Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

Lecture 11. Diffusion rate of Ehrenfest windtree billiards and Lyapunov exponents of Teichmüller flow

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Lyapunov exponents for quadratic differentials

Lyapunov exponents for strata of quadratic differentials

Consider a stratum $\mathcal{Q}$ of meromorphic quadratic differentials with at most simple poles. As in the case of Abelian differentials, the bundle $H^1_{\mathbb{R}}$ over $\mathcal{Q}$ is endowed with the flat Gauss–Manin connection. Denote the corresponding Lyapunov exponents with respect to the action of the Teichmüller geodesic flow on the flat vector bundle $H^1_{\mathbb{R}}$ by $\lambda_1^+ \geq \cdots \geq \lambda_g^+$.

For every $(S, q) \in \mathcal{Q}$ consider the canonical double cover $p : \hat{S} \to S$ such that $p^*q = (\hat{\omega})^2$, where $\hat{\omega}$ is an Abelian differential on $\hat{S}$. This double cover has ramification points at all zeroes of odd orders of q and at all simple poles, and no other ramification points. By construction the double cover $\hat{S}$ is endowed with a natural involution $\sigma : \hat{S} \to \hat{S}$ interchanging the two sheets of the cover. We can decompose the vector space $H^1(\hat{S}, \mathbb{R})$ into a direct sum of subspaces $H^1_+(\hat{S}, \mathbb{R})$ and $H^1_-(\hat{S}, \mathbb{R})$ which are correspondingly invariant and anti-invariant with respect to the involution $\sigma^* : H^1(\hat{S}, \mathbb{R}) \to H^1(\hat{S}, \mathbb{R})$ on cohomology. We get two flat vector bundles H^1_+ and H^1_- over $\mathcal{Q}$ equivariant with respect to the $\mathrm{PSL}(2, \mathbb{R})$-action. The bundle H^1_+ is canonically isomorphic to $H^1_{\mathbb{R}}$.

Sum of Lyapunov exponents for quadratic differentials

Let $g_{\mathit{eff}} := \hat{g} - g$. Denote the top g_{eff} Lyapunov exponents of the subbundle H^1_- by $\lambda_1^- \geq \cdots \geq \lambda^-_{g_{\mathit{eff}}}$.

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014) *Consider a stratum $\mathcal{Q}(d_1, \ldots, d_n)$ in the moduli space of quadratic differentials with at most simple poles, where $d_1 + \cdots + d_n = 4g - 4$. Let $\mathcal{M}$ be any regular $\mathrm{PSL}(2, \mathbb{R})$-invariant suborbifold of $\mathcal{Q}(d_1, \ldots, d_n)$.*
The Lyapunov exponents $\lambda_1^+ \geq \cdots \geq \lambda_g^+$ of the invariant subbundle H^1_+ of the Hodge bundle over $\mathcal{M}$ along the Teichmüller flow satisfy the following relation:

$$\lambda_1^+ + \cdots + \lambda_g^+ \; = \; \frac{1}{24} \sum_{j=1}^{n} \frac{d_j(d_j+4)}{d_j+2} + \frac{\pi^2}{3} \cdot c_{area}(\mathcal{M})$$

where $c_{\text{area}}(\mathcal{M})$ is the Siegel–Veech constant corresponding to the suborbifold $\mathcal{M}$. By convention the sum in the left-hand side of equation is defined to be equal to zero for $g = 0$.

Relation between the two sums

Theorem (A. Eskin, M. Kontsevich, A. Zorich, 2014) *The Lyapunov exponents $\lambda_1^- \geq \cdots \geq \lambda_{g_{eff}}^-$ of the anti-invariant subbundle H_-^1 of the Hodge bundle over $\mathcal{M}$ along the Teichmüller flow satisfy the following relation:*

$$\left(\lambda_1^- + \cdots + \lambda_{g_{eff}}^-\right) - \left(\lambda_1^+ + \cdots + \lambda_g^+\right) = \frac{1}{4} \cdot \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2}.$$

The leading Lyapunov exponent λ_1^- is equal to one.

Still Conjecture (A. Eskin, M. Kontsevich, A. Zorich, 2014). *For any $\mathrm{SL}(2,\mathbb{R})$-invariant suborbifold $\mathcal{M}$ in any stratum of Abelian differentials the corresponding Siegel–Veech constant $\pi^2 \cdot c_{area}(\mathcal{M})$ is a rational number.*

The conjecture is open even for most of the strata of quadratic differentials (especially for those, which have zeroes of even degrees).

"You, my forest and water! One swerves, while the other shall spout
Through your body like draught; one declares, while the first has a doubt."
J. Brodsky

Ты, мой лес и вода, кто объедет, а кто, как сквозняк,
проникает в тебя, кто глаголет, а кто обиняк...
И. Бродский

Periodic billiards

Central limit theorem

Let $X_1, ..., X_n$ be a sequence of independent and identically distributed random variables (heads or tails, measurements in uncorrelated experiments, etc). Assume that the variance σ^2 is finite and that the expected value is 0. Let $S_n := X_1 + \cdots + X_n$. Clearly, with probability one one has

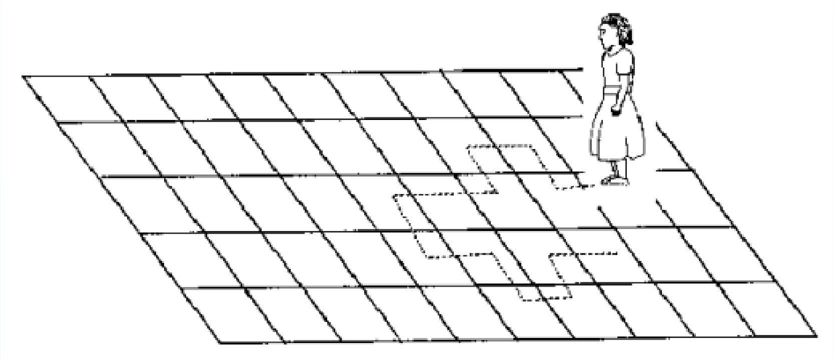
$$\frac{X_1 + \cdots + X_n}{n} = \frac{S_n}{n} \to 0 \quad \text{as } n \to +\infty \, .$$

The Central Limit Theorem describes the expected deviation of the sum S_n from 0. In a sense, it is one of the fundamental laws of Nature:

Cental Limit Theorem. *The distribution of the the sum S_n normalized by the factor $\frac{1}{\sqrt{n}}$ tends to the normal distribution with mean 0 and variance σ^2.*

Random walk and brownian motion in the plane

Random walk. For every step you flip two coins; depending of the combination you go one step forward, one step backward, one step right, or one step left.



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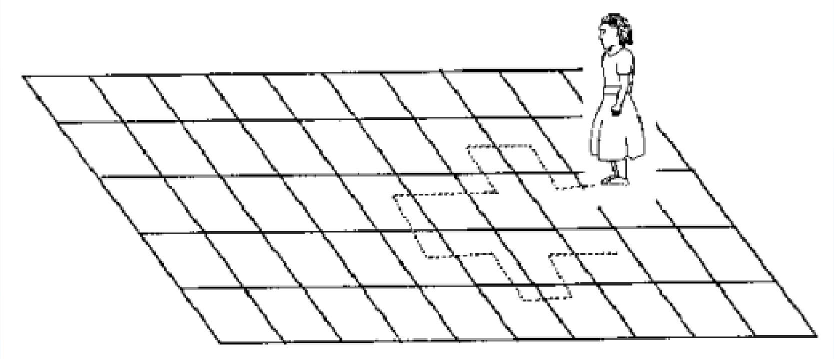
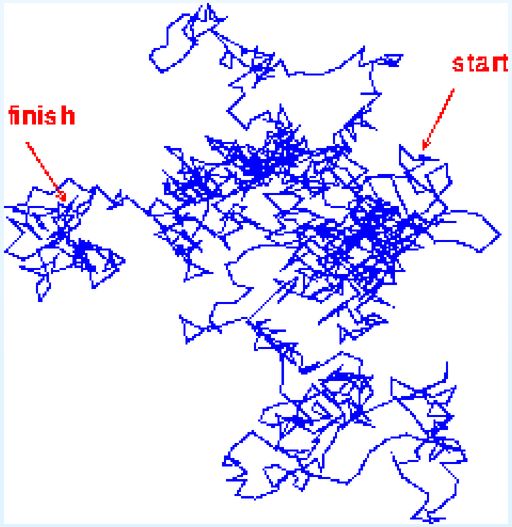
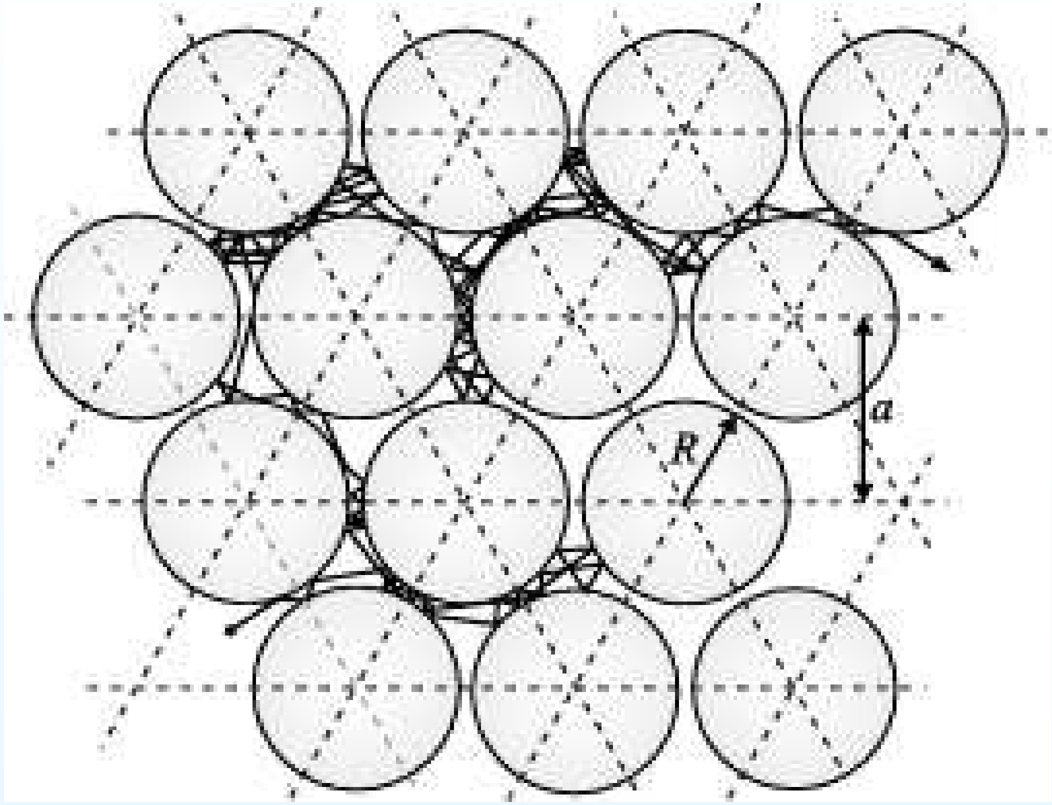


Corollary of the Central Limit Theorem. *The root of the mean square of the translation distance after n steps of a random walk with zero mean is*

$$\sqrt{E|S_n^2|} = \sigma\sqrt{n} = \sigma \cdot n^{\frac{1}{2}} .$$



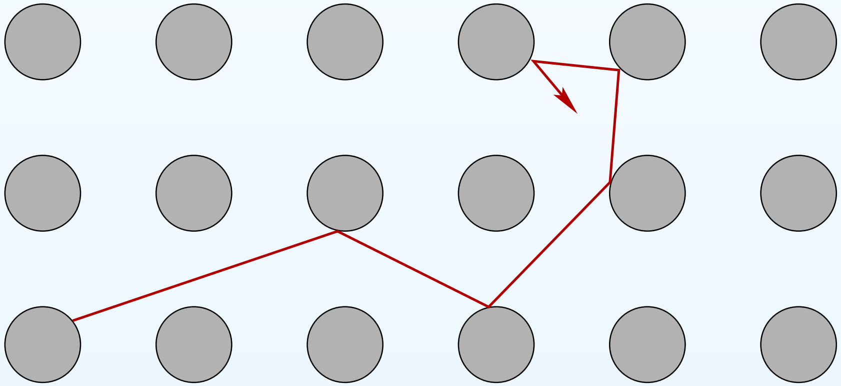
Lorentz gas and Sinai billiard. Finite horizon.



Theorem (Bunimovich, Chernov, Sinai, 1991). *For periodic configuration of convex scatterers on the plane the particle after scaling by* $\sqrt{t}$ *satisfies the Central Limit Theorem if the horizon is finite (that is, if any ray intersects a scatterer).*

Lorentz gas and Sinai billiard. Infinite horizon.

Theorem (Szász, Varjú, 2007; some ideas — Bleher, 1992).
In infinite horizon case, for example, for round scatterers placed at the lattice points, the Central Limit Theorem still holds but the scaling should be by $\sqrt{t \ln t}$*.*

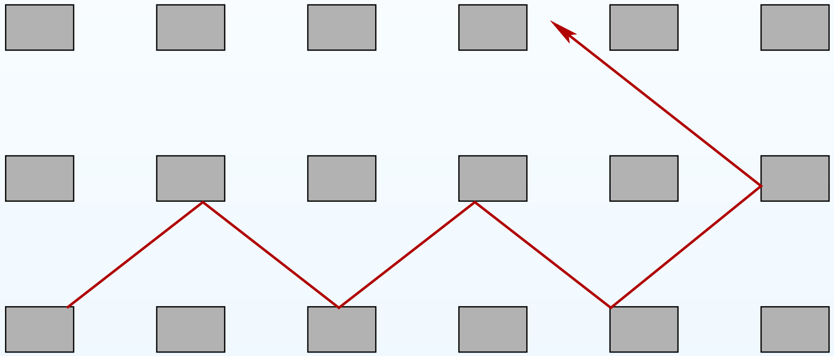


Chernov, Dolgopyat (2009): further interesting results in this direction.

