PART II. Deligne–Lusztig theory for finite reductive groups

Schedule.

- NO LECTURE on Friday, Dec. 16
- Two lectures:
  - → Wednesday, Dec. 21
  - → Friday, Dec. 23

Chapter 8. Deligne–Lusztig theory

- G connected reductive algebraic group
- $F : G \longrightarrow G$ Frobenius endomorphism $/ \mathbb{F}_q$
- $G^F = \{ g \in G \mid F(g) = g \}$ (finite reductive group)
- B_0 F-stable Borel subgroup

  $\cup$

  T_0 F-stable maximal torus
- $W = N_G(T_0)/T_0$: Weyl group
- $S = \{ s \in W \mid \dim B_0 s B_0 - \dim B_0 = 1 \}$

Example. $G = GL_n(\mathbb{F})$

8. A. Deligne-Lusztig varieties.

Let P be a parabolic subgroup of G admitting an F-stable Levi complement. Let $V = R_u(P)$; $P = L \ltimes V$.

THE GROUP P IS NOT ASSUMED TO BE F-STABLE !!!

We set

$$\begin{array}{ccl} Y_P^G = Y_P & = & \{ gV \in G/V \mid g^{-1}F(g) \in V\cdot F(V) \} \\ \Big\downarrow \pi & & \quad \Big\downarrow \\ X_P^G = X_P & = & \{ gP \in G/P \mid g^{-1}F(g) \in P\cdot F(P) \} \end{array}$$

(Deligne-Lusztig varieties)

- G^F acts on the left : if $g_0 \in G^F$, $(g_0 g)^{-1} F(g_0 g) = g^{-1}F(g)$ (on Y_P and X_P)
- L^F acts on the right on Y_P because L normalizes V : $(gl)^{-1}F(gl) = l^{-1}g^{-1}F(g)\,l$
- π is G^F-equivariant.

Proposition 8.1. The varieties Y_P and X_P are smooth, quasi-affine, of pure dimension $\dim V\cdot F(V) - \dim V$.
The map π is a quotient of Y_P by L^F.

Proof. $V\cdot F(V)$ is smooth : it is the $(V\times V)$-orbit of 1 for the action on G given by $(v, v')\cdot g = v\,g\,F(v')$.

$\mathcal{L}^{-1}(V\cdot F(V)) \subset G$ is smooth because $\mathcal{L}$ is a Galois unramified covering. But $Y_P = \mathcal{L}^{-1}(V\cdot F(V))/V$ is the quotient of a smooth variety by a free action : so it is smooth.

The dimension also follows.

Also G/V is quasi-affine because V is unipotent (ref ??).

- Let us prove that π is an orbit map.

Let $g \in G$ be such that $g^{-1}F(g) \in P\cdot F(P)$

$$P\cdot F(P) = L\cdot V\cdot L\cdot F(V) = L\cdot V\cdot F(V).$$

We can write $g^{-1}F(g) = l\cdot x$, $x \in V\cdot F(V)$.
By Lang Theorem applied to L, there exists $a \in L$ such that $a\,F(a)^{-1} = l$.

Now $(ga)^{-1} F(ga) = a^{-1} g^{-1} F(g) F(a)$

$$= a^{-1} \ell \; x \; F(a)$$

$$= \underbrace{a^{-1} \ell F(a)}_{=1} \cdot \underbrace{F(a)^{-1} x F(a)}_{\in V \cdot F(V)}$$

So $gaV \in Y_P$ and $\pi(gaV) = gaP = gP$

(since $a \in L \subset P$).

$\Rightarrow$ π is surjective.

Also, fibers of π are L^F-orbits (use the fact that $V \cdot F(V) \cap L = \{1\}$: see Digne-Michel book).

Proving that π is a quotient requires some more work... ∎

Example 8.2. Assume here that P is F-stable.

So V is F-stable :

$$Y_P = \{ gV \in G/V \mid g^{-1} F(g) \in V \}$$
$$= (G/V)^F = G^F / V^F \text{ because } V \text{ is connected.} \; ∎$$

Example 8.3. Assume that $G = SL_2(\overline{\mathbb{F}})$ and $G^F = SL_2(\mathbb{F}_q)$. Let

$$B_0 = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}_{\substack{a \in \overline{\mathbb{F}}^\times \\ b \in \overline{\mathbb{F}}}} \; ; \; T_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, a \in \overline{\mathbb{F}}^\times \right\}$$

$$U_0 = R_u(B_0) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$$

Let $s \in \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $g_0 \in G$ be such that $g_0^{-1} F(g_0) = s$.

