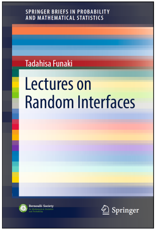
Lecture No 21 May 20, 2022 (Fri)

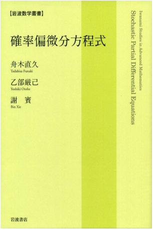
Part III. Applications of Stochastic Analysis

In this part, we discuss stochastic partial differential equations (SPDEs) as an application of stochastic analysis.

Textbooks:

[4] T. Funaki, Lectures on Random Interfaces, SpringerBriefs, 2016.
[7] T. Funaki, Y. Otobe, B. Xie (舟木直久, 乙部厳己, 謝賓):
確率偏微分方程式, 岩波書店, 2019.





随机偏微分方程

23.1 White noise $\dot{B}_t$ in time (formal understanding)

- SDE was formally introduced as an ODE with Gaussian noise $\dot{B}_t$ (innovation) which is produced independently at each t:
$$\dot{X}_t = b(X_t) + \alpha(X_t)\dot{B}_t \tag{1}$$
- Mathematically, (1) was defined in an integrated form.
- Gaussian variables are characterized by their mean and covariance.
- At least formally, the covariance of $\dot{B}_t$ is given by
$$E[\dot{B}_s \dot{B}_t] = \delta(t-s), \tag{2}$$
where $\delta(t) \equiv \delta_0(t)$ is the δ-function at $t=0$.
($\rightarrow$ See the next page)
- δ-correlation (covariance) of Gaussian variables $\dot{B}_t$ implies their independence for different t, since the covariance is 0 if $t \neq s$.

[Formal proof of (2)] • First recall that Brownian motion $B = (B_t)_{t \geq 0}$ is a Gaussian process with mean 0 and covariance

$$E[B_s B_t] = \min\{s, t\}(\equiv s \wedge t), \quad s, t \geq 0. \tag{3}$$

• Then, differentiate both sides of (3) in t and then in s (in generalized functions' sense).
• For the RHS, first for a fixed s,

$$\frac{\partial}{\partial t}(t \wedge s) = 1_{[0,s]}(t) = 1_{[t,\infty)}(s)$$

and then differentiating in s, we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}(t \wedge s) = \delta(t - s).$$

• Assuming we can interchange the differentiation and the expectation $E[\cdot]$, we obtain (2). (Of course, this is not true, since $\dot{B}_t$ does not exist.) $\square$

23.2 Space-time Gaussian white noise

- In PDE setting (instead of ODE setting), similarly to $\dot{B}_t$, it is natural to consider a space-time noise
$$\dot{W}(t,x) = \dot{W}(t,x,\omega)$$
which is independent for different (t,x), that is,
- Extending the relation (2) for $\dot{B}_t$ to space-time setting, it is natural to consider a Gaussian noise $\dot{W}(t,x)$ with mean 0 and covariance
$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y), \qquad (4)$$
for $t,s \geq 0,\ x,y \in \mathbb{R}^d$ or $\in D\,(\subset \mathbb{R}^d)$.
- Such noise is called the space-time Gaussian white noise.
- RHS of (4) $= 0$ if $t \neq s$ or $x \neq y$, and this implies the independence.

23.3 Formal construction of space-time Gaussian white noise on a domain D

- Let D be $\mathbb{R}^d$ or a bounded domain of $\mathbb{R}^d$ with smooth boundary ∂D or d-dimensional torus $\mathbb{T}^d = [0,1)^d$ (identifying 0 and 1).
- We consider real L^2-space $L^2(D) \equiv L^2(D, dx)$ with the Lebesgue measure dx on D.
- Let $\{\psi_k\}_{k=1}^{\infty}$ be a complete orthonormal system (CONS) of $L^2(D)$, i.e. $(\psi_i, \psi_j)_{L^2(D)} = \delta_{ij}$ and the set of linear combinations of $\{\psi_k\}$ is dense in $L^2(D)$.
- Let $\{B_t^k = B_t^k(\omega)\}_{k=1}^{\infty}$ be independent 1-dimensional Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$.
- For example, for $D = \mathbb{R}$, one can take Hermite functions

$$h_k(x) = \left(2^k k! \sqrt{\pi}\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_k(x), \quad k = 0, 1, 2, \ldots,$$

  where $H_k(x)$ are Hermite polynomials defined by

$$H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

  For $D = \mathbb{R}^d$, we may consider products $\prod_{i=1}^d h_{k_i}(x_i)$.