In all cases the *diffusion rate* is again $\frac{1}{2}$ as for the random walk.

Diffusion in a periodic billiard (Ehrenfest "Windtree model")

Consider a billiard on the plane with $\mathbb{Z}^2$-periodic rectangular obstacles.



Theorem (V. Delecroix, P. Hubert, S. Lelièvre, 2014). *For all parameters of the obstacle, for almost all initial directions, and for any starting point, the billiard trajectory spreads in the plane with the speed $\sim t^{2/3}$. That is,*

$$\lim_{t\to+\infty} \log \text{(diameter of trajectory of length } t)/\log t = \frac{2}{3} \neq \frac{1}{2}.$$

The diffusion rate $\frac{2}{3}$ is given by the Lyapunov exponent of certain renormalizing dynamical system associated to the initial one.

Remark. Changing the height and the width of the obstacle we get quite different billiards, but this does not change the diffusion rate!

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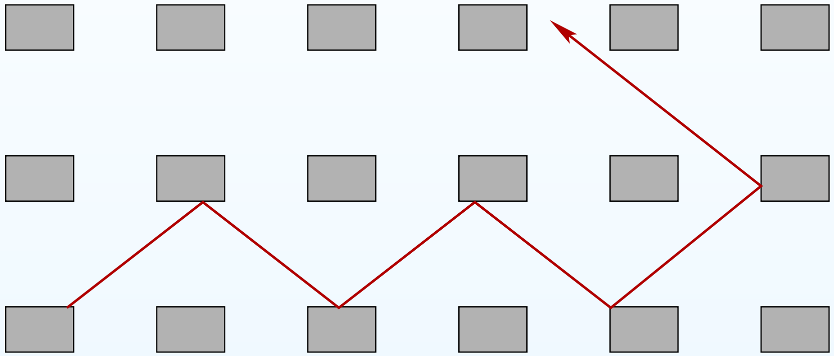

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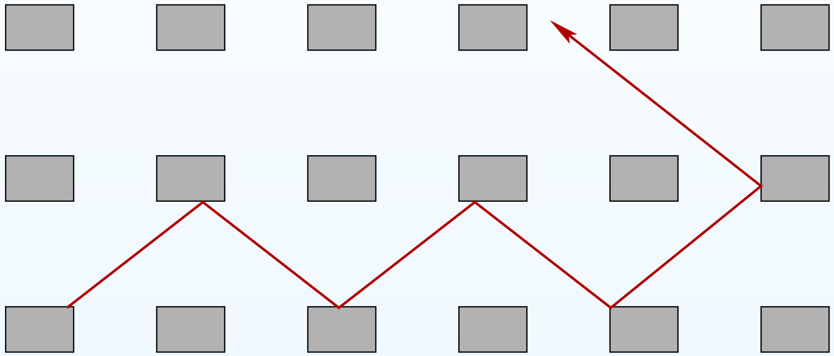

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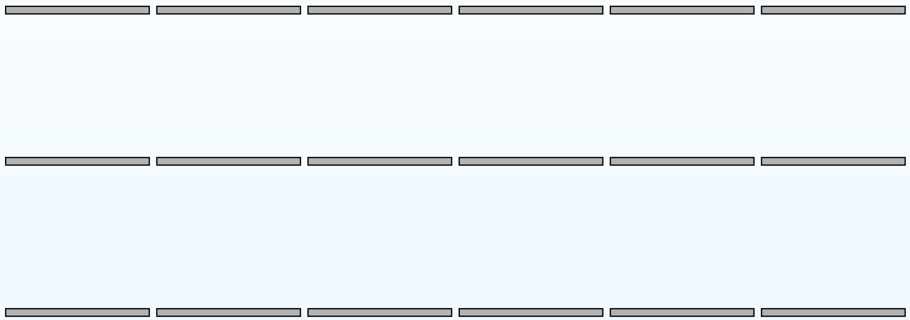

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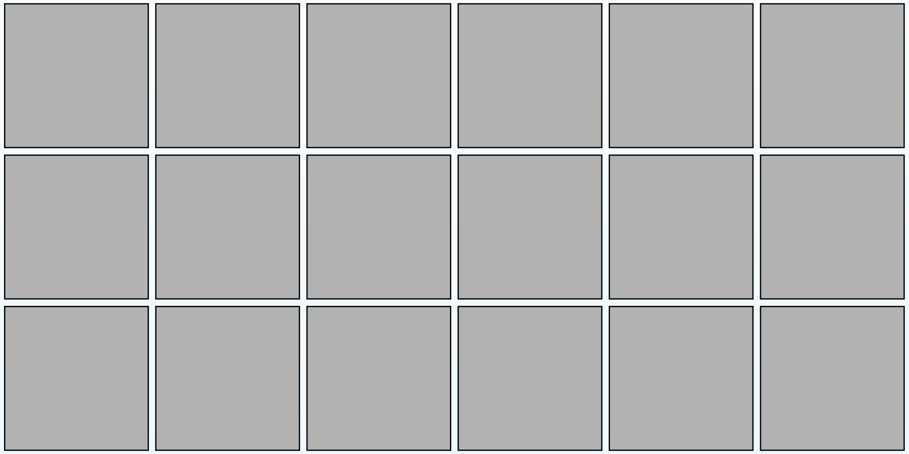

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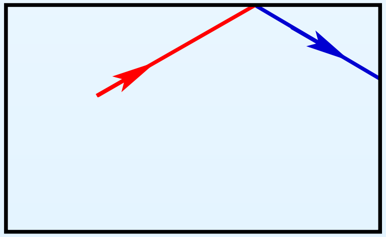
From billiards to surface foliations

Consider a rectangular billiard. Instead of reflecting the trajectory we can reflect the billiard table. The trajectory unfolds to a straight line. Folding back the copies of the billiard table we project this line to the original trajectory. At any moment the ball moves in one of four directions defining four types of copies of the billiard table. Copies of the same type are related by a parallel translation.



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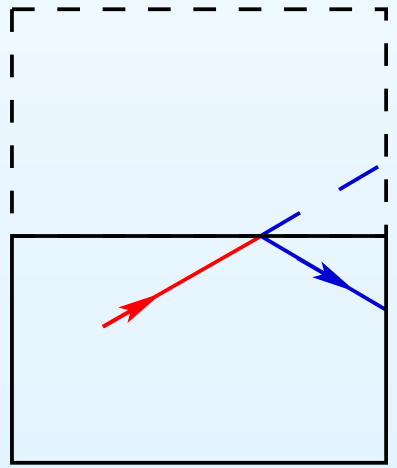

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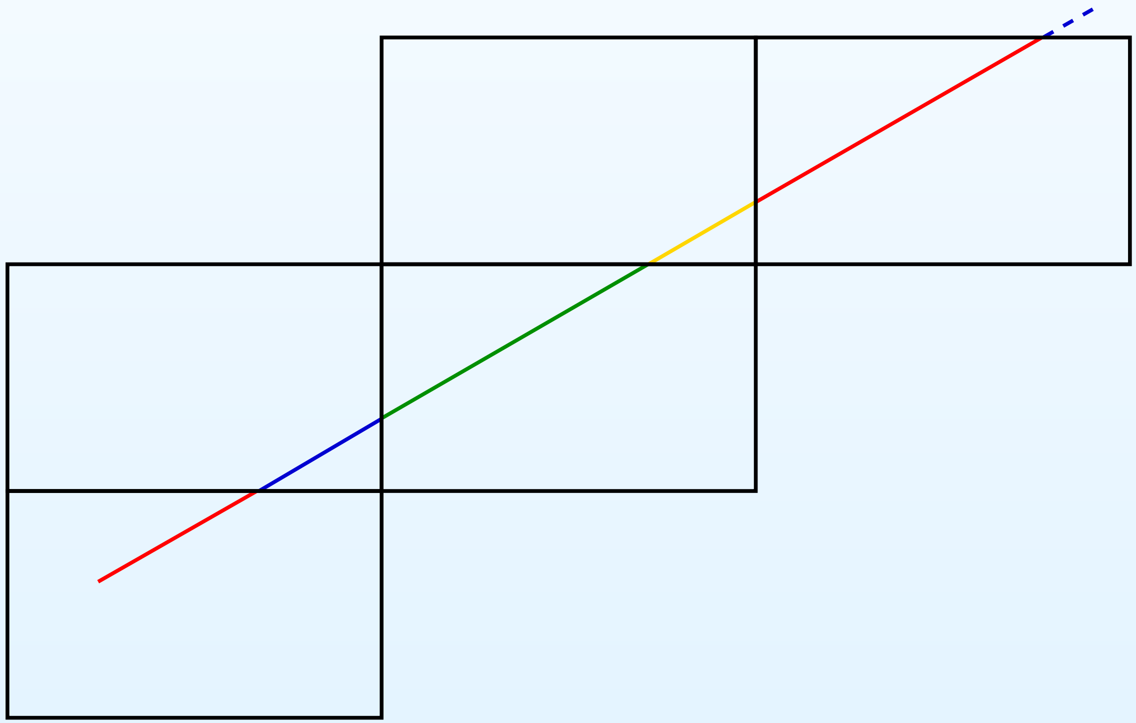

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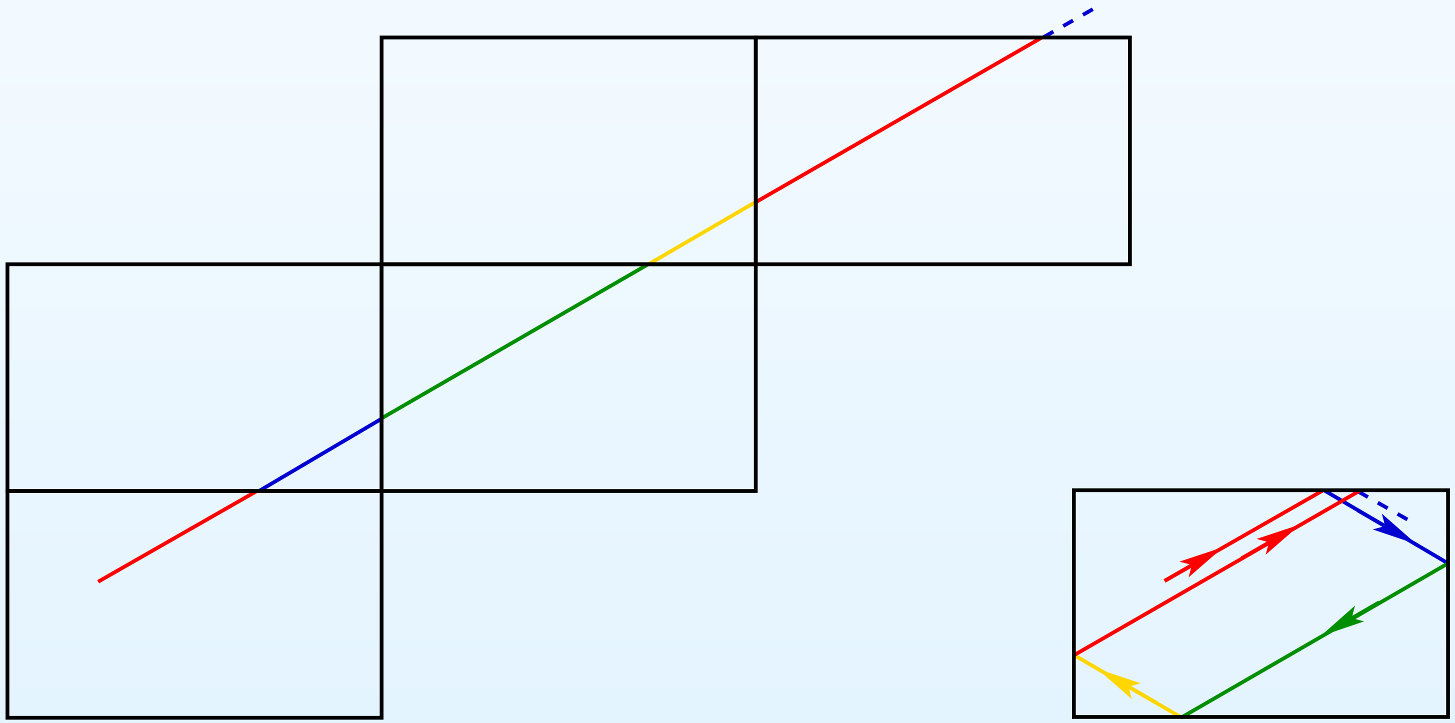

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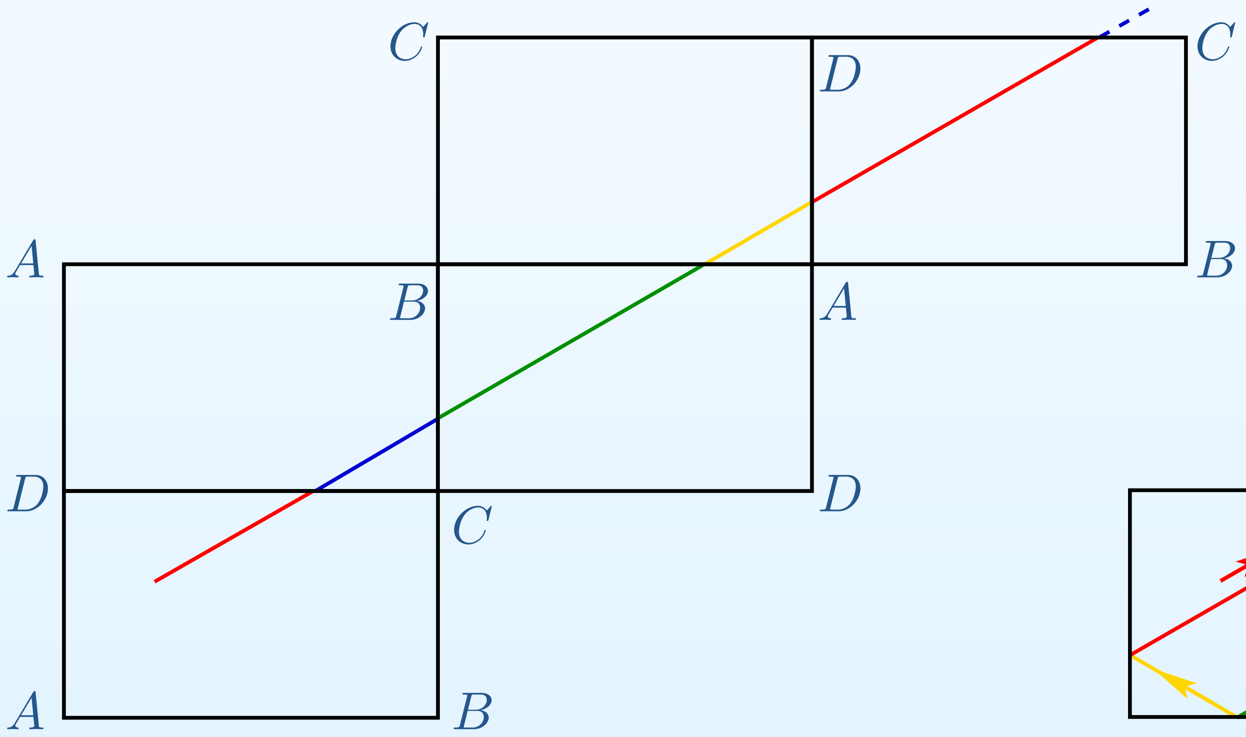

