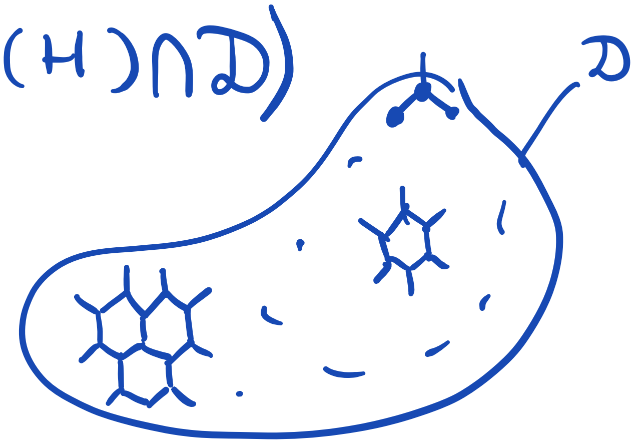
Thermodynamic limit

$\exists\, N$ – "linear size" of $\Gamma \subset$ hexagonal lattice H. Natural realization:

- $\varphi_\varepsilon : H \hookrightarrow \mathbb{R}^2$, step ε
- $\mathcal{D} \subset \mathbb{R}^2$ connected, simply connected region
- $\underbrace{\mathcal{D}_\varepsilon}_{\Gamma_\varepsilon} = \text{interiour}\left(\varphi_\varepsilon(H) \cap \mathcal{D}\right)$

$N = \frac{1}{\varepsilon}$,

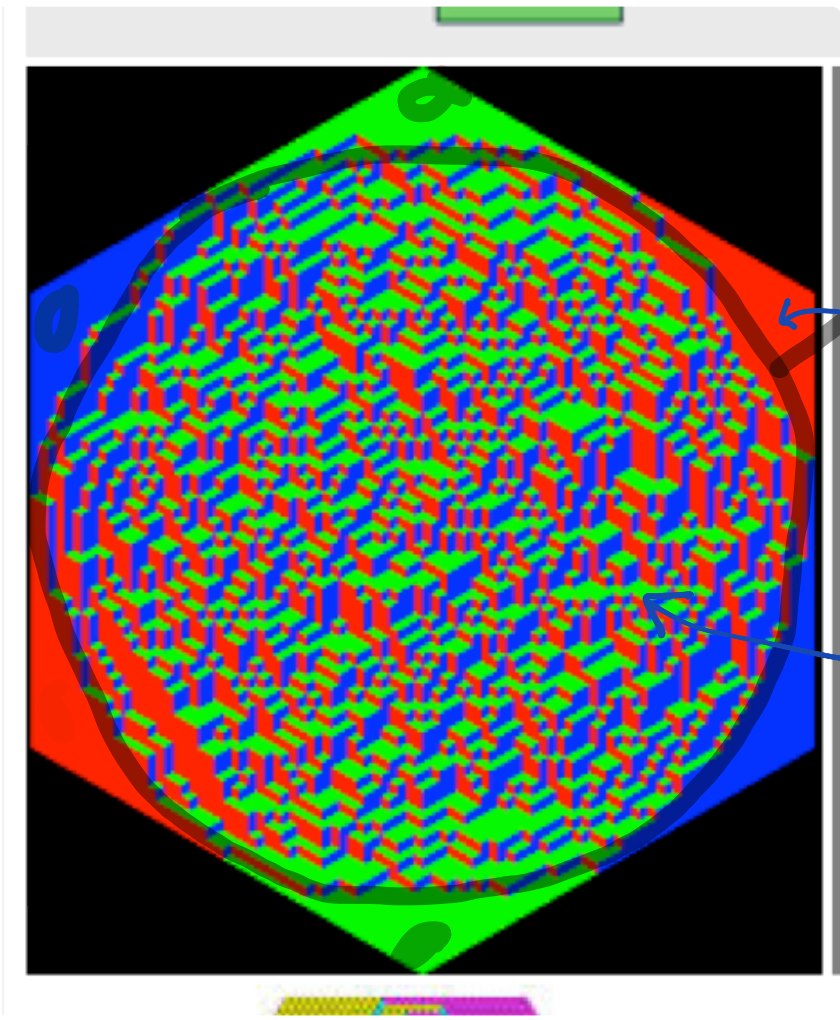


Problem: How the weight distribution on dimers behave as $N \to \infty$? $\varepsilon \to 0$

Equivalently, how height function distribution behave as $N \to \infty$? $\varepsilon \to 0$

Hexagonal region, uniform distribution

$w(e)=1$, $q_f = 1$, counting problem.



