Lecture No 12 April 15, 2022 (Fri)

- Wiener measure is a probability measure P on $(W_0, \mathcal{B}(W_0))$ (recall $\mathcal{B}(W_0) = \mathcal{B}_K(W)$) such that

$$P(C_{t_1,\ldots,t_n;A}) = \int_A \prod_{i=1}^{n} p(t_i - t_{i-1}, y_{i-1}, y_i) dy_1 dy_2 \cdots dy_n,$$

  for any cylinder set

$$C_{t_1,\ldots,t_n;A} := \{w \in W_0 \,;\, (w(t_1), \ldots, w(t_n)) \in A\},$$

  with $0 \leq t_1 < \cdots < t_n,\ A \in \mathcal{B}(\mathbb{R}^n)$.
- It is a distribution of the Brownian motion on the path space W_0.
- Wiener measure is unique, if it exists.

11.2 Construction of a Brownian motion

[Theorem 11.5] Brownian motion exists on a certain probability space. □

- In Lebesgue's theory of measures and integrals, we start with the construction of Lebesgue measure on the interval $[0,1]$, $\mathbb{R}$, $\mathbb{R}^n$, which is fundamental for further discussions.
- In our setting, Wiener measure is a measure on an infinite-dimensional space W_0. This theorem plays most fundamental role in stochastic analysis.
- Brownian motion has several aspects such as Markov process, Gaussian process, martingale and others.
- Due to which property we notice, we can give several different proofs to Theorem 11.5 ($\rightarrow$ next page).

(1) Proof noting Markov property (we will give details later) : This method is applicable to the construction of general Markov processes.
(2) Construction as Fourier series expansion of a formal derivative $\dot{B}$ of the Brownian motion: Wiener, Paley-Wiener, Itô-Nisio. Lévy-Ciesielski took a system of Haar functions (a kind of wavelets) as a CONS of $L^2([0,1])$. By the independence of $\dot{B}$, the uniform convergence of a formal Fourier series (in a.s.-sense) is rather easy under this choice.
(3) Construction by reducing to the existence theorem of Gaussian system ($X = \{X_\lambda\}_\lambda$ is called Gaussian if its any (finite) linear combination is Gaussian) associated with a given covariance structure $v_{\lambda\mu}$: if $(v_{\lambda\mu})$ is symmetric and (its any finite subset is) non-negative definite, then X exists.
(4) Donsker's invariance principle: derivation as a space-time scaling limit of a random walk.

We state the outline of the proof of Theorem 11.5 by the method (1) noting the Markov property, which is applicable to the construction of more general Markov processes.

Step 1: Kolmogorov's extension theorem

$\mathbb{R}^{[0,\infty)}$: Product space i.e. the set of all maps: $[0,\infty) \to \mathbb{R}$.
(This is a path space in a wide sense without continuity.)
$\mathcal{B}(\mathbb{R}^{[0,\infty)})$: Borel field determined by the product topology.
For $\omega \in \mathbb{R}^{[0,\infty)}$, we denote $\omega = (\omega(t))_{t\geq 0}$ as before.

Assume

P: a probability measure on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$

is given.

Set $\mathcal{T} := \{\underline{t} = (t_1, \dots, t_n); 0 \le t_1 < \cdots < t_n < \infty, n \in \mathbb{N}\}$.
For $\underline{t} \in \mathcal{T}$, define the finite-dimensional distribution of P by

$$\begin{aligned} Q_{\underline{t}}(A) :=& P\big(\{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \dots, \omega(t_n)) \in A\}\big) \qquad (\star) \\ \equiv& P\big(C_{t_1,\dots,t_n;A}\big), \qquad A \in \mathcal{B}(\mathbb{R}^n). \end{aligned}$$

Then, obviously $\{Q_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ is consistent i.e.

If $\underline{s}, \underline{t} \in \mathcal{T}$ satisfies $\underline{s} = \underline{t} \setminus \{t_i\}$, then $Q_{\underline{s}}(A) = Q_{\underline{t}}(A \times \mathbb{R})$ holds $^\forall A \in \mathcal{B}(\mathbb{R}^{n-1})$, where $A \times \mathbb{R} := \{(x_1, \dots, x_n) \in \mathbb{R}^n; (x_1, \dots, \underset{\hat{i}}{\dots}, x_n) \in A\}$ (i.e., x_i is excluded)

- Kolmogorov's extension theorem claims that the converse is also true, that is, if $\{Q_{\underline{t}}\}$ satisfies the consistency condition (minimal condition), then one can construct P, which is σ-additive and satisfies the condition $(\star)$.

[Theorem 11.6] Let a family of probability measures $\{Q_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ be given and satisfy the consistency condition. Then, a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, which satisfies the condition $(\star)$, exists uniquely. □

[Outline of proof] • Let $\mathcal{C}$ be a family of all cylinder sets of $\mathbb{R}^{[0,\infty)}$. Note that $\mathcal{C}$ is a π–system and $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^{[0,\infty)})$ holds. (The uniqueness of P follows by π-λ theorem.)