Let $T' = {}^{g_0}T_0$, $B' = {}^{g_0}B_0$.

Then $F(T') = {}^{F(g_0)}T_0 = {}^{g_0 g_0^{-1} F(g_0)}T_0$

$$= {}^{g_0 s}T_0$$
$$= {}^{g_0}T_0 = T'$$

$$F(B') = \cdots = {}^{g_0 s}B_0 \neq {}^{g_0}B_0 = B'.$$

$\cdot \; \underline{T'^F}$? Let $c_{g_0} : T_0 \longrightarrow T'$

$$t \longmapsto g_0 t g_0^{-1}$$

$$c_{g_0}(sF(t)) = F(c_{g_0}(t))$$

so $T'^F \simeq T_0^{sF} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T_0 \mid a = a^{-q} \text{ and } a^q = a^{-1} \right\} \simeq \mu_{q+1}.$

· $\underline{Y_{B'}}$? $\quad G/U' \ni gU'$ where $U' = R_u(B')$

$\wr\!\wr$

$G/U_0 \ni gU'g_0 = gg_0\, g_0^{-1}U'g_0 = gg_0U_0$

So $Y_{B'} \simeq \{ gU_0 \in G/U_0 \mid (gg_0^{-1})^{-1}F(gg_0^{-1}) \in U' \cdot F(U')\}$

$\iff \cancel{g_0}\, g^{-1}F(g)\, F(\cancel{g_0})^{-1}$
$\in \cancel{g_0}\, U_0 \underbrace{g_0^{-1}F(g_0)}_{= s} U_0\, \cancel{F(g_0)}^{-1}$

$Y_{B'} \simeq \{ gU_0 \in G/U_0 \mid g^{-1}F(g) \in U_0 s U_0 \}$

The map $\quad G/U_0 \xrightarrow{\sim} \mathbb{A}^2(\mathbb{F}) \setminus \{(0,0)\}$

$$gU_0 \longmapsto g\cdot(1,0)$$

$$\begin{pmatrix} x & z \\ y & t \end{pmatrix} U_0 \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$$

$U_0 s U_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}) \;\middle|\; c = 1 \right\}$

(exercise).

$Y_{B'} \simeq \left\{ (x,y) \in \mathbb{A}^2(\mathbb{F}) \setminus \{(0,0)\} \;\middle|\; \begin{pmatrix} x & z \\ y & t \end{pmatrix}^{-1} F\begin{pmatrix} x & z \\ y & t \end{pmatrix} \in U_0 s U_0 \right\}$

$$= \begin{pmatrix} t & -z \\ -y & x \end{pmatrix} \begin{pmatrix} x^q & z \\ y^q & t \end{pmatrix}$$

$Y_{B'} = \{ (x,y) \in \mathbb{A}^2(\mathbb{F}) \setminus \{(0,0)\} \mid xy^q - yx^q = 1 \}$

(!!!).

Conjecture. Y_P is affine.

Theorem (Deligne – Lusztig). True if q is large enough (for GL_n, $q > n$ is ok).

Remark.

- He, Orlik-Rapoport, B.-Rouquier (2008)
- Harashita (2013): $|S| \leq 2$.
- It is known that $H^i_c(Y_P) = 0$ if $i \notin [\dim Y_P, 2\dim Y_P]$ (Digne-Michel-Rouquier, 2006, coro 3.3.22). ∎

8.B. Deligne-Lusztig induction and restriction.

