Energy estimates for non-linear wave equations in mathematical General Relativity

1 Lecturer:

Sari Ghanem (Assistant Professor, BIMSA).

2 Mode of the discussion, venue and time, audience:

- Offline, in English, at C658 in the Shuangqing Complex Building, at Tsinghua University.
- During the Fall semester 2025, starting September 17, 2025. Two sessions per week: every Wednesday, at 11:00–12:30 and then at 14:30–16:00. No class on Wednesday October 15, 2025.
- Audience: Graduate, Postdoc, Researcher.

3 Prerequisite:

Basic knowledge from my previous courses on "The Cauchy problem in mathematical General Relativity", on "Non-linear wave equations in General Relativity", and on "Dispersive estimates for non-linear waves in mathematical General Relativity", graduate level knowledge in differential geometry and in Riemannian geometry, and basic knowledge in partial differential equations and analysis.

4 Introduction:

This course introduces mathematical tools of analysis for partial differential equations to prove uniform bounds and decay for solutions of non-linear wave equations arising in General Relativity. The course material builds on a series of courses that I gave in Spring–Fall 2024, and in Spring 2025, on the Cauchy problem in mathematical General Relativity, on non-linear wave equations in General Relativity, and on dispersive estimates for non-linear waves in mathematical General Relativity. The goal of this course is to explain the vector field method and how to obtain energy estimates for solutions of tensorial coupled non-linear hyperbolic partial differential equations, in order to prove decay for solutions of non-linear wave equations provided that one exploits the non-linear structure of the wave equations. We shall exhibit how this can be applied to the Einstein equations coupled to non-linear matter such as the Yang-Mills fields, by studying the simpler case of higher dimensions.

5 Keywords:

Energy estimates, non-linear wave equations, hyperbolic partial differential equations, Minkowski vector fields, Klainerman-Sobolev inequality, weighted energy norms, bootstrap argument, decay estimates, Hardy type inequality, commutator term, Grönwall type inequality, Einstein equations, Yang-Mills fields, Einstein-Yang-Mills system, gauge transformations, Minkowski metric, wave coordinates, Lorenz gauge.

6 Syllabus:

1. **Reminders of prerequisites:**
   The Einstein equations, the Yang-Mills equations, the coupled Einstein-Yang-Mills system, wave coordinates, the Lorenz gauge, recasting the Einstein-Yang-Mills system as a coupled system of non-linear hyperbolic partial differential equations, the hyperbolic Cauchy problem, the constraint equations, the gauges invariance of the equations.

2. **Set-up of analysis for proving decay for solutions of non-linear wave equations:**
   - The Minkowski vector fields.
   - Weighted Klainerman-Sobolev inequality.
   - Definition of the norms.
   - The energy norm.
   - The bootstrap argument.
   - The bootstrap assumption.
   - The big O notation.

3. **À priori decay estimates:**
   - The spatial asymptotic behaviour of the fields on the initial hypersurface.
   - Estimates on the time evolution of the fields.

4. **Looking at the structure of the source terms of the coupled non-linear wave equations for the Einstein-Yang-Mills system in the Lorenz gauge and in wave coordinates.**

5. **Using the bootstrap assumption to exhibit the structure of the source terms of the Einstein-Yang-Mills system in higher dimensions:**
   - Using the bootstrap assumption to exhibit the structure of the source terms for the Yang-Mills potential.
   - Using the bootstrap assumption to exhibit the structure of the source terms for the metric.
   - The source terms in higher dimensions $n \geq 5$.

6. **Energy estimates for non-linear wave equations.**

7. **A Hardy type inequality.**

8. **The commutator term for $n \geq 4$:**
   - Using the Hardy type inequality to estimate the commutator term.

9. **The energy estimate for the Einstein-Yang-Mills fields in higher dimensions $n \geq 4$.**

10. **Closing the bootstrap argument for the Einstein-Yang-Mills fields in higher dimensions :**
    - Using the Hardy type inequality for the space-time integrals of the source terms for $n \geq 5$.
    - Grönwall type inequality on the energy for $n \geq 5$.
    - Decay estimates for the Einstein-Yang-Mills fields in higher dimensions $n \geq 5$.

7 References:

1. L. Andersson and V. Moncrief, Future complete vacuum spacetimes, in The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel (2004).

2. Y. C. Bruhat, Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non-linéaires, Acta Math. 88, 1952, 141–225.

3. Y. C. Bruhat and R. Geroch, Global Aspects of the Cauchy Problem in General Relativity. Comm. Math. Phys. 14, 1969, 329–335.

4. M. Dafermos and I. Rodnianski, Lectures on black holes and linear waves, Clay Math. Proc., 17:97–205, 2013.

5. M. P. Do Carmo, Riemannian Geometry, Birkhäuser (1992).

6. D. Eardley and V. Moncrief, The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. I. Local existence and smoothness properties, Comm. Math. Phys. 83 (1982), no. 2, 171-191.

7. D. Eardley and V. Moncrief, The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. II. Completion of proof, Comm. Math. Phys. 83 (1982), no. 2, 193-212.

8. A. Einstein, Zur Elektrodynamik bewegter Körper, Annalen der Physik und Chemie 17, 1905, 891–921.

9. A. Einstein, Der Feldgleichungen des Gravitation, Preuss. Akad. Wiss. Berlin, Sitzber., 1915, 844–847.

10. S. W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time Cambridge University Press, 1973.

11. L. Hörmander, Lectures on nonlinear hyperbolic differential equations, volume 26 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer-Verlag, Berlin, 1997.

12. J. Isenberg, The Initial Value Problem in General Relativity, in: Ashtekar, A., Petkov, V. (eds) Springer Handbook of Spacetime. Springer Handbooks. Springer, Berlin, Heidelberg (2014).

13. J. Leray, Hyperbolic differential equations, The Institute for Advanced Study, Princeton, N. J., 1953.

14. H. Lindblad and I. Rodnianski, Global existence for the Einstein vacuum equations in wave coordinates, Commun. Math. Phys. 256:43-110, 2005.

15. H. Minkowski, "Raum und Zeit", Physikalische Zeitschrift, 10. Jahrgang, 1909, 104–115.

16. P. Petersen, Riemannian Geometry, Graduate Texts in Mathematics, 171, Springer, 1998.

17. B. Riemann, Über die Hypothesen, welche der Geometrie zugrunde liegen, Habilitationsschrift, 1854, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 13 (1868).

18. H. Ringström, The Cauchy Problem in General Relativity, ESI Lectures in Mathematics and Physics. European Mathematical Society (EMS), Zürich, 2009.

19. C. D. Sogge, Lectures on Non-Linear Wave Equations, Monographs in Analysis, International Press, 2008.

20. C. N. Yang and R. L. Mills, Conservation of Isotopic Spin and Isotopic Gauge Invariance, Phys. Rev., 96:191–195, Oct 1954.

21. R. Wald, General Relativity The University of Chicago Press, 1984.