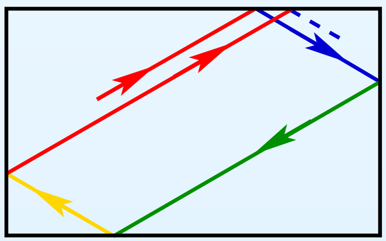

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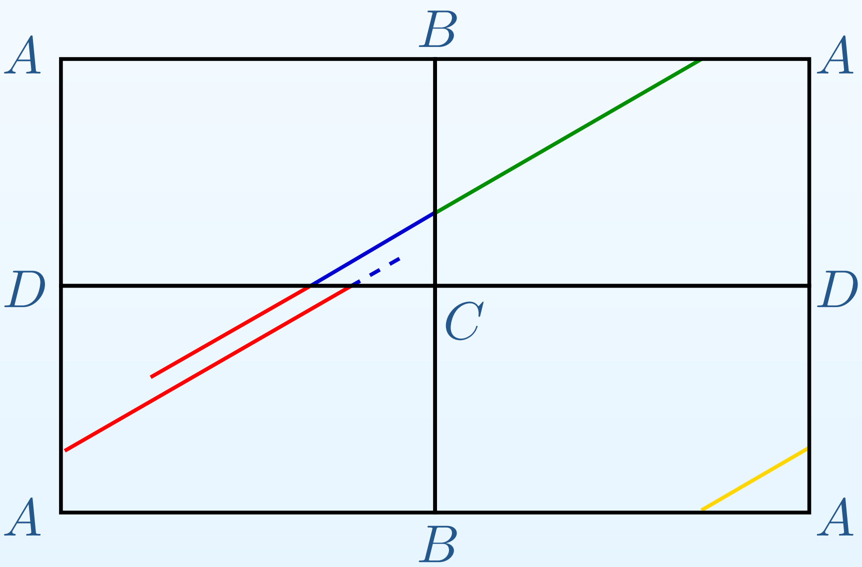

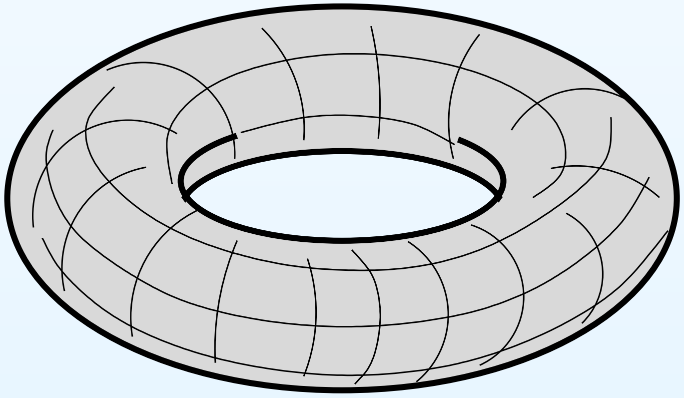

Identifying the equivalent patterns by a parallel translation we obtain a torus; the billiard trajectory unfolds to a "straight line" on the corresponding torus.

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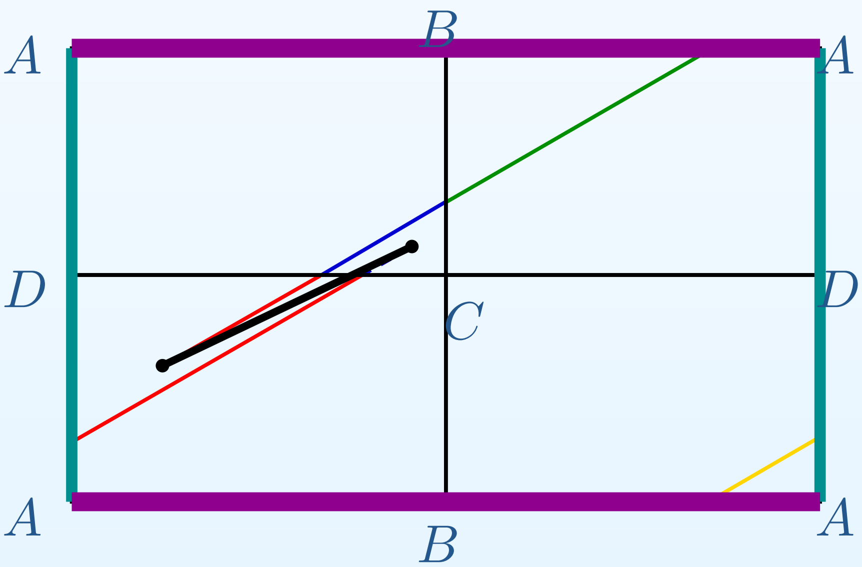
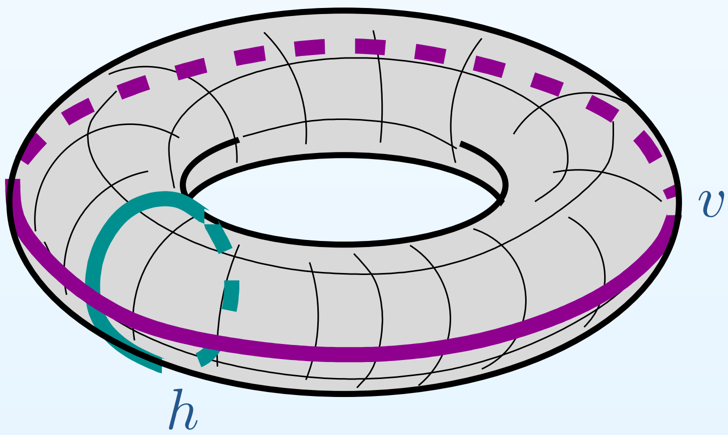


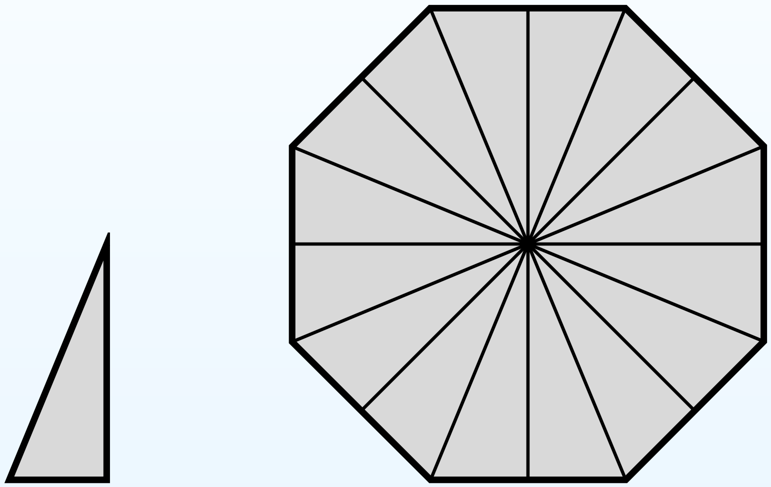


Join the endpoints of a piece of trajectory after time t to obtain a closed loop $c(t)$ on the torus. Vertical and horizontal displacement after time t of the unfolded billiard trajectory is described by the intersection numbers $c(t) \circ h$ and $c(t) \circ v$ with a parallel h and a meridian v of the torus.

From billiards to surface foliations

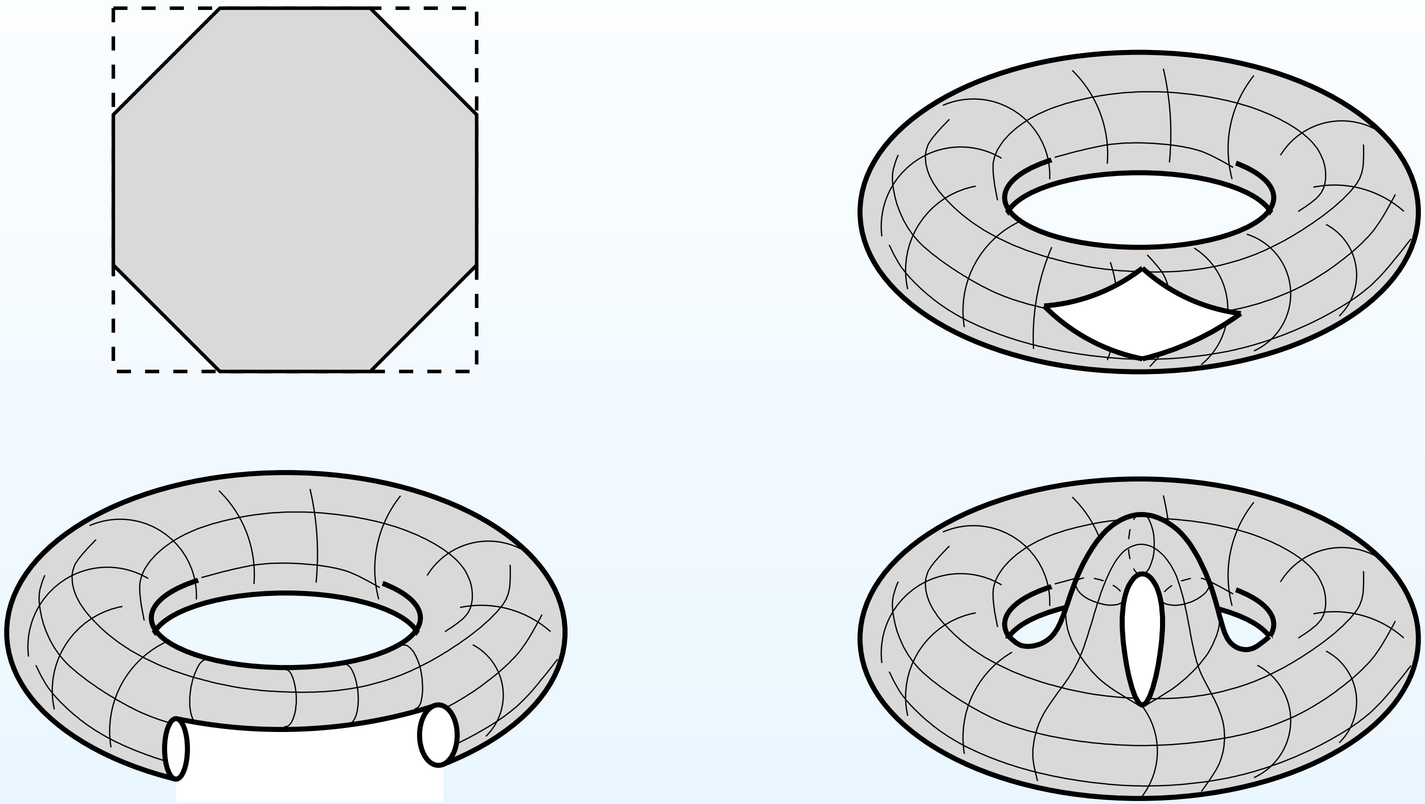
Unfolding rational billiards

We can apply an unfolding procedure procedure (which was already applied to a rectangular billiard) to *any* rational billiard.



Consider, for example, a triangle with angles $\pi/8, 3\pi/8, \pi/2$. It is easy to check that a generic trajectory in such billiard table has 16 directions (instead of 4 for a rectangle). Using 16 copies of the triangle we unfold the billiard into a regular octagon with opposite sides identified by parallel translations.

Very flat surface of genus 2



Identifying the opposite sides of a regular octagon we get a flat surface of genus two. All the vertices of the octagon are identified into a single conical singularity. We always consider such a flat surface endowed with a distinguished (say, vertical) direction. By construction, the holonomy of the flat metric is trivial. Thus, the vertical direction at a single point globally defines vertical and horizontal foliations.