"Arctic curve"

"frozen" corners

disordered interiour of the limit shape

Result of Markov sampling

The limit shape phenomenon:

Random surfaces (height fncns) are fluctuating around most probable limit shape $h_0(\vec{x})$

$$\mathrm{Prob}(h) \propto \exp\left(\frac{1}{\varepsilon^2}\left(S(h_0) - S(h)\right)\right) \qquad \leq 0$$

with Gaussian fluctuations

$$h = h_0 + \varepsilon \varphi$$

φ is a Gaussian (conformally invariant) fluctuation field.

The limit shape h_0:

$S(h)$ = large deviation rate function, can be explicitly computed using Kasteleyn solution (Cohn, Kenyon, Propp, 2000)

$$S(h) = \iint_D \sigma(\vec{\nabla} h)\, d^2x$$

$$\sigma(s,t) = \max_{H,V} \left(Hs + Vt - f(H,V) \right)$$

$$f(H,V) = \left(\frac{1}{2\pi}\right)^2 \iint_{|z|=e^H,\, |w|=e^V} \ln|1+z+w| \frac{dz}{z} \frac{dw}{w}$$

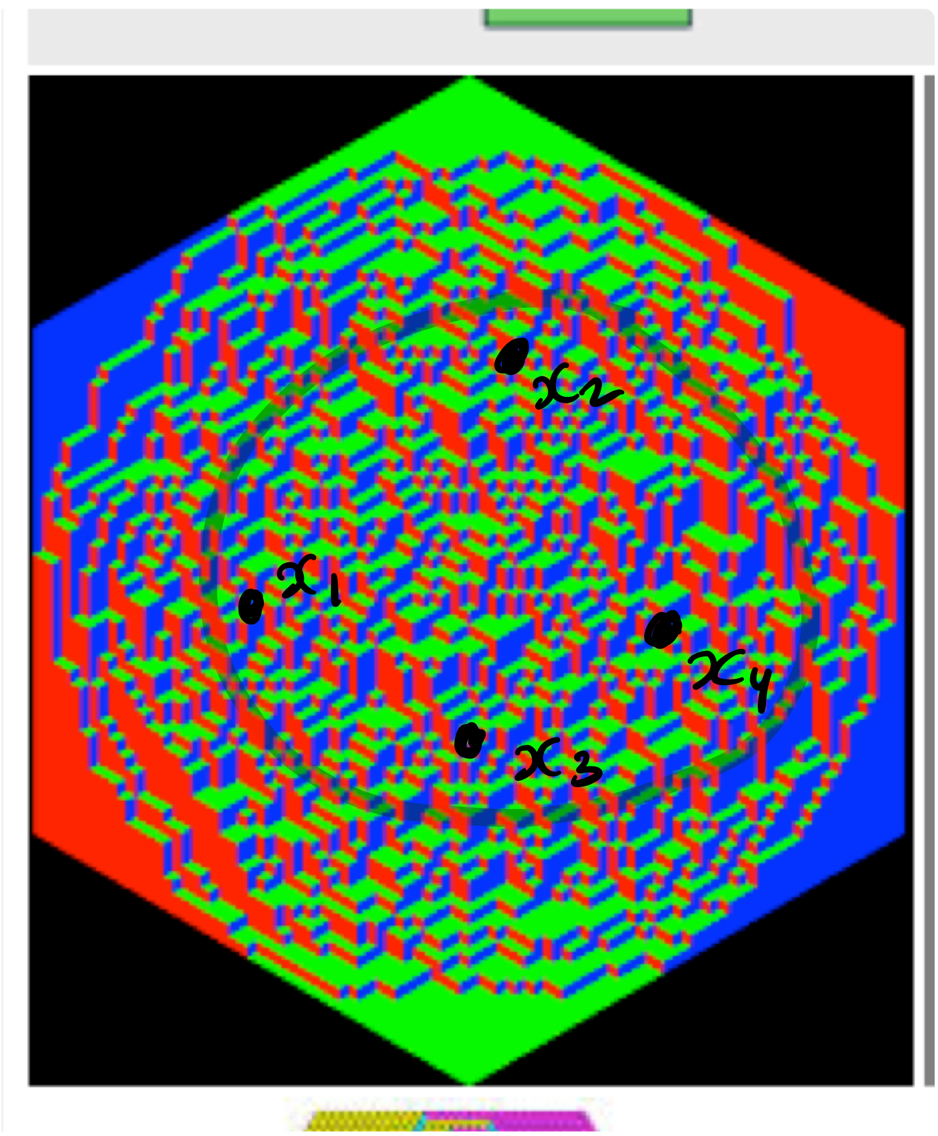
h_0 = the unique minimizer of $S[h]$

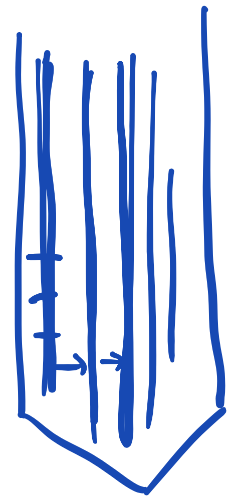
For the hexagon can be explicitly computed (CKP).

Gaussian fluctuations

$\varepsilon \to 0$,

$$\mathbb{E}[h(f_1)\, h(f_2)] \longrightarrow h_0(x_1)\, h_0(x_2) + \varepsilon^2\, \mathbb{E}(\varphi(x_1)\, \varphi(x_2)) + \dots$$





$Z =$

$= (\Omega, \prod_i V_i \, \Omega)$

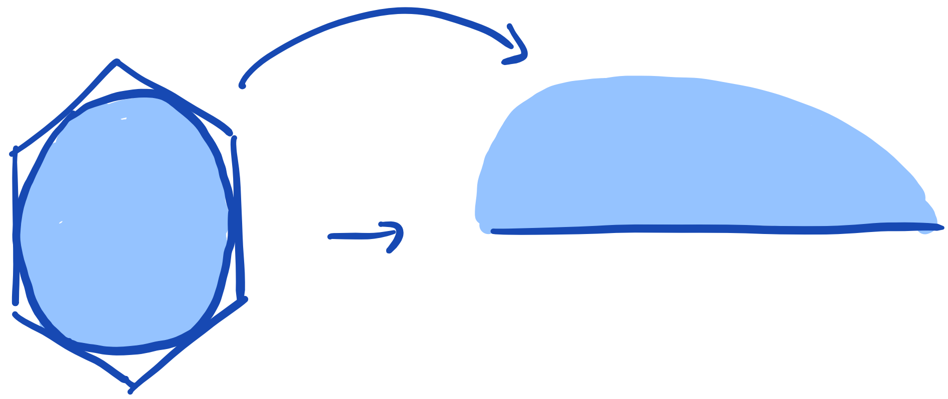
gl_∞

$\widehat{sl_2}$

Points $\{x_i\}$ are macroscopically separated.

$$\mathbb{E}(\varphi(x_1)\varphi(x_2)) = -\frac{1}{4\pi} \ln \left| \frac{z(x_1) - z(x_2)}{z_1(x_1) - \overline{z}(x_2)} \right|$$

z :



