$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[rr, bend right, "r"'] & Y \arrow[r, "g"] & X \end{tikzcd}$$

From MMP, we know that $f^*(K_Y+\Lambda)+G \equiv (K_{\tilde{X}}+\tilde{\Delta})$ with $G>0$, supp G is exactly the f-exceptional locus.

$$K_Y+\Lambda \equiv g^*K_X+F$$

Since (Y, Λ) is a minimal model of $(\tilde{X}, \tilde{\Delta})$, $K_Y+\Lambda$ is nef over X.

Thus by negativity lemma, $F \leq 0$

Then $K_{\tilde{X}}+\tilde{\Delta} \equiv f^*(K_Y+\Lambda)+G_{\geq 0}$

$$\equiv f^*(g^*K_X+F_{\leq 0})+G_{\geq 0}$$

$$\equiv r^*K_X+f^*F_{\leq 0}+G_{\geq 0}$$

$K_{\tilde{X}}+\tilde{\Delta} \equiv K_{\tilde{X}}+\tilde{D}+\varepsilon\tilde{D}$

$\equiv r^*K_X+\underbrace{\varepsilon\tilde{D}'_{\geq 0}}_{\varepsilon\sum_{i=1}^{k} C_i}$

Hence $F_{\leq 0}$ is zero

Then $G_{\geq 0}=\varepsilon\tilde{D}'$

Thus f-exceptional locus is exactly

$$\bigcup_{i \in J} C_i$$

Exercise: X proj klt surface $\quad \tilde{r}: \tilde{X} \to X$ minimal res

Write $K_{\tilde{X}} = \tilde{r}^* K_X + \sum_{i=1}^{s} a_i C_i$

Let $\tilde{\Delta} = -\sum_{i=1}^{s} a_i C_i + \varepsilon \sum_{i=1}^{s} C_i$, $0 < \varepsilon \ll 1$

We consider the pair $(\tilde{X}, \tilde{\Delta})$

It is a klt pair

Show that Any minimal model of $(\tilde{X}, \tilde{\Delta})$ is a minimal model of X.

This can be easili extends to the case when (X, Δ) is a klt pair.

It means that if $(\tilde{X}, \tilde{\Delta}) \to (Y, \Gamma)$ is a minimal model, then

$\Gamma = 0$ and it factorizes throug

$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & Y \\ & X \arrow[ur] & \end{tikzcd}$$

, and $X \to Y$ is a minimal model

X minimal model.

VI 2 3-fold case.

There aise some portial classification on 3-fold terminal sing (KM98)

V3 Higher dimension.

Thm: X a variety with klt singularit

Then X has quotient sing in codim 2

That is, $\exists X^\circ \subseteq X$ open with $\operatorname{codim} X - X^\circ \geq 2$

s.t. $\forall x \in X^\circ$, $\exists$ analytic open $x \in U \subseteq X^\circ$

s.t. $x \in U$ is analytically locally iso to $0 \in \mathbb{C}^n/G$, G is a finite group in $GL_n(\mathbb{C})$.

Proof (Greb-Kebekus-Kovac-Peternell Diff forms on sing spaces, ~2012)

Note: There much more klt sing than quotient sing.

Bertini Result: Assume (X, Δ) klt.

$|H|$ a base point-free linear system

Then for a general member $H \in |H|$,

the pair $(H, \Delta|_H)$ is klt

(we can replace klt by lc, terminal, can.)

Proof: Prove it by fixing one res $\tilde{X} \to X$.

Behavior under finite morphisms.

Let $g: X' \to X$ be a finite morphism

between normal varieties.

Let Δ', Δ be divisors such that

$$K_{X'} + \Delta' = g^*(K_X + \Delta)$$

Then ① $K_{X'} + \Delta'$ is $\mathbb{Q}$-Cartier iff so is $K_X + \Delta$

② (X', Δ') is klt (lc)

iff so is (X, Δ)

Proof: (KM98, 5.2)

VI.4 Inversion of adjunction

Recall the adjunction formula:

$H \subseteq X$ is a prime divisor.