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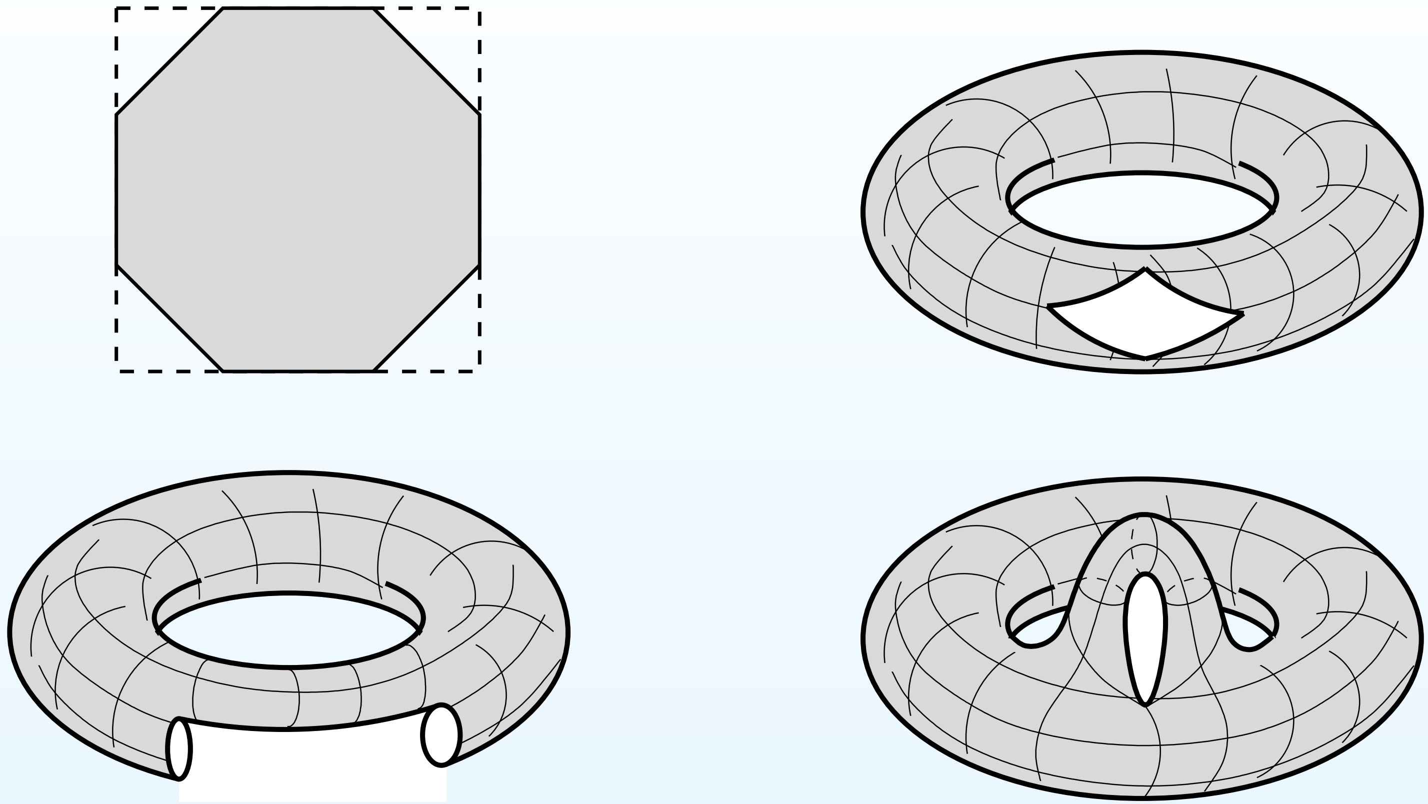

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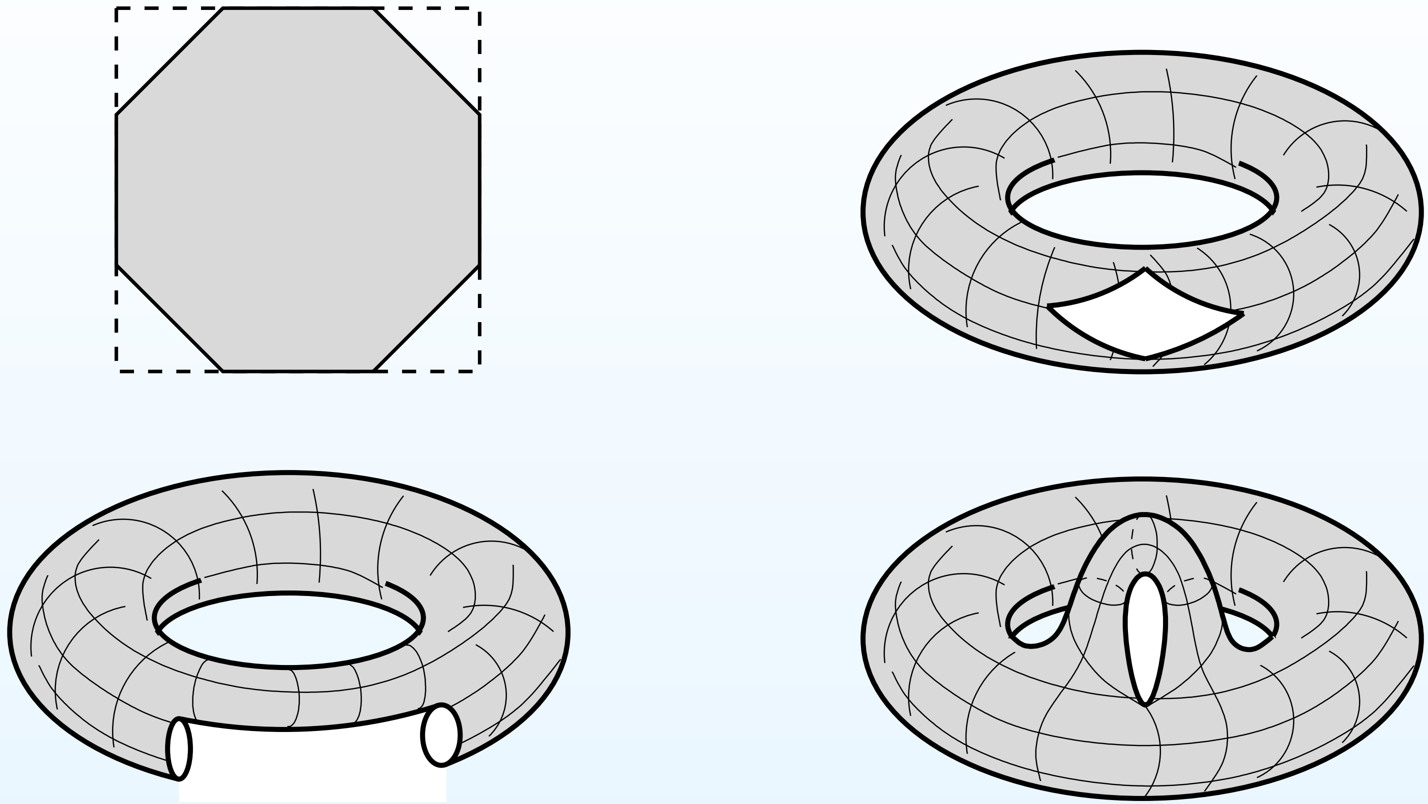

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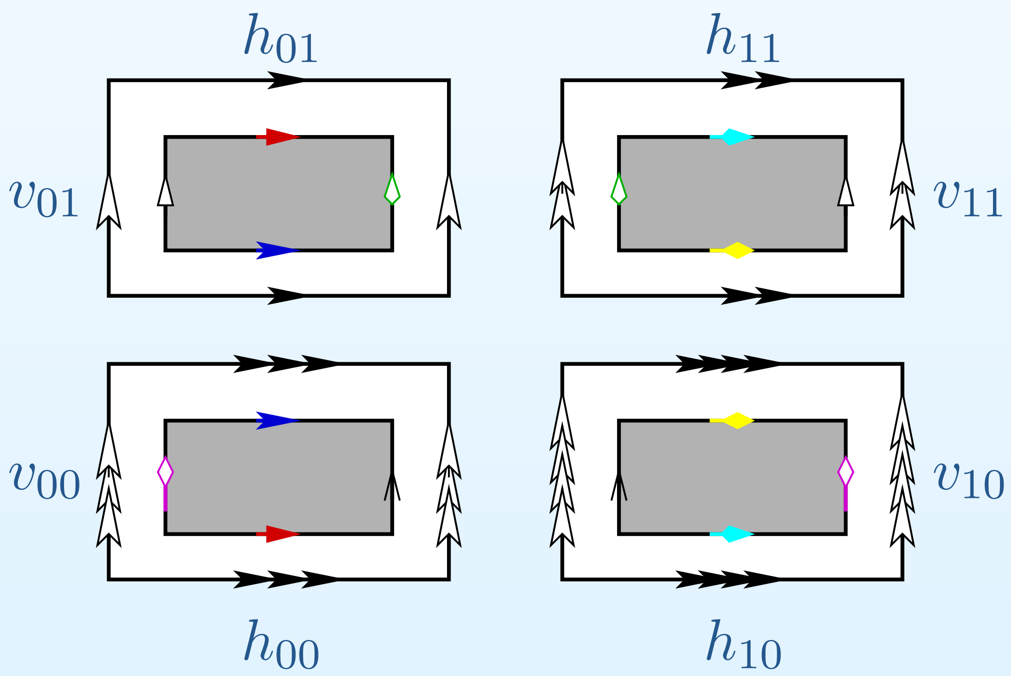
Windtree flat surface

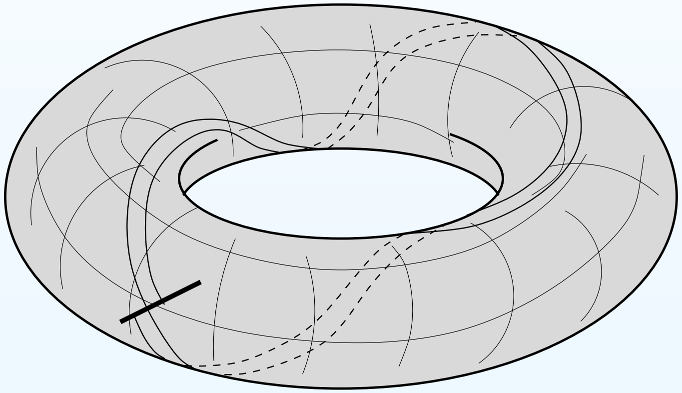
Similarly, taking four copies of our $\mathbb{Z}^2$-periodic windtree billiard we can unfold it to a foliation on a $\mathbb{Z}^2$-periodic surface. Taking a quotient over $\mathbb{Z}^2$ we get a compact surface endowed with a measured foliation. Vertical and horizontal displacement (and thus, the diffusion) of the billiard trajectories is described by the intersection numbers $c(t) \circ v$ and $c(t) \circ h$ of the cycle $c(t)$ obtained by closing up a piece of leaf with the cycles $h = h_{00} + h_{10} - h_{01} - h_{11}$ and $v = v_{00} - v_{10} + v_{01} - v_{11}$. The intersection number of the asymptotic cycle with h and v is null. We need to study deviations of the ergodic sums from the mean value.

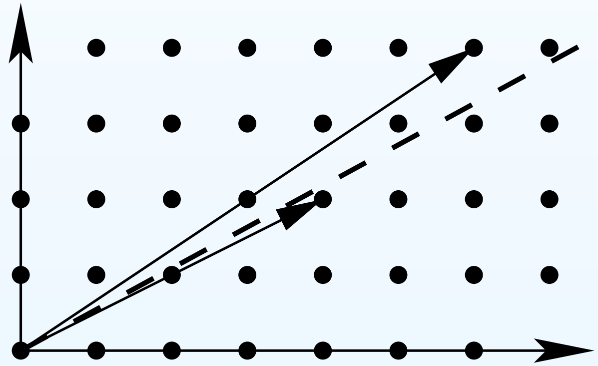


Asymptotic flag of an orientable measured foliation

Asymptotic cycle for a torus

Consider a leaf of a measured foliation on a surface. Choose a short transversal segment X. Each time when the leaf crosses X we join the crossing point with the point x_0 along X obtaining a closed loop. Consecutive return points $x_1, x_2, \ldots$ define a sequence of cycles $c_1, c_2, \ldots$.





The *asymptotic cycle* is defined as $\lim_{n\to\infty} \frac{c_n}{n} = c \in H_1(\mathbb{T}^2; \mathbb{R})$.

Theorem (S. Kerckhoff, H. Masur, J. Smillie, 1986.) *For any flat surface directional flow in almost any direction is uniquely ergodic.*

This implies that for almost any direction the asymptotic cycle exists and is the same for all points of the surface.

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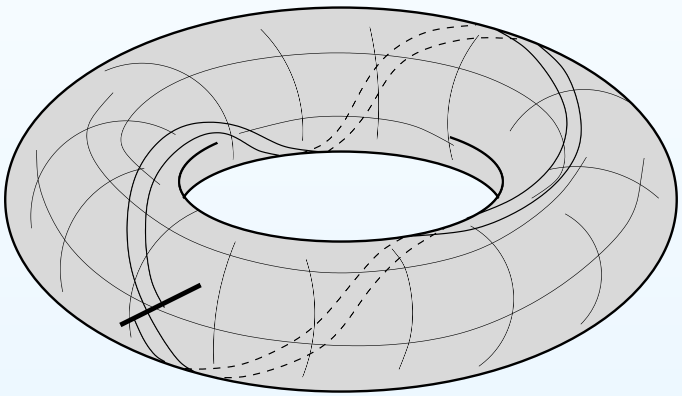