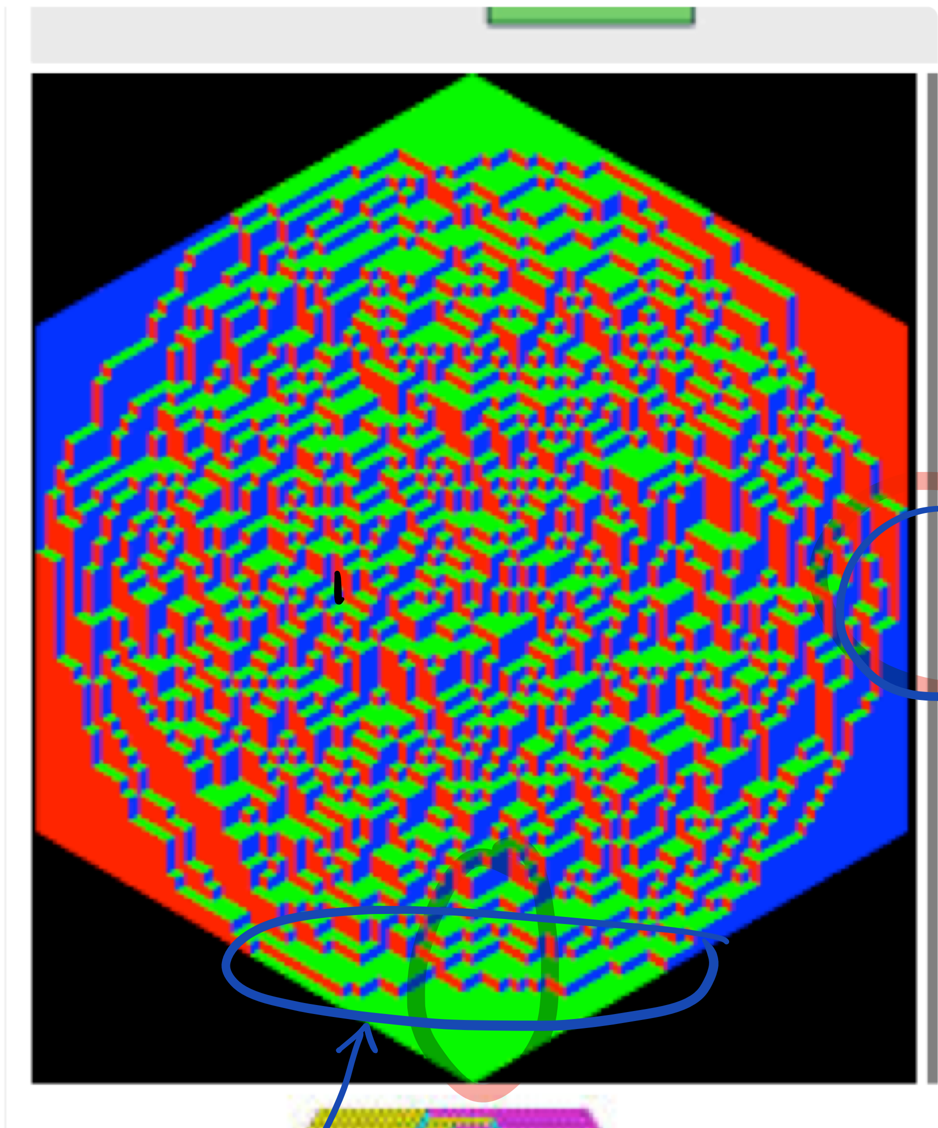
(Kenyon, Okounkov, Sheffield, 2003)

$$\mathbb{E}(\varphi(x_1)\cdots\varphi(x_n)) = \int e^{-S^{(2)}[\varphi]}\,\varphi(x_1)\cdots\varphi(x_n)\,\mathcal{D}\varphi$$

$$S^{(2)}[\varphi] = \frac{1}{2}\iint_{\text{ellips}} \partial_i\varphi\,\partial_j\varphi\,\frac{\partial^2\sigma}{\partial s_i\partial s_j}(\vec\nabla h_0(x))\,d^2x$$

$$= \frac{1}{2}\iint_{\text{half plane}} \partial_z\varphi(z,\bar z)\partial_{\bar z}\varphi(z,\bar z)\,dz\,d\bar z$$

Conformal quantum field theory on $\mathrm{Im}\, z > 0$ with Dirichlet boundary condition on $z \in \mathbb{R}$.



