}(M) = R^{<0}(M)/\mathrm{Diff}_0(M)\,[h] \end{array}$$

$[h \times M] \cong \uparrow \pi|_{h\times M}$ $(h\times M, h)$ → equipped with push-forward metric of h.

(Need to verify if $\pi(h\times M) = \pi(g\times M)$,

i.e. $\exists f \in \mathrm{Diff}_0(M)$ s.t. $g = f\cdot h = (f^{-1})^* h$

then $(h\times M, h) \longrightarrow (g\times M, g)$ is isometry:

$(h, x) \longmapsto (f\cdot h, f(x))$

$f^* g = h$)

Claim: $\eta: R^{<0}(M) \times_{\mathrm{Diff}_0(M)} M \longrightarrow T^{<0}(M)$ is a universal neg. curved M-bundle with FHT.

/ universal neg. curved M-bundle with structure grp $\mathrm{Diff}_0(M)$.

i.e. for any neg. curved M-bundle

$$\xi : E \xrightarrow{\pi} B$$

with structure grp $\mathrm{Diff}_0(M)$,

$\exists$ cont. map $F : B \longrightarrow T^{<0}(M)$ s.t.

$F^*\eta \equiv \xi$ as neg. curved M-bundle.

part of Pf: Suppose each M_x equipped with neg. curved h_x s.t. h_x vari cont. over $x \in B$.

$\forall x \in B$, $\exists\, x \in U \underset{\text{open}}{\subseteq} B$ and

$$\begin{tikzcd} U \times M \arrow[rr, "\varphi", "\text{fiberwise diffeom.}"'] \arrow[dr] & \circlearrowleft & \pi^{-1}[U] \arrow[dl] \\ & U & \end{tikzcd}$$

$$\varphi_x : \underset{\substack{\| \\ M}}{x \times M} \longrightarrow \pi^{-1}(x) = M_x$$

$$\varphi_x^* h_x \in R^{<0}(M)$$

Claim: The correspondence $x \longmapsto [\varphi_x^* h_x] \in T^{<0}(M)$ doesn't depend on choice of φ, and hence induce a cont. map: $B \to T^{<0}(M)$

Pf: if $\exists\, x \in V \underset{\text{open}}{\subseteq} B$ and

$$\begin{tikzcd} V \times M \arrow[rr, "\psi", "\cong"'] \arrow[dr] & \circlearrowleft & \pi^{-1}[V] \arrow[dl] \\ & V & \end{tikzcd}$$

$$\psi_x : \underset{\substack{\| \\ M}}{x \times M} \longrightarrow \pi^{-1}(x) = M_x$$

$$[\psi_x^* h_x] = [\varphi_x^* h_x] \in R^{<0}(M) / \mathrm{Diff}_0(M)$$

$$\Uparrow$$

$$\psi_x^{-1} \circ \varphi_x \in \mathrm{Diff}_0(M)$$

$\square$

3. Connection and difference between $T^{<0}(M)$ and $T(M)$

M: neg curved mfd.

$R^{<0}(M) \hookrightarrow R(M)$

$\rightsquigarrow$ $R^{<0}(M) \xrightarrow{F} R(M)$

$Diff_0(M)$ — forgetful map — $Diff_0(M)$

$\| \qquad\qquad\qquad\qquad \|$

$T^{<0}(M) \qquad\qquad T(M)$

$T^{<0}(M)$: classify neg. curved bundle with FHT

$T(M)$: $\|$ when M is neg. curved

$BDiff_0(M)$: classify smooth M-bundle with FHT

Some information for $\pi_i T(M)$ has been known.

Aim:

① Understand difference between neg. curved M-bundle and smooth M-bundle (with FHT)

i.e. study $F_*: \pi_i T^{<0}(M) \longrightarrow \pi_i T(M)$

(trivial? surjective? injective?)

$\Updownarrow$

② $\pi_i T^{<0}(M) = ?$

3.1 $\pi_i T(M)$

Th 3.1 ([FH78], [FJ89]) Let M be a neg. curved mfd of dim n. Then

$$\pi_i(Diff_0(M), id) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$= \pi_i(Diff(M), id) \otimes \mathbb{Q}$$

→ commutative for $i>0$ since $Diff(M)$ and $Diff_0(M)$ are top. grps.

$$= \begin{cases} \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M, \mathbb{Q}), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

→ homology of grps $H_*(\pi_i M, \mathbb{Q})$

for $0 < i < \min\left\{\frac{n-7}{2}, \frac{n-4}{3}\right\}$

can be replaced by "concordance stable range".

└ obtained by Igusa, may be improved.

Cor 3.2 M = neg. curved mfd of dim n.

$$\begin{aligned}\pi_{i+1}T(M)\otimes\mathbb{Q} &= \pi_{i+1}B\mathcal{D}if\!f_\partial(M)\otimes\mathbb{Q}\\ &= \pi_i(\mathcal{D}if\!f_\partial(M), id)\otimes\mathbb{Q}\\ &= \pi_i(\mathcal{D}if\!f(M), id)\otimes\mathbb{Q}\\ &\neq 0 \quad \text{for } i\equiv 1 \pmod 4,\\ &\quad n \text{ odd and } 0<i<\min\left\{\frac{n-7}{2}, \frac{n-4}{3}\right\}.\end{aligned}$$

3.2 $F_*: \pi_i T^{<0}(M) \longrightarrow \pi_i T(M)$ and $\pi_i T^{<0}(M)$.

3.2.1 The case: M is a closed real hyp. mfd.

a smooth closed mfd admitting Riem. metric of sec $=-1$ everywhere

Def: a smooth closed mfd Σ of dim n is called an exotic sphere if $\Sigma \approx S^n$ but $\Sigma \not\cong S^n$

Th 3.3 ([FO09]) Let M be a closed real hyp. mfd of dim $n \geq 6$. Assume that $\exists$ exotic sphere of dim $n+1$. Then $\exists$ a finite sheeted cover N of M s.t.

$F_*: \pi_1 T^{<0}(N) \longrightarrow \pi_1 T(N)$ is nontrivial

$\Downarrow$

$\pi_1 T^{<0}(N) \neq 0 \Rightarrow T^{<0}(N)$ is not contractible.

Rk: N is a finite sheeted cover of M iff

$\exists$ covering map $N \xrightarrow{\pi} M$ with $\pi^{-1}(m)$ is finite for $m \in M$

Th 3.4 ([FO09]) For $\forall\ k_0 \in \mathbb{Z}_{>0}$, $\exists\ n_0 = n_0(k_0) \in \mathbb{Z}$ s.t. the following holds for any closed real hyp. mfd M^n of dim $n \geq n_0$:

M^n has a finite sheeted cover N s.t. for $1 \leq k \leq k_0$ and $n+k \equiv 3 \bmod 4$

$F_* : \pi_k T^{<0}(N) \longrightarrow \pi_k T(N)$ is nontrivial

$(\Rightarrow \pi_k T^{<0}(N) \neq 0 \Rightarrow T^{<0}(N) \not\simeq *)$.