Let $H^\bullet_c(Y_P) = \bigoplus_{i \geq 0} H^i_c(Y_P)$

$\rotatebox{90}{$\in$}$

(KG^F, KL^F)-grbimod

graded

Deligne-Lusztig functors

Deligne-Lusztig maps.

$$\mathcal{R}^G_{L\subset P} : KL^F\text{-grmod} \longrightarrow KG^F\text{-grmod}$$

$$M \longmapsto H^\bullet_c(Y_P) \otimes_{KL^F} M$$

$$\bigoplus_i M_i \qquad \bigoplus_i \left(\bigoplus_j H^j_c(Y_P) \otimes_{KL^F} M_{i-j} \right)$$

$$^*\mathcal{R}^G_{L\subset P} : KG^F\text{-grmod} \longrightarrow KL^F\text{-grmod}$$

$$M \longmapsto H^\bullet_c(Y_P)^* \otimes_{KG^F} M$$

If $M \in KL^F$-grmod, we set

$$\chi^{eu}_M = \sum_i (-1)^i \chi_{M_i} \in \mathbb{Z}\operatorname{Irr} L^F$$

We set

$$R^G_{L\subset P} : \operatorname{Class}(L^F) \longrightarrow \operatorname{Class}(G^F)$$

$$\lambda \longmapsto \left(g \longmapsto \frac{1}{|L^F|} \sum_{l \in L^F} \operatorname{Tr}^*_{Y_P}(g,l)\, \lambda(l^{-1}) \right)$$

$$^*R^G_{L\subset P} : \operatorname{Class}(G^F) \longrightarrow \operatorname{Class}(L^F)$$

$$(8.4) \begin{cases} R^G_{L\subset P}\, \chi^{eu}_M = \chi^{eu}_{\mathcal{R}^G_{L\subset P}(M)} \\ ^*R^G_{L\subset P}\, \chi^{eu}_N = \chi^{eu}_{^*\mathcal{R}^G_{L\subset P}(N)} \end{cases}$$

Computing $\mathcal{R}^G_{L\subset P}$ is very hard

Computing $R^G_{L\subset P}$ is easier

Proposition 8.5 (adjunction).

$$\langle \gamma, R^G_{L\subset P}\lambda\rangle_{G^F} = \langle {}^*R^G_{L\subset P}\gamma, \lambda\rangle_{L^F}$$

Examples 8.6. (1) If P is F-stable, then $Y_P = G^F/V^F$ (example 8.2(1))

so $H^\bullet_c(Y_P) = K[G^F/V^F]$. In particular, Deligne-Lusztig induction coincides with Harish-Chandra induction.

(2) If $G = SL_2(\mathbb{F})$:

$$\mathcal{R}^G_{T'\subset B'} = \mathcal{R}'_1 \oplus \mathcal{R}'_2$$

(see Part I)

and $R^G_{T'\subset B'} = -R'$. ∎

Transitivity 8.7. $(L', P') \subset L$

$$\begin{cases} \mathcal{R}^G_{L\subset P}\circ \mathcal{R}^L_{L'\subset P'} = \mathcal{R}^G_{L'\subset P'V} \\ {}^*\mathcal{R}^L_{L'\subset P'}\circ {}^*\mathcal{R}^G_{L\subset P} = {}^*\mathcal{R}^G_{L'\subset P'V} \end{cases}$$

Corollary 8.8.

$$\begin{cases} R^G_{L\subset P}\circ R^L_{L'\subset P'} = R^G_{L'\subset P'V} \\ {}^*R^L_{L'\subset P'}\circ {}^*R^G_{L'\subset P'V} = {}^*R^G_{L'\subset P'V} \end{cases}$$

Proof of 8.7. By the Künneth formula 3.7(c), it is sufficient to prove that

$$\left(Y^G_P \times Y^L_{P'}\right)/L^F \simeq Y^G_{P'V} \qquad (?)$$

$$Y^G_P \times Y^L_{P'} \xrightarrow{\varphi} Y^G_{P'V}$$

$$(gV, \ell V') \longmapsto g\ell VV' = g\ell V'V$$

$$(g\ell)^{-1}F(g\ell) = \ell^{-1}g^{-1}F(g)F(\ell)$$

$$= \underbrace{\ell^{-1}F(\ell)}_{\in V'\cdot F(V')}\, F(\ell)^{-1}\underbrace{(g^{-1}F(g))}_{V\cdot F(V)}F(\ell)$$