- Consider a formal Fourier series with B_t^k as its Fourier coefficients:

$$W(t,x) = \sum_{k=1}^{\infty} B_t^k \psi_k(x). \tag{5}$$

- Since $\sum_{k=1}^{\infty}(B_t^k(\omega))^2 = \infty$ a.s. ω ${}^{\star)}$, the condition $\sum_{k=1}^{\infty}(B_t^k)^2 < \infty$ a.s. for the a.s.-convergence of (5) in $L^2(D)$ is not satisfied.
- Thus, the series (5) does not converge in $L^2(D)$ in a.s.-sense. (it determines only so-called cylindrical Brownian motion).
- We leave rigorous discussion later.

${}^{\star)}$ By strong law of large numbers, $\frac{1}{N}\sum_{k=1}^{N}(B_t^k(\omega))^2 \to t$ a.s.

- The covariance of $W(t,x)$ for $t,s \geq 0, x,y \in D$ is formally computed as

$$E[W(t,x)W(s,y)] = \sum_{k,j=1}^{\infty} E[B_t^k B_s^j]\psi_k(x)\psi_j(y)$$
$$= (t \wedge s) \sum_{k=1}^{\infty} \psi_k(x)\psi_k(y)$$
$$\underset{(\star)}{=} (t \wedge s)\delta(x-y) \tag{6}$$

- The last identity $(\star)$ is seen from

$$\int_D \Big(\sum_{k=1}^{\infty} \psi_k(x)\psi_k(y) \Big) \varphi(y)dy = \sum_{k=1}^{\infty} \psi_k(x)(\varphi, \psi_k)_{L^2(D)}$$
$$= \varphi(x) = \int_D \delta(x-y)\varphi(y)dy$$

for ${}^\forall \varphi \in C(D) \cap L^2(D)$.

- Since $\frac{\partial}{\partial s}\frac{\partial}{\partial t}(t \wedge s) = \delta(t-s)$ as we saw above,
$$\dot{W}(t,x) := \frac{\partial}{\partial t} W(t,x)$$
would have the desired covariance (4):
$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y), \quad x, y \in D.$$
- Formally, $\dot{W}(t,x)$ has a representation
$$\dot{W}(t,x) = \sum_{k=1}^{\infty} \dot{B}_t^k \psi_k(x) \tag{7}$$
- $W(t,x)$ is called the white noise process ("white" means $\lambda_k = 1$ for all k in colored noise stated below).
- It is constructed by means of infinitely many Brownian motions so that it is considered as an infinite-dimensional Brownian motion.

[Remark] We can also formally construct $\dot{W}(t,x)$ as a product

$$\dot{W}(t,x) = \dot{B}(t)\dot{W}^1(x_1)\cdots\dot{W}^d(x_d),$$

where $B, W^1, \ldots, W^d$ are independent (two-sided) Brownian motions.

Here, "two-sided" means that, for example for $W^1(x_1)$, preparing two independent Brownian motions $W^{1,+}(x_1), x_1 \geq 0$ and $W^{1,-}(x_1), x_1 \geq 0$ starting at 0: $W^{1,\pm}(0) = 0$, it is defined by $W^1(x_1) = W^{1,+}(x_1)$ for $x_1 \geq 0$ and $W^1(x_1) = W^{1,-}(-x_1)$ for $x_1 \leq 0$. $\square$

23.4 Stochastic integrals and colored noises

- $\dot{W}(t,x)$ has no rigorous meaning.
- As we saw in the relation between $\dot{B}_t$ and B_t, the integrated form $W(t,x)$ in t is better, but still has no rigorous meaning.
- So we integrate also in space by multiplying a test function $\varphi = \varphi(x)$ and consider

$$W_t(\varphi) \equiv \langle W(t), \varphi \rangle \left(= (W(t), \varphi)_{L^2(D)} = \int_D W(t,x)\varphi(x)dx \right)$$

$$:= \sum_{k=1}^{\infty} B_t^k (\varphi, \psi_k)_{L^2(D)}.$$

- Then, $W_t(\varphi)$ has a rigorous meaning, since

$$E[W_t(\varphi)^2] = t \sum_{k=1}^{\infty} (\varphi, \psi_k)^2_{L^2(D)} \underset{\text{Parseval}}{=} t\|\varphi\|^2_{L^2(D)}.$$

- More generally, we can introduce stochastic integrals with respect to $W(t)$.