(X, Δ) a pair s.t. $K_X + \Delta + H$ is $\mathbb{Q}$-Cartier.

$H \notin$ Supp of Δ.

If H is Cartier, H is normal, then

$$K_H + \Delta_{|H} = (K_X + H + \Delta)_{|H}$$

In general, we need to correct a little.

If H is just normal

Then there is a divisor $\textcircled{H}$ on H

s.t. $K_H + \Delta_{|H} + \textcircled{H} = (K_X + H + \Delta)_{|H}$

$\textcircled{H}$ is called the difference

(Asterisque, flips and Abundance for alg 3-folds)
Section 16

With this Adjunction formula, we simply write $K_H + \Delta_H = (K_X + H + \Delta)_{|H}$

$$\Delta_H = \Delta_{|H} + \textcircled{H}$$

Thm (Inversion of Adjunction):

Assume that $H \subseteq X$ normal varieties, H hypersurface

Δ a boundary on X s.t. $K_X + H + \Delta$ is $\mathbb{Q}$-Cartier.

and $H \not\subseteq \operatorname{supp} \Delta$.

Then ① $(X, H+\Delta)$ is plt $\Longleftrightarrow$ (H, Δ_H) is klt.

② $(X, H+\Delta)$ is lc $\Longleftrightarrow$ (H, Δ_H) is lc

Proof: (KM98, 5.4)

Thm: (X, Δ) is a dlt pair. Then

① (X, Δ) is plt $\Updownarrow$

② $\lfloor \Delta \rfloor$ is normal

(It means its irr components are disjoint and all normal)

(KM98, section 5.4)

VII BCHM Paper.

Existence of log minimal model for varieties of log general type

Birkar, Cascini, Hacon, M^{c}Kernan

~ 2010

Main Thm: klt flips exists.

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & U & \end{tikzcd}$$

ϕ is a flip if it is a log terminal model of X over U (f-relative)

They first prove that pl-flip exist, this is a flip for plt pair,

$$\begin{tikzcd} (X,\Delta) \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & (X^+,\Delta^+) \arrow[dl] \\ & U & \end{tikzcd}$$

ϕ is a pl-flip if it is a flip, and (X,Δ) is plt and $\lfloor\Delta\rfloor = S$ is a prime divisor. and $-S$ is f-ample

For this pl-flip, we can use induction on dimension.

The pl-flip exist $\iff$ $R(X,\Delta)$ is finitely generated

$$\| \qquad \bigoplus f_* \mathcal{O}_X(mr(K_X+\Delta))$$

$$= \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mr(K_X+\Delta))$$

To attack this finite generation, see [HM, extension thm and existence of flip]

Write $\Delta = S + B$, $S = \lfloor \Delta \rfloor$

Then there is a natural map

$$R(X,\Delta) \xrightarrow{\text{res}} R(S, B_S)$$

induced by $\mathcal{O}_X(r(K_X+S+B)) \longrightarrow \mathcal{O}_S(r(K_S+B_S))$

Let R_{res} be the image of res

Then shokurov $R(X,\Delta)$ is f.g. $\Longleftrightarrow$ R_{res} is finitely gen.

In HM, they construct birational (Y, Λ) of (X,Δ) with $\lfloor \Lambda \rfloor = T$ s.t. $\Lambda = T + D$

$R(X,\Delta) \cong R(Y,\Lambda)$ and $R(Y,\Lambda) \to R(T, D_T)$ is surj.

Since (T, D_T) is klt, if we know f.g. for klt pair in dim $n-1$, then we are done.

Finit generation of $R(X,\Delta)$ for (X,Δ) klt.

Assume existence of MMP, then (X,Δ) has a minimal model (X',Δ')

$R(X,\Delta) = R(X',\Delta')$, what we gain is $K_{X'}+\Delta'$ is nef.

Abundance conj: If $K_{X'}+\Delta'$ is nef, (X',Δ') klt then $\exists$ ~~proj~~ morphism $g: X' \to Y$ s.t. $K_{X'}+\Delta' = g^*A$, A ample on Y.