Random matrices near touching points

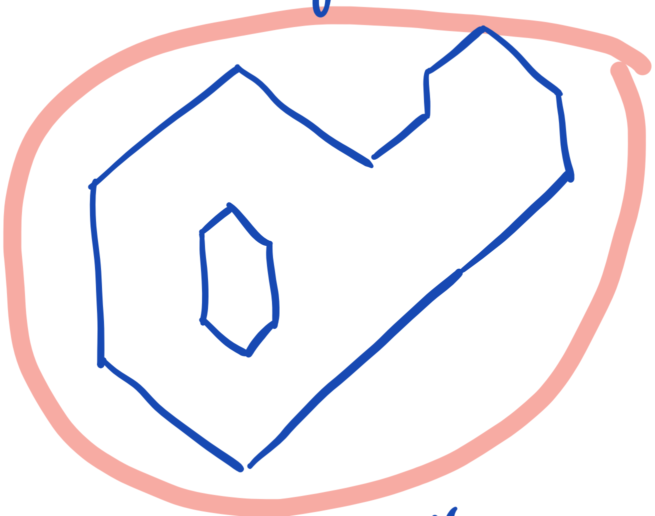
(Okounkov, R. 2005)

Airy correlations near the boundary

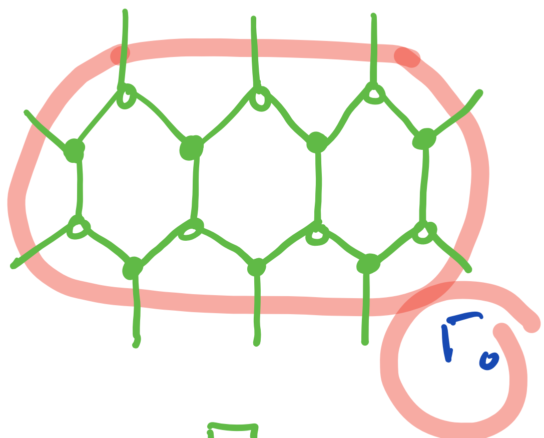
(Johansson, 2002)

Other region, other weights

Natural class of regions (on $\Gamma^\vee$, $\Gamma \subset$ hexagonal lattice).