Proof of 8.7. By the Künneth formula 3.7(c), it is sufficient to prove that

$$\left(Y_p^G \times Y_{p'}^L\right)/_{L^F} \simeq Y_{p'Y}^G \qquad (?)$$

$$Y_p^G \times Y_{p'}^L \xrightarrow{\varphi} Y_{p'Y}^G$$

$$(gV, \ell V') \longmapsto g\ell\, YV' = g\ell\, V'V$$

$$(g\ell)^{-1}F(g\ell) = \ell^{-1} g^{-1} F(g)\, F(\ell)$$

$$= \underbrace{\ell^{-1}F(\ell)}_{\in V'\cdot F(V')} \; {}^{F(\ell)^{-1}}\big(\underbrace{g^{-1}F(g)}_{V\cdot F(V)}\big)$$

$$\in V'\cdot F(V')\cdot {}^{F(\ell)^{-1}}\big(V\cdot F(V)\big)$$

$$= V'\cdot F(V')\cdot V\cdot F(V)$$

$$= V'\cdot V\cdot F(V')\cdot F(V)$$

$$= (V'V)\cdot F(V'V)$$

Exercise. Prove that φ is a quotient map by L^F. ∎

Mackey formulas for Harish-Chandra induction and restriction (7.6 and 7.7)

$$\begin{cases} {}^*\mathcal{R}^G_{L\subset P} \circ \mathcal{R}^G_{L'\subset P'}(M) \simeq \bigoplus_{g\in L^F\backslash S(L,L')^F/L'^F} \mathcal{R}^L_{L\cap {}^gL' \subset L\cap {}^gP'} \circ {}^*\mathcal{R}^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}({}^gM) \\ \langle R^G_{L\subset P}\lambda, R^G_{L'\subset P'}\lambda'\rangle_{G^F} = \sum_{g\in L^F\backslash S(L,L')^F/L'^F} \langle {}^*R^L_{L\cap {}^gL'\subset L\cap {}^gP'}\lambda, {}^*R^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}{}^g\lambda'\rangle_{L^F\cap {}^gL'^F} \end{cases}$$

Mackey formula for Deligne-Lusztig induction and restriction (CONJECTURE)

$$(\mathcal{M}^G_{L,P,L',P'}) \qquad \langle R^G_{L\subset P}\lambda, R^G_{L'\subset P'}\lambda'\rangle_{G^F} = \sum_{g\in L^F\backslash S(L,L')^F/L'^F} \langle {}^*R^L_{L\cap {}^gL'\subset L\cap {}^gP'}\lambda, {}^*R^{{}^gL'}_{L\cap {}^gL'\subset P\cap {}^gL'}{}^g\lambda'\rangle_{L^F\cap {}^gL'^F}$$

Theorem 8.9. Mackey formula holds in the following cases:

(1) If P and P' are F-stable (Deligne, 1979)

(2) If L or L' is a maximal torus (Deligne-Lusztig, 1976)

(3) If $q > 2$

(4) If G is classical

} (B.-Michel, 2011)

(classical means type A, B, C, D; i.e GL_n, SL_n, PGL_n, Sp_{2n}, SO_n, $Spin_n$, ...)

Corollary 8.10. Assume $q > 2$, or L is a maximal torus, or G is classical. Then $R^G_{L\subset P}$ does not depend on P.

Remark 8.11. In general, $\mathcal{R}^G_{L\subset P}$ does depend on P.

For instance, there is an F-stable maximal torus $T \subset GL_3(\overline{\mathbb{F}})$ such that $T^F \simeq \mathbb{F}_{q^2}^\times \times \mathbb{F}_q^\times$ admitting two Borel subgroups B and B' such that $T \subset B, B'$ and $\dim Y_B = 1$ and $\dim Y_{B'} = 3$.

$$\{\text{partitions of } 3\} \xleftrightarrow{\sim} HC(G^F, T_0, 1_{T_0^F})$$

$$\ydiagram{3} \longmapsto 1_{G^F}$$

$$\ydiagram{1,1,1} \longmapsto St$$

$$\ydiagram{2,1} \longmapsto \rho$$

Then $$\mathcal{R}^G_{T\subset B}(K_{T^F}[0]) = (St \oplus \rho)[-1] \oplus (1_{G^F} \oplus \rho)[-2]$$

and $$\mathcal{R}^G_{T\subset B'}(K_{T^F}[0]) = St[-3] \oplus 1_{G^F}[-6]$$

$$R^G_{T\subset B}(1_{T^F}) = 1_{G^F} - St = R^G_{T\subset B'}(1_{T^F}). \blacksquare$$