- Stochastic integrals w.r.t. $W(t,x)$: For $f = f(t,x,\omega)$: $(\mathcal{F}_t)$-adapted, $\in L^2([0,T] \times D \times \Omega)$ (for every $T > 0$),

$$M_t(f) \equiv \int_0^t \int_D f(s,x) W(dsdx)$$
$$:= \sum_{k=1}^{\infty} \int_0^t \left(f(s,\cdot), \psi_k\right)_{L^2(D)} dB_s^k,$$

  where $\mathcal{F}_t = \sigma\{W(s,\cdot); s \le t\}$. (or we may assume $\{B_t^k\}$ are, in general, independent $(\mathcal{F}_t)$-Brownian motions.)

- Then, $M(f) \in \mathcal{M}_c^2$, i.e. square integrable continuous $(\mathcal{F}_t)$-martingale and cross variation is given by

$$\langle M(f), M(g) \rangle_t = \int_0^t (f(s), g(s))_{L^2(D)} ds.$$

  ∵ Since $\{B^k\}$ are independent, by taking the limit,

$$\langle M(f), M(g) \rangle_t = \sum_{k=1}^{\infty} \int_0^t \left(f(s), \psi_k\right)_{L^2(D)} \left(g(s), \psi_k\right)_{L^2(D)} ds$$
$$= \int_0^t (f(s), g(s))_{L^2(D)} ds.$$

- In particular, we have Itô isometry:

$$\|M_T^2(f)\|_{L^2(\Omega)} = \|f\|_{L^2([0,T]\times D\times\Omega)}$$

i.e.,

$$E\left[M_T^2(f)\right] = \int_0^T \int_D E[f(t,x)^2]dtdx$$

- Burkholder-Davis-Gundy's inequality:
$^\forall p > 0$, $^\exists C = C_p > 0$ s.t.

$$E\left[\sup_{0\le t\le T} |M_t(f)|^p\right] \le C\, E\left[\left(\int_0^T \int_D f(t,x)^2 dtdx\right)^{p/2}\right].$$

$\because$ (LHS) $\le C_p E[\langle M(f)\rangle_T^{p/2}]$.

- Though the series (5) does not converge in $L^2(D)$, if we introduce a damping factor $\{\lambda_k \geq 0\}_{k=1}^{\infty}$ s.t. $\mathrm{Tr}Q \equiv \sum_{k=1}^{\infty} \lambda_k < \infty$ and consider

$$W(t,x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} B_t^k \psi_k(x), \tag{8}$$

  then it converges in $L^2(D)$ a.s. (since $E[\sum_{k=1}^{\infty}(\sqrt{\lambda_k} B_t^k)^2] < \infty$) and defines Q-Brownian motion, where Q is a linear operator s.t. $Q\psi_k = \lambda_k \psi_k$.
- Especially, Q is a self-adjoint nonnegative linear operator of nuclear type (trace class).
- Its time derivative $\dot{W}(t,x)$ is called a colored noise or smooth noise.
- If $\lambda_1 = 1, \psi_1 \equiv 1$ and $\lambda_k = 0$ for $k \geq 2$, $W(t,x)$ is nothing but a 1-dimensional Brownian motion depending only on time.

[Remark] General framework is known.

- Let H be a separable real Hilbert space (instead of $L^2(D)$).
- Then, one can introduce H-valued Brownian motion (Q-Brownian motion) for self-adjoint nonnegative linear operator Q of nuclear type.
- If $Q = I$ or more generally for self-adjoint nonnegative bounded linear operator Q, one can define Q-cylindrical Brownian motion, which is not H-valued process in general, but $W_t(\varphi) = (W(t), \varphi)$ has meaning for $\varphi \in H$.
- Stochastic integrals with respect to cylindrical Brownian motions can be defined for H-valued $(\mathcal{F}_t)$-adapted integrands or operator-valued processes.