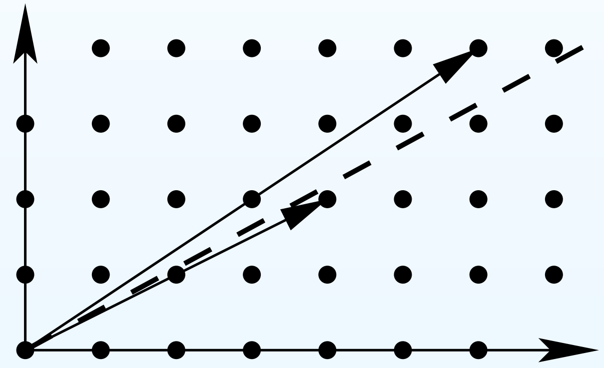

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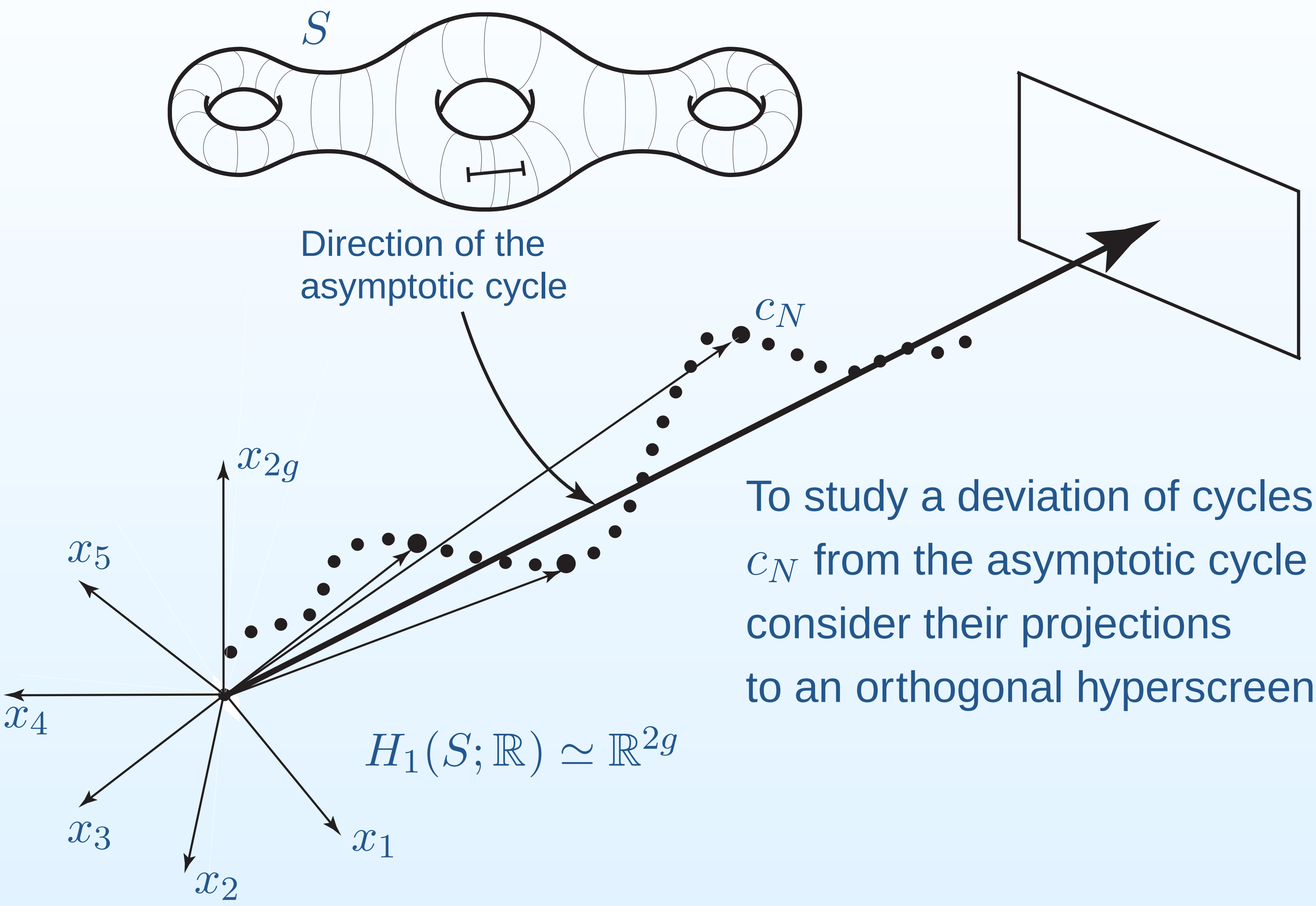
Asymptotic flag: empirical description



To study a deviation of cycles c_N from the asymptotic cycle consider their projections to an orthogonal hyperscreen

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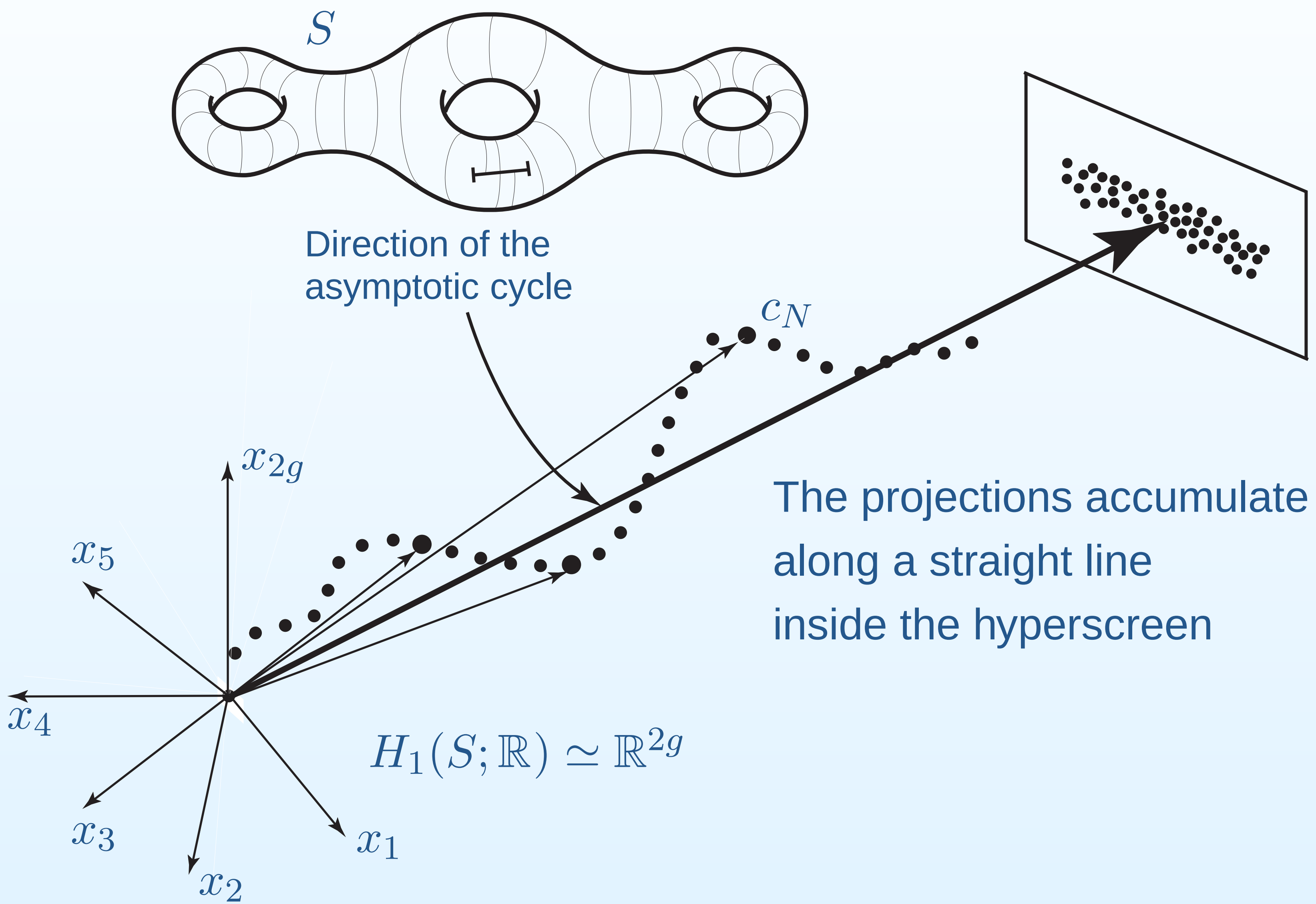
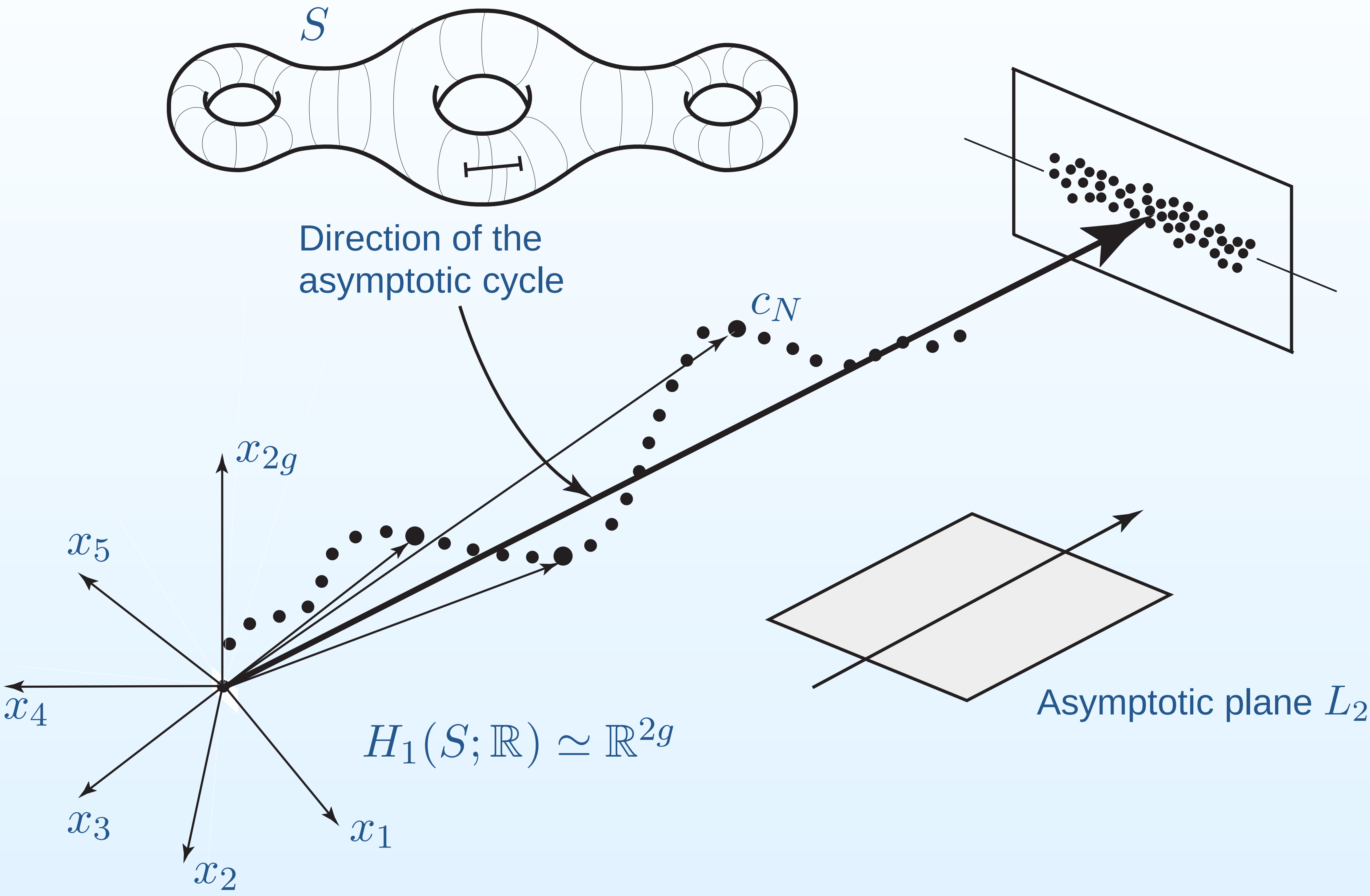


The projections accumulate along a straight line inside the hyperscreen

$H_1(S;\mathbb{R}) \simeq \mathbb{R}^{2g}$

Asymptotic flag: empirical description



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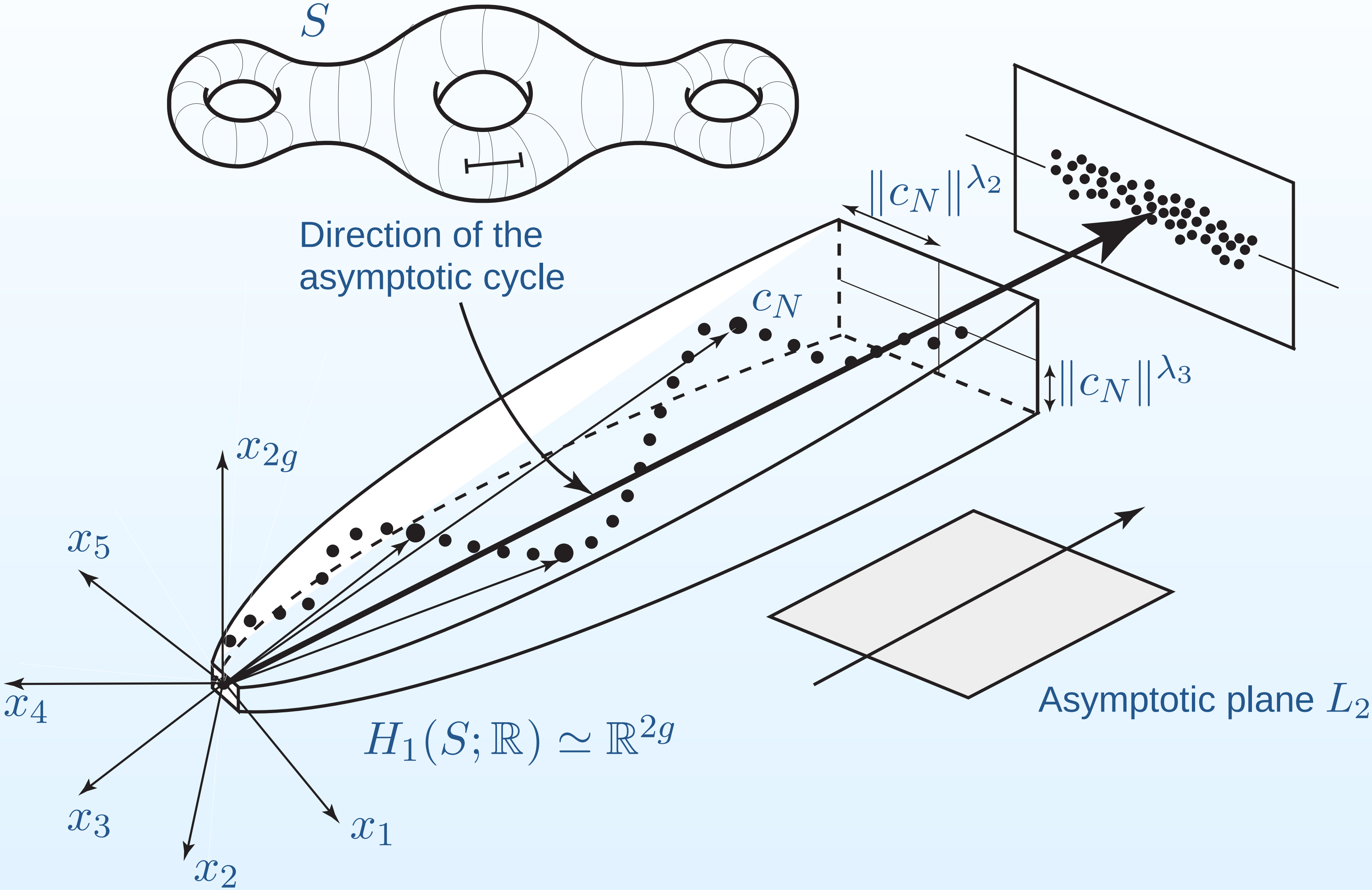

Asymptotic flag: Theorem

Theorem (A. Zorich, 1999) *For almost any surface S in any stratum $\mathcal{H}_1(d_1, \dots, d_n)$ there exists a flag of subspaces $L_1 \subset L_2 \subset \cdots \subset L_g \subset H_1(S;\mathbb{R})$ such that for any $j = 1, \dots, g-1$*

$$\limsup_{N\to\infty} \frac{\log \operatorname{dist}(c_N, L_j)}{\log N} = \lambda_{j+1}$$

and

$$\operatorname{dist}(c_N, L_g) \le const,$$

where the constant depends only on S and on the choice of the Euclidean structure in the homology space.