• Define Q on $(\mathbb{R}^{[0,\infty)}, \mathcal{C})$ by $Q(C) := Q_{\underline{t}}(A)$ for $C \in \mathcal{C}$ expressed by $\underline{t}$ and A. Consistency shows that Q is well-defined.

• To extend Q on $(\mathbb{R}^{[0,\infty)}, \mathcal{C})$ to $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, we apply Carathéodory-Hopf extension theorem (i.e., If Q is σ-additive on $\mathcal{C}$, then it can be extended to $\sigma(\mathcal{C})$ being σ-additive).

• All what we have to show is "Q is σ-additive on $\mathcal{C}$", but we leave it to other textbooks. □

- Recalling the formula obtained from the condition (3) of Brownian motion, set
$$Q_{\underline{t}}(A) := \int_A \prod_{i=1}^n p(t_i - t_{i-1}, y_{i-1}, y_i) dy_1 dy_2 \cdots dy_n$$
for $\underline{t} \in \mathcal{T}, A \in \mathcal{B}(\mathbb{R}^n)$, where we set $t_0 = 0, y_0 = 0$ and, in case $t_1 = 0$, we regard $p(0, 0, y)dy = \delta_0(dy)$.
- The consistency of $\{Q_{\underline{t}}\}_{\underline{t} \in \mathcal{T}}$ follows from Chapman-Kolmogorov equation for $p(t, x, y)$
$$p(t+s, x, z) = \int_{\mathbb{R}} p(t, x, y) p(s, y, z) dy, \quad t, s > 0,\ x, z \in \mathbb{R}.$$
- Therefore, by Theorem 11.6 (Kolmogorov's extension theorem), one can construct P on $\mathcal{B}(\mathbb{R}^{[0,\infty)})$.
- If we define $B_t(\omega) := \omega(t), \omega \in \mathcal{B}(\mathbb{R}^{[0,\infty)})$, then its all finite-dimensional distributions coincide with those of Brownian motion.
- However, B_t is not continuous in t. Moreover, we even have $C([0, \infty)) \notin \mathcal{B}(\mathbb{R}^{[0,\infty)})$.

Step 2: Kolmogorov's regularization theorem

- We need to modify B_t to make it continuous in t without changing its all finite-dimensional distributions.

[Theorem 11.7] If a stochastic process $(X_t)_{t\in[0,T]}$ satisfies: ${}^{\exists}C, \alpha, \beta > 0$ s.t.

$$E[|X_t - X_s|^{\alpha}] \leq C|t-s|^{1+\beta}, \quad {}^{\forall}s, t \in [0, T].$$

Then, X_t has a continuous modification $(\tilde{X}_t)_{t\in[0,T]}$, i.e.

· For ${}^{\forall}t \in [0, T]$, $P(X_t = \tilde{X}_t) = 1$.

· $\tilde{X}_t$ is continuous in $t \in [0, T]$ (a.s.) Indeed, Hölder-continuous with exponent γ for ${}^{\forall}\gamma \in (0, \frac{\beta}{\alpha})$. □

[Proof] See Le Gall [2] or Karatzas and Shreve [3]. □

[Completion of the proof of Theorem 11.5]

- By Theorem 11.7, we can show that B_t constructed in Step 1 has a continuous modification and this concludes the proof of Theorem 11.5.
- Indeed, by the Gaussian property (or by property ① in Section 11.3 below), we have

$$E[(B_t - B_s)^{2n}] = (2n-1)!!(t-s)^n, n \in \mathbb{N}.$$

  Therefore, we can take $\alpha = 2n, \beta = n-1$ in Theorem 11.7 and see that B_t has a Hölder continuous modification on $[0, T]$ for ${}^\forall\gamma < \frac{n-1}{2n}$. (By making n large, one can show Hölder continuity for ${}^\forall\gamma < \frac{1}{2}$.) Since $T > 0$ is arbitrary, we obtain the conclusion. □

- [Advantage of Kolmogorov's method] As we already pointed, the method due to Kolmogorov's extension theorem+regularization theorem is applicable to the construction of general continuous Markov process (diffusion).
- Note that the fundamental solution $p(t, x, y)$ of linear parabolic PDE of second order is non-negative and satisfies Chapman-Kolmogorov equation (semigroup property). Moment estimate as above type is also available.
- [Itô's method to construct diffusion] Kolmogorov's method is useful to construct processes in distribution's sense. As we will see later, Itô introduced stochastic differential equation. His method pathwisely modifies Brownian motion to construct diffusion processes.

• Kolmogorov's regularization theorem holds in case that the time parameter runs over a higher dimensional space.