23.5 Another way to introduce white noise process

- (Discrete approximations) We can construct white noise process also as follows: Let $D \subset \mathbb{R}^d$ be a domain with smooth boundary, and discretize it as:

  $$D_N := D \cap \frac{1}{N}\mathbb{Z}^d.$$

  Prepare independent Brownian motions $\{B_t(y)\}_{y \in D_N}$ and define

  $$W^N(t,\cdot) := \frac{1}{N^{d/2}} \sum_{y \in D_N} \delta_y(\cdot) B_t(y),$$

  or

  $$\tilde{W}^N(t,x) := N^{d/2} \sum_{y \in D_N} 1_{\Lambda(y,\frac{1}{N})}(x) B_t(y),$$

  where $\Lambda(y,\frac{1}{N})$ is a box with center y and side-length $\frac{1}{N}$.

- Then, both W^N and $\tilde{W}^N$ converge as $N \to \infty$ to $W(t,x)$ in law. Indeed, for $\varphi \in C_b(D)$,

$$
\begin{aligned}
E[\langle W^n(t,\cdot),\varphi\rangle^2] &= E\left[\left(N^{-d/2}\sum_{y\in D_N}\varphi(y)B_t(y)\right)^2\right] \\
&= tN^{-d}\sum_{y\in D_N}\varphi^2(y) \to t\|\varphi\|^2_{L^2(D)}, \\
E[\langle \tilde{W}^n(t,\cdot),\varphi\rangle^2] &= tN^d\sum_{y\in D_N}\langle 1_{\Lambda(y,\frac{1}{N})},\varphi\rangle^2 \\
&= tN^d\sum_{y\in D_N}\left(\varphi(y)N^{-d}\right)^2 + o(1) \to t\|\varphi\|^2_{L^2(D)}.
\end{aligned}
$$

§24 Simple example of SPDE

24.1 Heat equation with random external field

- As a simple example of SPDE, consider a heat equation on $\mathbb{R}^d$ with an external force $F = F(t, x, u, \omega)$:

$$\frac{\partial u}{\partial t} = \Delta u + F, \quad x \in \mathbb{R}^d \tag{9}$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian.

- For example, if $F = a(\omega)u$ is linear in u with random coefficient $a = a(\omega)$, (9) is written as

$$\frac{\partial u}{\partial t} = \Delta u + a(\omega)u. \tag{10}$$

- (10) is easily solved as $u(t, x, \omega) = v(t, x)e^{a(\omega)t}$, where $v(t, x)$ is the solution of the heat equation $\frac{\partial v}{\partial t} = \Delta v$.
- (10) is solved for each fixed ω as a usual PDE so that it is called random partial differential equation.

- Recalling the SDE $\dot{X}_t = b(X_t) + \alpha(X_t)\dot{B}_t$, it is natural to consider the heat equation (PDE) with $F = \dot{W}(t,x)$, which is independent for different (t,x)

$$\frac{\partial u}{\partial t} = \Delta u + \dot{W}(t,x), \quad x \in \mathbb{R}^d \text{ (or } x \in D \subset \mathbb{R}^d) \quad (11)$$

- This is a stochastic heat equation sometimes called Edwards-Wilkinson equation.

- Let $H^s_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) = W^{s,2}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d)$ be the Sobolev space with (mostly negative) exponent s and set

$$H^{s-}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d) := \bigcap_{\delta>0} H^{s-\delta}_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^d),$$

where $\mathbb{R}_+ = [0,\infty)$.

- It is known that $\dot{W}(t,x) \in H_{\mathrm{loc}}^{-\frac{d+1}{2}-}(\mathbb{R}_+ \times \mathbb{R}^d)$ a.s. ω.
- In particular, $\dot{W}(t,x)$ is not a usual function but a generalized function of (t,x).
- Since $B_t \in H_{\mathrm{loc}}^{\frac{1}{2}-}(\mathbb{R}_+)$ a.s. ω is known for Brownian motion, we see $\dot{B}_t \in H_{\mathrm{loc}}^{-\frac{1}{2}-}(\mathbb{R}_+)$ and this coincides with the above result when $d = 0$.
- Compared to the usual PDE theory, in a sense, we need to consider PDEs with external forces having very bad regularity.
- The equation such as (11), which is not defined in ω-wise sense (defined only integrated in t and also in x), is called stochastic partial differential equation (SPDE).