The numbers $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g$ are the top g Lyapunov exponents of the Hodge bundle along the Teichmüller geodesic flow on the corresponding connected component of the stratum $\mathcal{H}(d_1, \dots, d_n)$.

The strict inequalities $\lambda_g > 0$ and $\lambda_2 > \cdots > \lambda_g$, and, as a corollary, strict inclusions of the subspaces of the flag, are difficult theorems proved later by G. Forni (2002) and by A. Avila–M. Viana (2007).

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Solution of the windtree problem

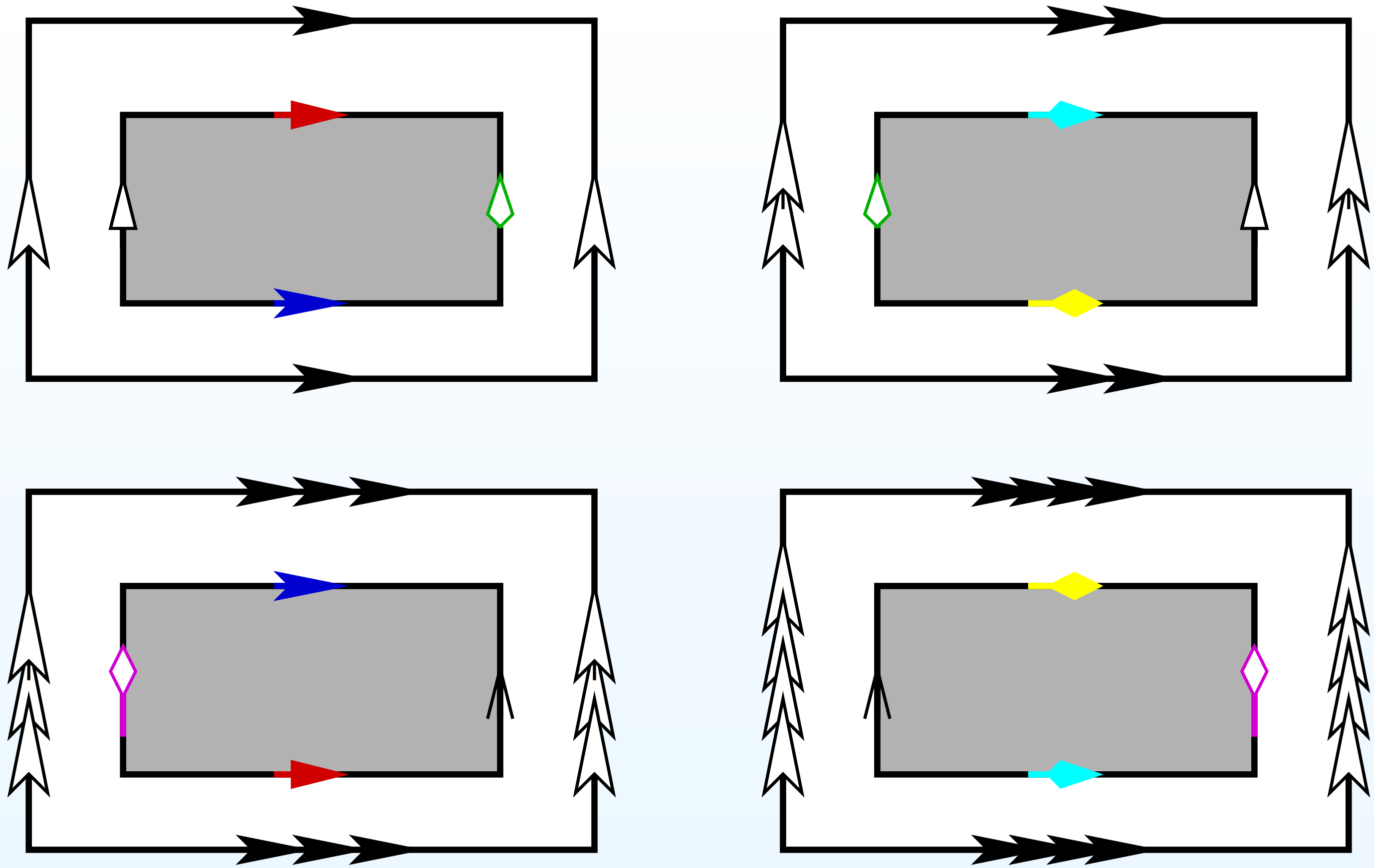
Scheme of solution

Theorem (J. Chaika–A. Eskin, 2014). *For any flat surface* S *almost all directions define a Lyapunov-generic point in the orbit closure* $\overline{\mathrm{SL}(2,\mathbb{R})\cdot S}$.

Scheme of solution of a generalized windtree problem

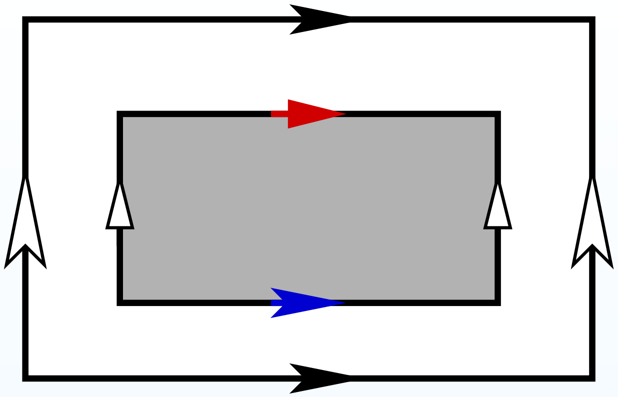
1. Find the family of flat surfaces $\mathcal{B}$ associated to the original family of rational billiards;
2. Find the orbit closure $\mathcal{L} = \overline{\mathrm{SL}(2,\mathbb{R})\cdot \mathcal{B}}$ of $\mathcal{B}$ inside the ambient moduli space (stratum).
3. Compute or estimate the relevant Lyapunov exponent of the bundle $H^1_{\mathbb{R}}$ along the Teichmüller geodesic flow on $\mathcal{L}$.

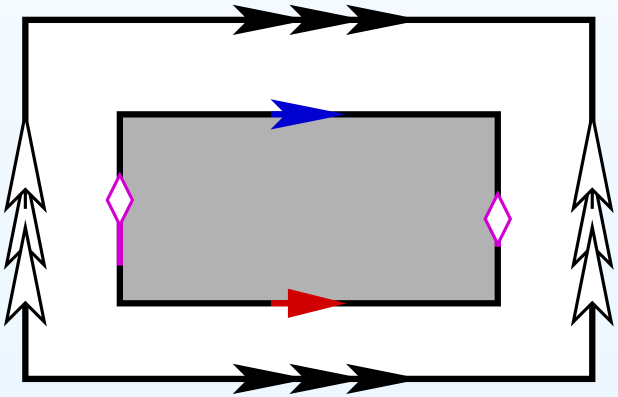
Symmetries



Any wind-tree surface X as in the picture admits the group $(\mathbb{Z}/2\mathbb{Z})^3$ of symmetries preserving the flat structure and nonoriented vertical and horizontal directions. We can choose the isometries τ_h and τ_v interchanging the pairs of flat tori with holes in the same rows (columns) by parallel translations and the isometry ι acting on each of the four tori with holes as a central symmetry centered in the center of the hole as three generators of $(\mathbb{Z}/2\mathbb{Z})^3$.

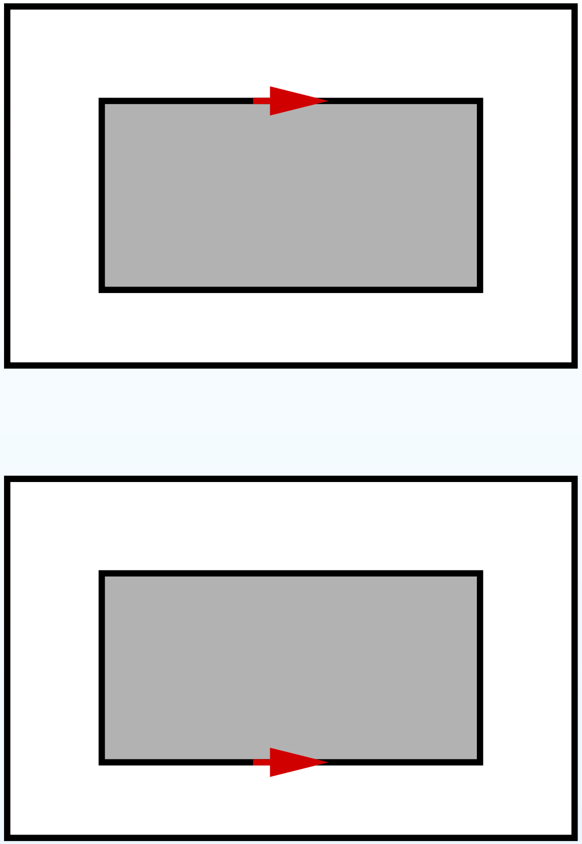
Symmetries





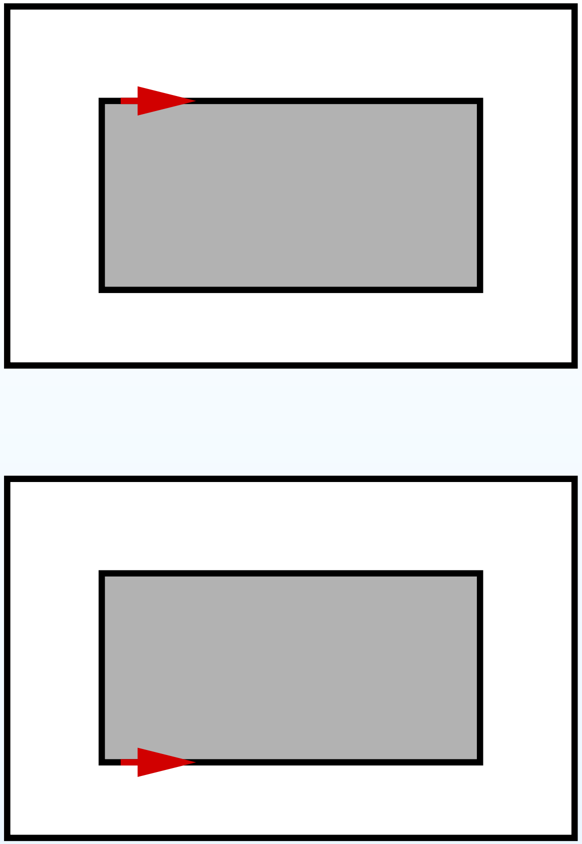
After the quotient over the first isometry τ_h, interchanging the pairs of flat tori with holes in the same rows, we get the surface with a fundamental domain as in the picture. The only detail to which we have to pay attention, is the new identifications of sides of the new fundamental domain. In our case we have to trace the new identifications of the vertical sides.

Symmetries



Consider the quotient of the latter surface over the composition $\iota \circ \tau_v$. Here τ_v interchanges the two flat tori with holes by parallel translations while ι acts on each of the two tori with holes as the central symmetry. Let us trace the new identifications of sides of this new fundamental domain paying particular attention to the horizontal sides.