[Theorem] If a random field $(X(x))_{x\in[0,1]^d}$ satisfies the condition: ${}^{\exists}C,\ \alpha, \beta_i > 0$ such that $b := \sum_{i=1}^{d} \frac{1}{\beta_i} < 1$ and

$$E[|X(x) - X(y)|^{\alpha}] \le C \sum_{i=1}^{d} |x_i - y_i|^{\beta_i}, \quad {}^{\forall}x, y \in [0,1]^d,$$

where $x = (x_i)_{i=1}^d$, $y = (y_i)_{i=1}^d$. Then, a modification $\tilde{X}(x)$ of $X(x)$ exists, and for ${}^{\forall}\gamma_i \in (0, \frac{\beta_i(\beta_0 - d)}{\beta_0 \alpha})$, $\beta_0 := \frac{d}{b}$,

$$|\tilde{X}(x) - \tilde{X}(y)| \le {}^{\exists}K(\omega) \sum_{i=1}^{d} |x_i - y_i|^{\gamma_i} \quad \text{a.s.}$$

holds. □

[Proof] See, e.g., Kunita "Stochastic flow and SDEs". □

[Definition 11.8] Let $\{B^i = (B^i_t)_{t\geq 0}\}_{i=1}^d$ be d independent Brownian motions (i.e. $\sigma\{B^i_t; t \geq 0\}$, $1 \leq i \leq d$ is independent). $\mathbb{R}^d$-valued stochastic process $B_t = \{B^i_t\}_{i=1}^d$ is called *d-dimensional Brownian motion*.

11.3 Property of Brownian motion

Let $(B_t)_{t\geq 0}$ be a 1-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Set $\mathcal{F}^B_t := \sigma\big\{\sigma\{B_s; s \leq t\} \cup \mathcal{N}\big\}$, where $\mathcal{N} := \{N \in \mathcal{F}; P(N) = 0\}$.

① Moments and Covariance:

$$E[B_t^{2n}] = (2n-1)!!t^n, \quad E[B_t^{2n-1}] = 0, \quad n \in \mathbb{N},$$

where $(2n-1)!! := (2n-1)(2n-3)\cdots 3 \cdot 1$.
Moreover, $E[B_t B_s] = t \wedge s$. □

∵ First two identities follow by the computation under the Gaussian distribution. To show the last, note $B_t - B_s \perp\!\!\!\perp B_s$ for $0 \leq s < t$ (by independent increments and $B_0 = 0$). Then,

$$E[B_t B_s] = E[(B_t - B_s)B_s] + E[B_s^2] = s. \quad □$$

② Generalization of independent increments property:

$$0 \leq s < t \Longrightarrow B_t - B_s \perp\!\!\!\perp \mathcal{F}_s^B \qquad \square$$

(∵) For $0 = s_0 \leq^{\forall} s_1 < \cdots <^{\forall} s_n \leq s < t$, ${}^{\forall}f \in C_b(\mathbb{R})$, ${}^{\forall}g \in C_b(\mathbb{R}^n)$, by rewriting both sides in terms of the transition probability, we obtain

$$\begin{aligned} &E[f(B_t - B_s)g(B_{s_1}, B_{s_2}, \ldots, B_{s_n})] \\ &\quad = E[f(B_t - B_s)]E[g(B_{s_1}, B_{s_2}, \ldots, B_{s_n})]. \end{aligned}$$

This implies that $B_t - B_s \perp\!\!\!\perp \{\text{cylinder sets by the time } s\}$ and π-λ theorem shows the conclusion. $\square$

③ $B = (B_t)_{t \geq 0}$ is $(\mathcal{F}_t^B)$-martingale and its quadratic variation is given by $\langle B \rangle_t = t$. (It is known that $(\mathcal{F}_t^B)$ is right continuous; see Karatzas-Shreve [3]) $\square$

(∵) • For the martingale property, it is enough to show $E[B_t, A] = E[B_s, A]$ for $0 \leq s < t$, $A \in \mathcal{F}_s^B$. However, by ②, $B_t - B_s \perp\!\!\!\perp A$, which implies $E[B_t - B_s, A] = E[B_t - B_s] \cdot P(A) = 0$.

• Next, we show $\langle B \rangle_t = t$. For any division $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of $[0, t]$, set

$$B^\Delta := \sum_{i=1}^{n} \left(B_{t_i} - B_{t_{i-1}}\right)^2$$

By denoting $Z_i := (B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})$, we have

$$E[(B^\Delta - t)^2] = E\left[\left\{\sum_{i=1}^{n} Z_i\right\}^2\right] \underset{(\star)}{=} \sum_{i=1}^{n} E[Z_i^2]$$

$$= \sum_{i=1}^{n} \left\{E[(B_{t_i} - B_{t_{i-1}})^4] - 2(t_i - t_{i-1})E[(B_{t_i} - B_{t_{i-1}})^2] + (t_i - t_{i-1})^2\right\}$$

$$= \sum_{i=1}^{n} 2(t_i - t_{i-1})^2 \leq 2t \cdot |\Delta| \underset{|\Delta| \to 0}{\longrightarrow} 0.$$

Here, $(\star)$ is shown by expanding the square and noting that $E[Z_i Z_j] = 0$ holds if $i \neq j$, since $Z_i \perp\!\!\!\perp Z_j$ and $E[Z_i] = 0$. Therefore, we obtain $B^\Delta \to t$ in L^2 so that also in probability. This shows $\langle B \rangle_t = t$ by the Fact we stated before. □