In this cas $R(K_{X'}+\Delta') = R(A)$, which f.g.

Abundance is known if $K_X+\Delta$ is big or Δ is big.

In BCHM, Δ is often big

Remark: In the following setting, bigness is Free.

When $f: X \to U$ is birational

Then every Cartier divisor in X is

f-relatively big.

Thm A_n: (Existence of pl-flip) in dim n.

Thm B_n: (Special finiteness in dim n)

"There are finitely many weak log can models under some condition"

Thm C_n: (Existence of log terminal model)

Existence of model

Thm D_n: (Non-Vanishing)

Thm E_n: (Finiteness of weak log can model)

Thm F_n: (Finite generation of $R(K_X+\Delta)$, (X,Δ) klt)

Sketch of Proof:

$F_{n-1} \Rightarrow A_n$, $E_{n-1} \Rightarrow B_n$

$A_n + B_n \Rightarrow C_n$; $B_n + C_n + D_{n-1} \Rightarrow D_n$

$C_n + D_n \Rightarrow E_n$; F_n

Type of model.

We have seen of $(X,\Delta) \dashrightarrow (X',\Delta')$

is a segence of div contracton or flips

then,
$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ (X,\Delta) \arrow[rr, dashed, "\phi"] & & (X',\Delta') \end{tikzcd}$$
$f^*(K_X+\Delta) = g^*(K_{X'}+\Delta') + F$

where $F \geq 0$, and $\mathrm{supp}(F)$ contains all divisors in (X,Δ) contracted during the segence.

Def: A map ϕ. $(X,\Delta) \dashrightarrow (X',\Delta')$ is $(K_X+\Delta)$-negative if $f^*(K_X+\Delta) = g^*(K_{X'}+\Delta') + F$

and F contains all divisor in X contracted by ϕ

Def: A log-terminal model $(X,\Delta) \overset{\phi}{\dashrightarrow} (X',\Delta')$ is a bir map s.t.

ϕ is $(K_X+\Delta)$-negative

and $K_{X'}+\Delta'$ is nef.

Finiteness problem:

Assume we have certain sequence of flips

$(X,\Delta) \dashrightarrow (X_1,\Delta_1) \dashrightarrow \cdots \dashrightarrow (X_k,\Delta_k)$

They are all different.

every $(X,\Delta) \dashrightarrow (X_i,\Delta_i)$ is $(K_X+\Delta)$-negative.

In BCHM, if this sequence is a MMP with scaling of some divisor C.

Then each $(X_i, \Delta_i + \lambda_i C_i)$ is a weak log can model for $(X, \Delta+\lambda_i C)$

, $\lambda_i \geq \lambda_{i+1} \geq \cdots\cdots$

Finiteness of model: Roughly speaking, (X,Δ) klt, there are finitely many bir model $\varphi_j : X \dashrightarrow Z_j$ such that If $\psi : X \dashrightarrow Y$ is an wlc model for some $(X, \Delta+D)$, $D \in \mathcal{L}$, then

$\exists Z_j$ s.t. $Y \simeq Z_j$

Here $\mathcal{L}$ is finite dim polygone in $\mathrm{Weil}_{\mathbb{R}}(X)$

Relation from n-1 to n dim

$(X, \overset{\Delta}{\overbrace{S+A+B}})$ a plt pair, A ample, $B \geq 0$

S irreducible.

$\phi : (X, \Delta) \dashrightarrow (Y, \Gamma)$ is a wlc model.

$(S, \Theta) = (K_X+S+A+B)|_S$

$\phi_* S = T$; $\tau : S \dashrightarrow T$

$K_T + \Psi = (K_Y+\Gamma)|_T$

Note that $\tau_* \Theta \neq \Psi$

Thm: If (S, Θ) is terminal, then

$\exists\ A|_S \leq \Xi \leq \Theta$ s.t.

(T, Ψ) is a wlc for

(S, Ξ)