Symmetries



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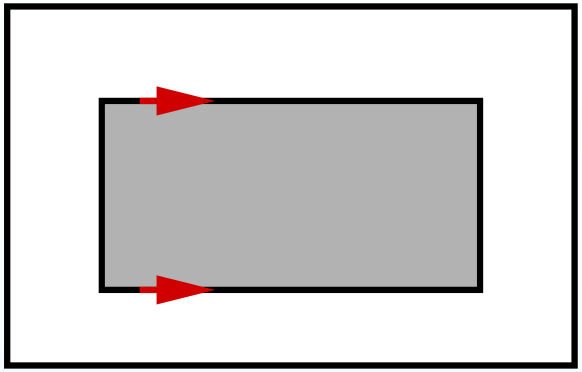

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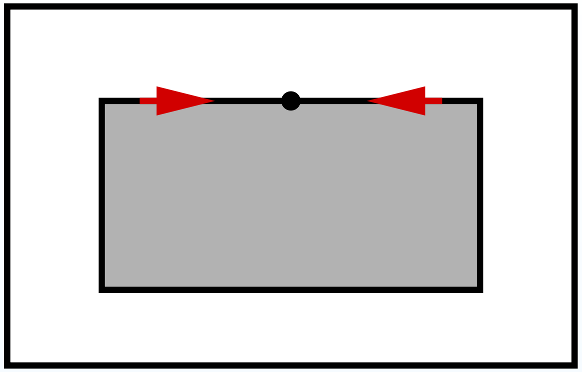

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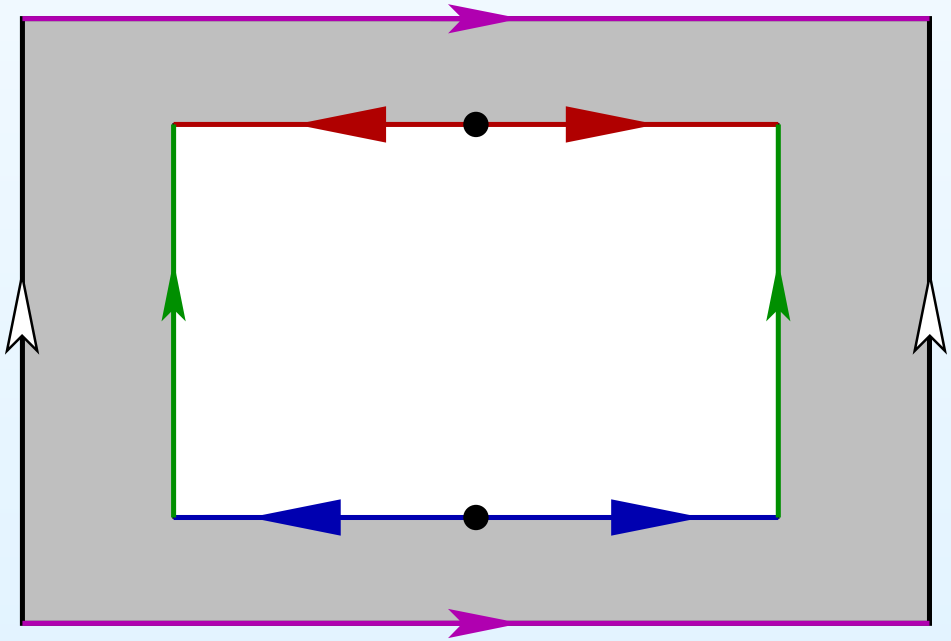
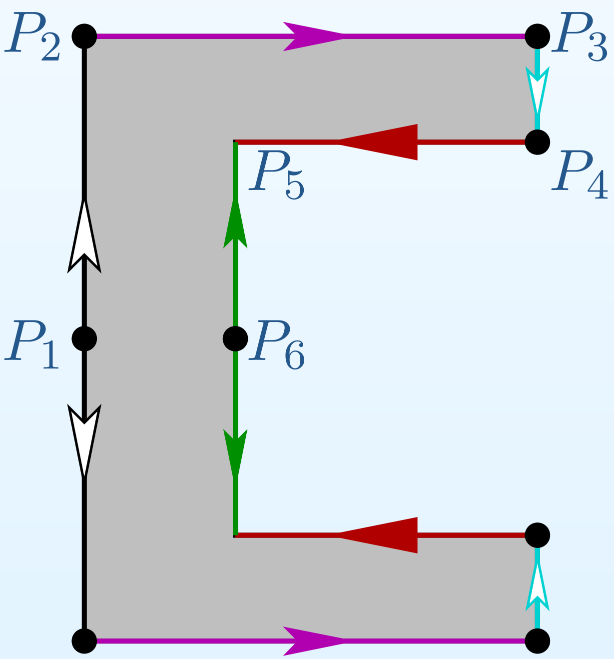
Symmetries

The resulting surface $\tilde{S}$ as on the left side of the picture below belongs to the stratum $\mathcal{Q}(1^2, -1^2)$; in particular, it has genus 1. The surface S obtained as the quotient of the original flat surface X over the entire group $(\mathbb{Z}/2\mathbb{Z})^3$ as on the right side of the picture below belongs to the stratum $\mathcal{Q}(1, -1^5)$; in particular, it has genus 0. It is easy to check that $\tilde{S}$ is a ramified double cover over S with ramification points at simple poles P_1, P_2, P_3, P_6 of S.





Diffusion rate as a Lyapunov exponent

The idea of renormalization suggests to study the asymptotic cycle $c(t)$ in the following way. We can pretend that there exists a pseudo-Anosov diffeomorphism $f_t : X \to X$ which sends a very long closed cycle $c(t)$ on the surface X to an integer cycle c of length comparable to 1. Then $|c(t) \circ h| = |f_t(c(t)) \circ f_t(h)| = |c \circ f_t(h)|$. Moreover, renormalization tells us how to relate the billiard flow time t to the Teichmüller geodesic flow time T. Thus, the asymptotic value of $|c \circ f_t(h)|$ is governed by the Lyapunov exponent of the element h^* in the fiber of the bundle $H^1_{\mathbb{R}}$.

The second observation is that the cycles $h = h_{00} + h_{10} - h_{01} - h_{11}$ and $v = v_{00} - v_{10} + v_{01} - v_{11}$ are induced from the corresponding cycles denoted by the same letters h and v on $\tilde{S}$ under the quadruple cover. The corresponding Lyapunov exponents for the locus of the surfaces X are the same as the exponents of h^* and of v^* “downstairs”, on the locus of the surface $\tilde{S}$.

Computation of the Lyapunov exponent

The flat surface $\tilde{S}$ belongs to the *(hyper)elliptic* locus $\mathcal{L} := \mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$. The family $\mathcal{B}$ of billiards has dimension larger than half dimension of the ambient locus $\mathcal{L}$ and is transversal to the unstable foliation in $\mathcal{L}$. This implies that for almost every billiard table Π in $\mathcal{B}$ the orbit closure $\mathcal{L}(S)$ of the associated flat surface $S(\Pi)$ coincides with the entire locus $\mathcal{L}$.

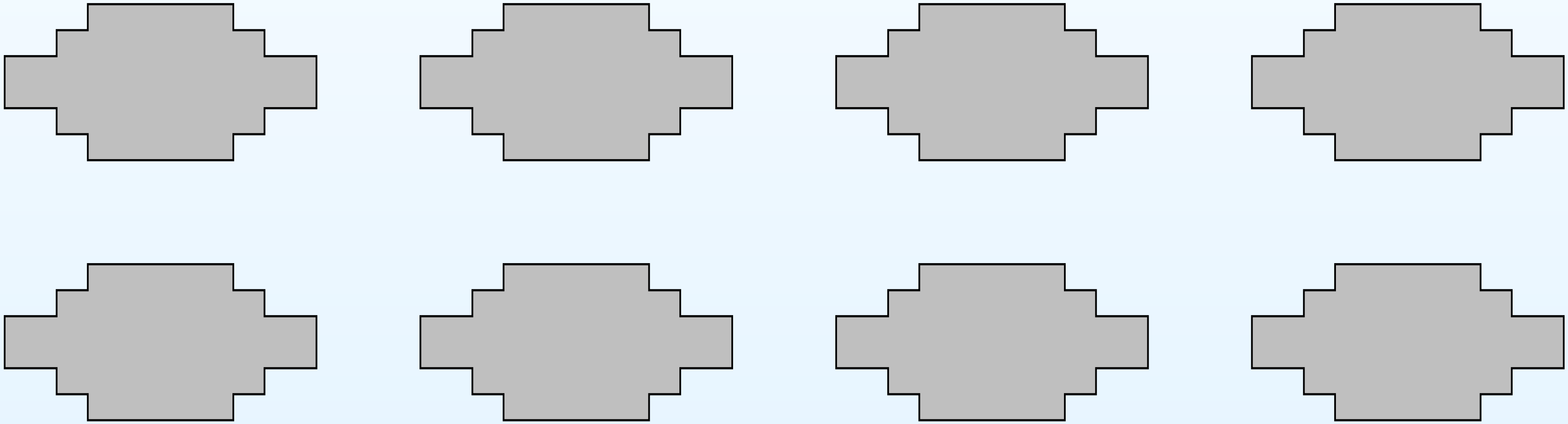
The flat surface $\tilde{S}$ has genus one, so the bundle $H^1_{\mathbb{R}} = H^1_+$ has single positive Lyapunov exponent λ_1^+. The fact that h^*, v^* are *integer* immediately implies that $\lambda^+(h^*) = \lambda^+(v^*) = \lambda_1^+$.

The sum $\sum_{i=1}^{g} \lambda_i^+$ of Lyapunov exponents of H^1_+ is expressed in terms of degrees of zeroes (and poles) in the ambient locus and in terms of the Siegel–Veech constant $c_{area}(\mathcal{L})$. In our case the genus of the surface is equal to one, $g = 1$, so we get a formula for the Lyapunov exponent λ_1^+ in terms of the Siegel–Veech constant $c_{area}(\mathcal{L})$ of the hyperelliptic locus $\mathcal{L} := \mathcal{Q}^{hyp}(1^{2m}, -1^{2m})$ over $\mathcal{Q}(1^m, -1^{m+4})$. This Siegel–Veech constant can be related to certain Siegel–Veech constants of the underlying stratum $\mathcal{Q}(1^m, -1^{m+4})$ computed by J. Athreya A. Eskin and A. Zorich.

Diffusion rate for obstacles with many corners

Theorem (V. Delecroix, A. Zorich, 2020). *For almost any symmetric obstacle with $4m-4$ angles $\frac{3\pi}{2}$ and $4m$ angles $\frac{\pi}{2}$, the diffusion rate equals the Lyapunov exponent $\lambda_1^+(m)$ of H_+^1 over the locus $\mathcal{Q}^{hyp}(1^{2m},-1^{2m}) \to \mathcal{Q}(1^m,-1^{m+4})$*

$$\lambda_1^+(m) = \frac{(2m)!!}{(2m+1)!!} \quad \sim \quad \frac{\sqrt{\pi}}{2\sqrt{m}} \quad \text{as } m \to \infty\,.$$



Note that once again the diffusion rate depends only on the number of the corners, but not on the (almost all) lengths of the sides, or other details of the shape of the obstacle.

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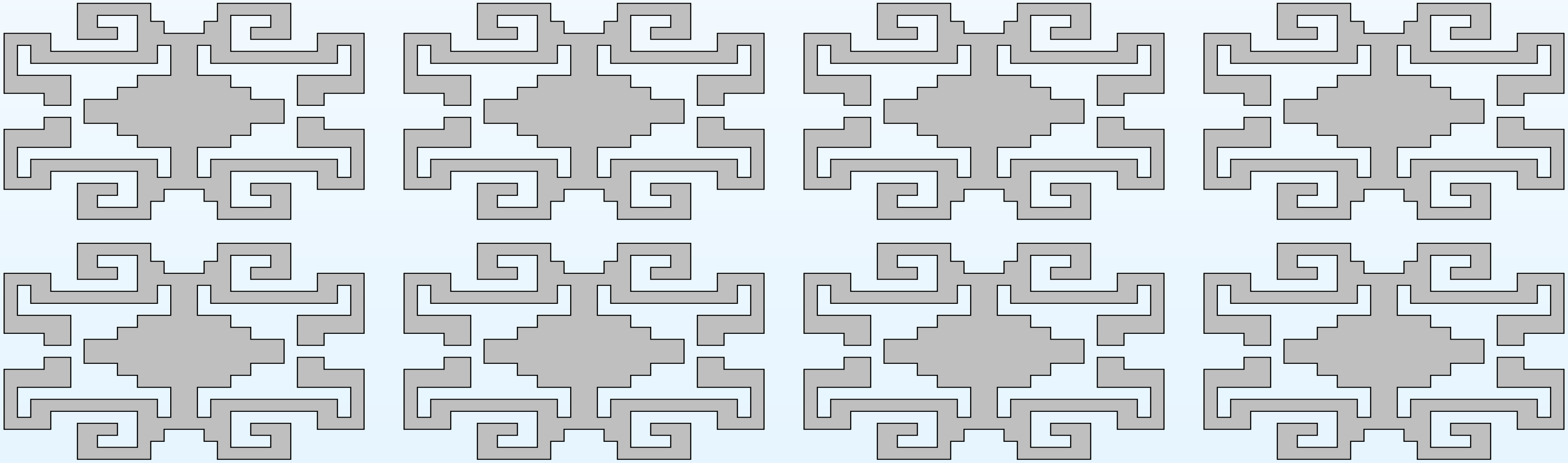

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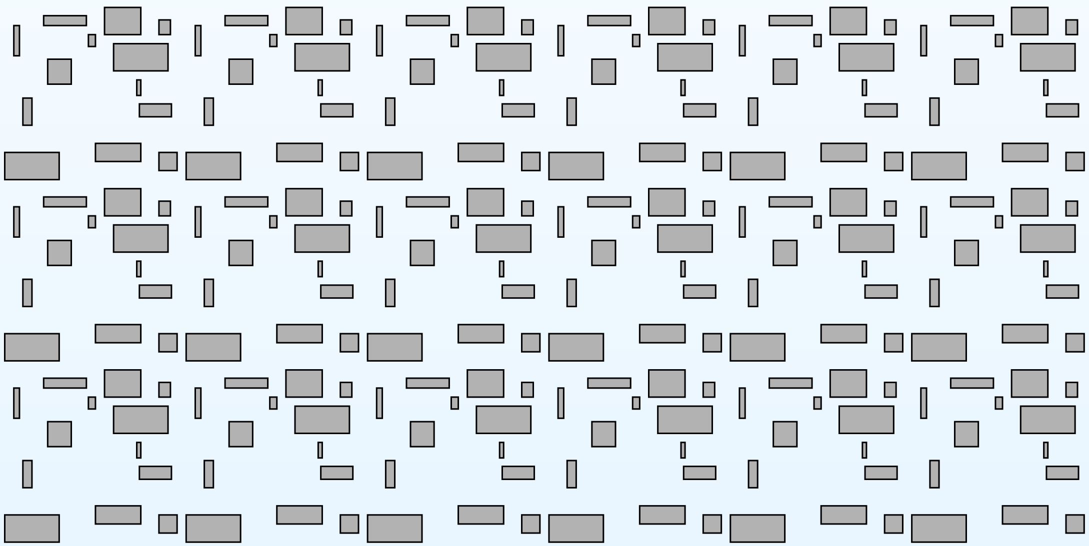
Random obstacles of high complexity

Theorem. (Ch. Fougeron, 2020) *The diffusion rate of the periodic billiard with $n \geq 2$ aligned small rectangular obstacles randomly placed in the fundamental domain is equal to the Lyapunov exponent λ_1^+ of the principal stratum $\mathcal{Q}(1^{4n})$ of holomorphic quadratic differentials in genus $n+1$.*

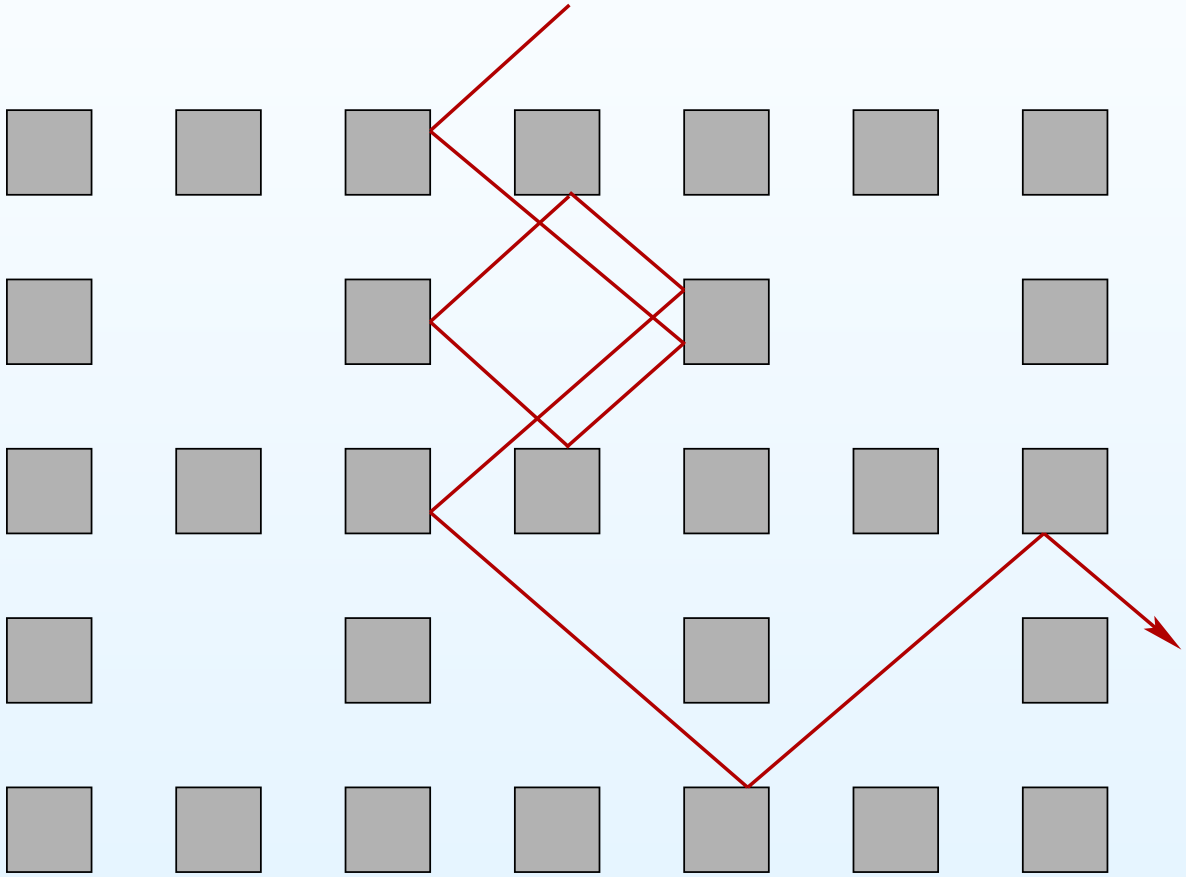


Conjecture (V. Delecroix, C. Fougeron) $\lim_{g\to+\infty} \lambda_1^+(\mathcal{Q}(1^{4n})) = \dfrac{1}{2}$.

Question. *What diffusion rate has a windtree billiard with "generic" (in any reasonable sense) irrational polygonal obstacles? Is it, by any chance, $\frac{1}{2}$?*

Removing part of the obstacles

How would change the diffusion rate if we remove periodically one out of four obstacles in every 2×2 group of squares?



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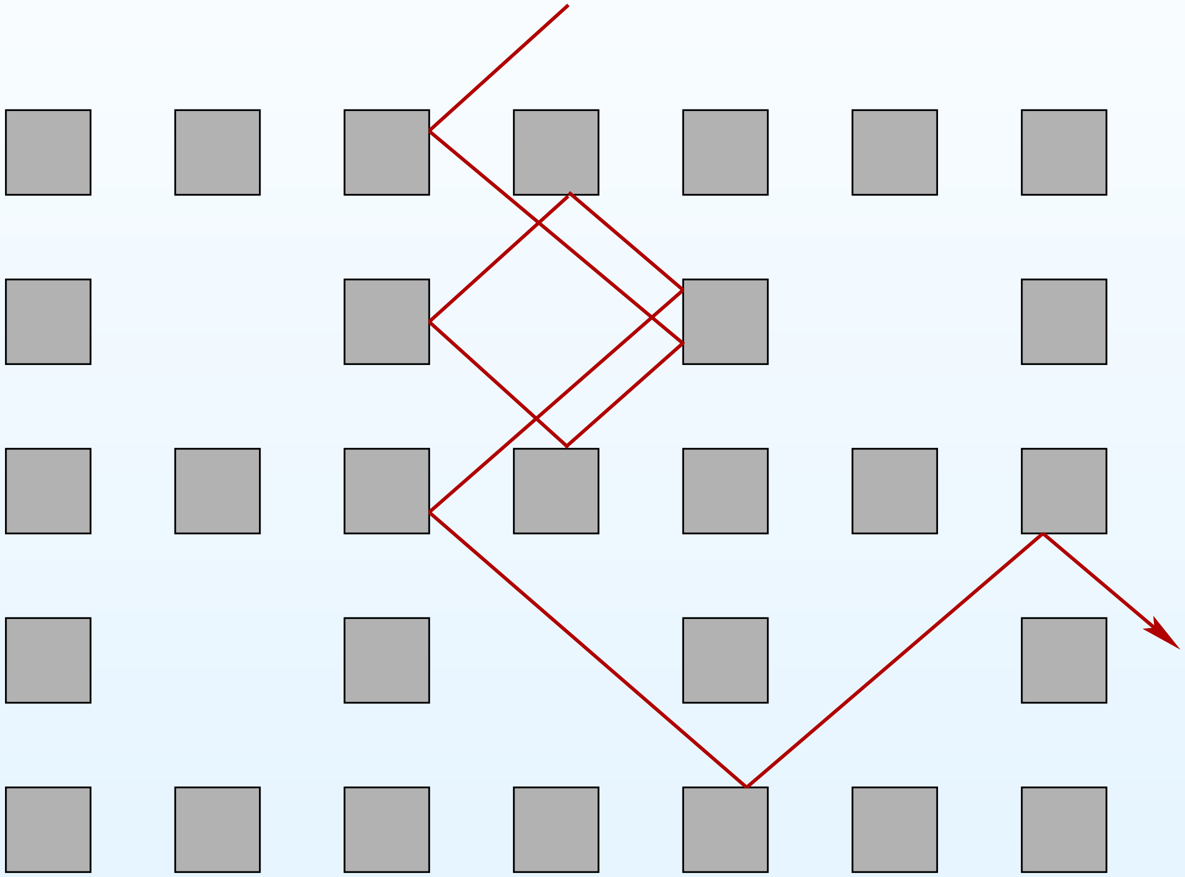
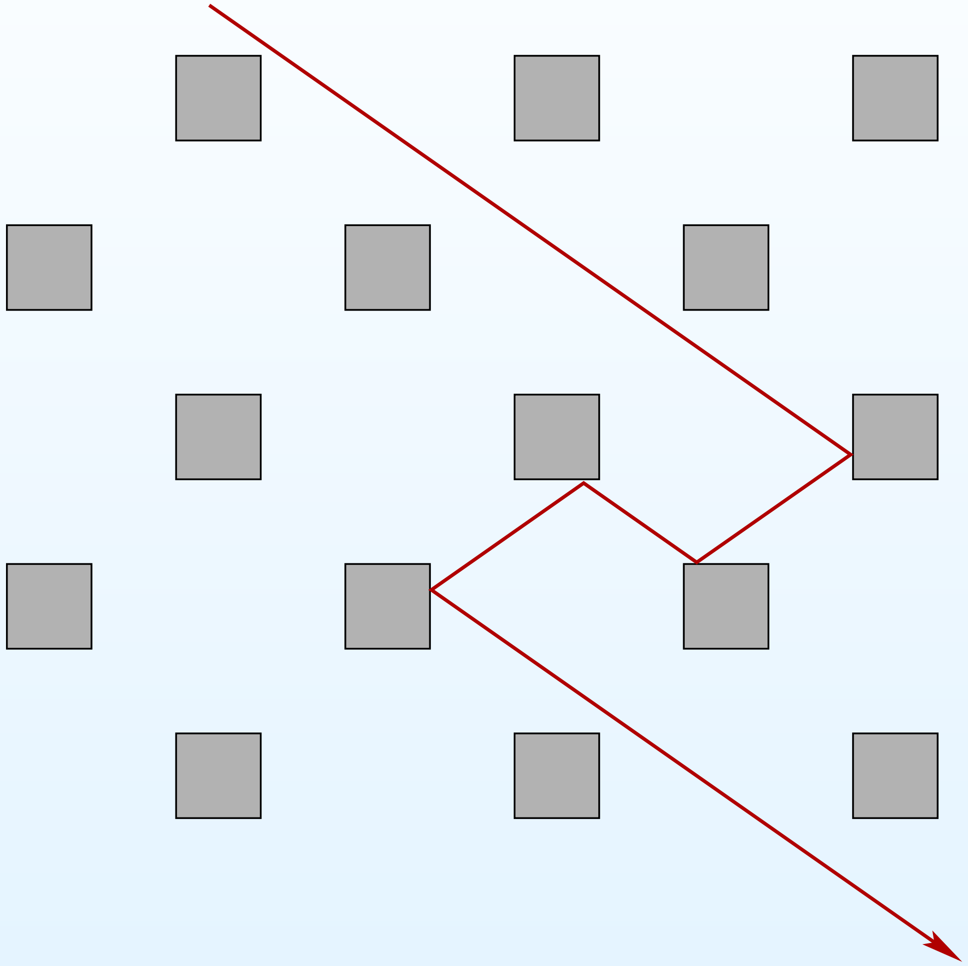


Lemma (V. Delecroix, A. Z., 2015). *Diffusion rate* $= \dfrac{491}{1053}$.

Removing part of the obstacles

And what about removing periodically two obstacles in every 2×2 group?



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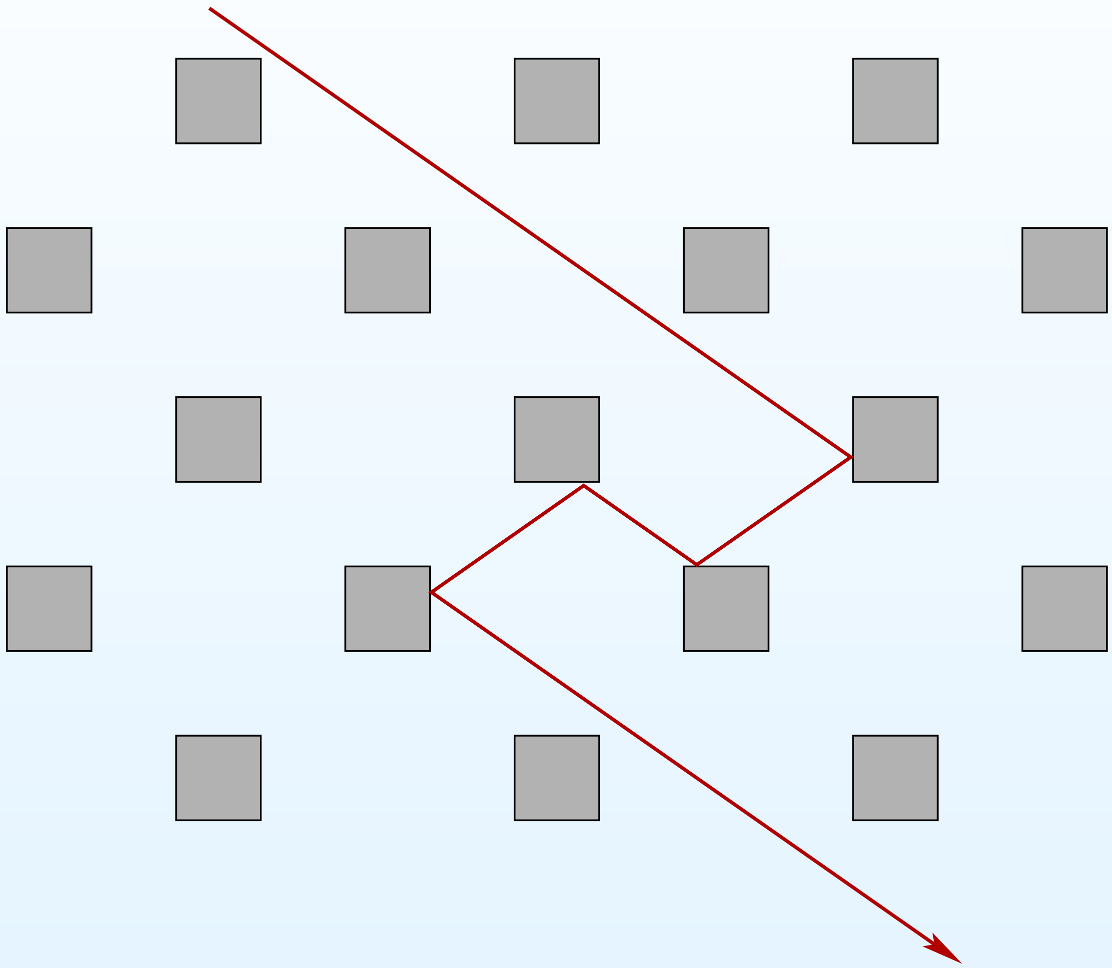

Lemma (V. Delecroix, A. Z., 2015). *Diffusion rate* $= \dfrac{2}{3}$.

Billiard in a polygon: artistic image

Varvara Stepanova. Joueurs de billard. Thyssen Museum, Madrid