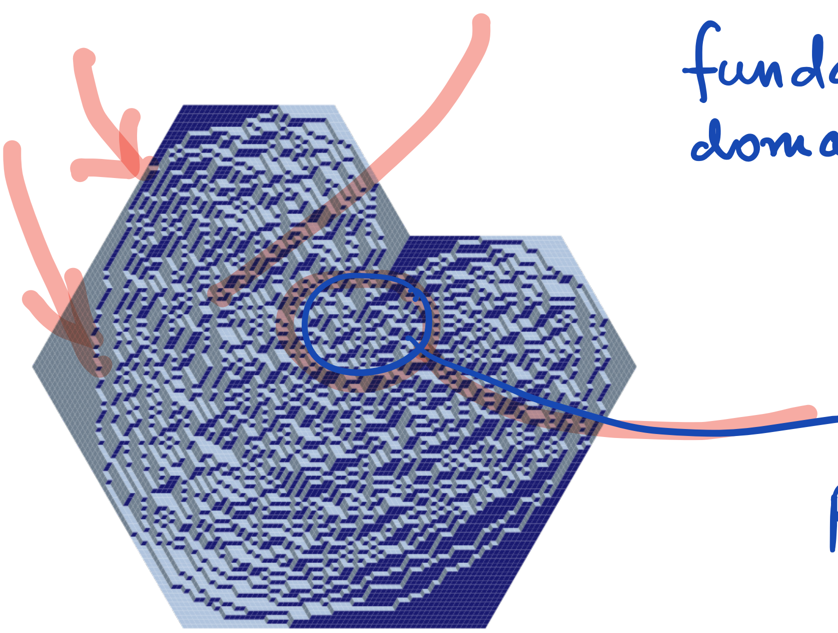


tiling, $\Gamma^\vee$

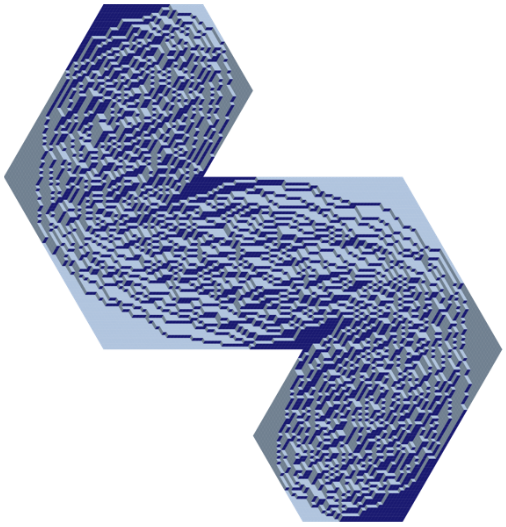


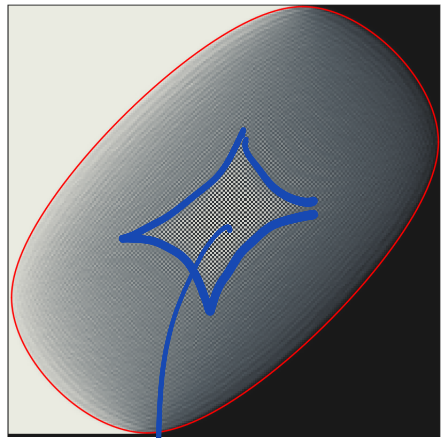
Γ

periodic weight
fundamental domain $\Gamma_0 \subset \Gamma$



Pearcey process





flat region
exponential decay
correlation functions

- Kenyon, Okounkov; Limit shapes and algebraic geometry 2003; Complex Burgers equation, 2005

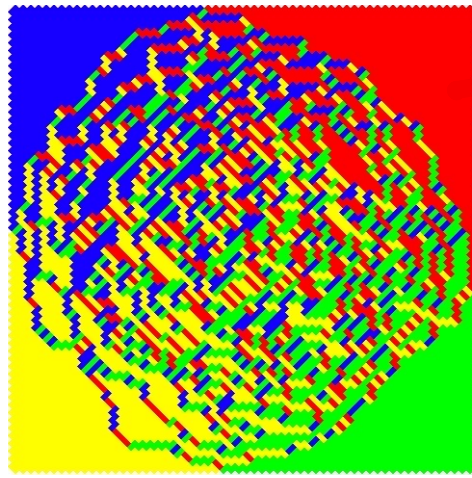
Correlation functions

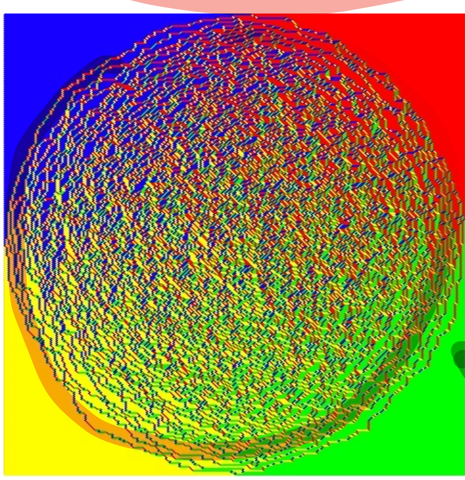
- Okounkov, R.; Pearcey processes & cusps in limit shapes, 2005
- Kenyon, Gaussian fluctuations, 2004

Other graphs

Region of square lattice "Aztec diamond"

on $\Gamma^\vee$

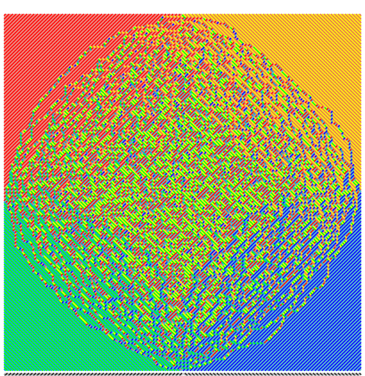




Uniform distribution

$w(e) = q(f) = 1$

frozen & disordered regions



two periodic weights

Chhita, Johansson, 2014

When $r_c \sim \ell$

new phenomenon

E. Bain, next week

Non-commutative, quantum, dimers

In physics $\quad w(e)\,w(e') = q^{\#}\, w(e')\, w(e)$

Solid state physics.

In mathematics

"Noncommutative geometry of random surfaces"
Okounkov, 2009.

"Dimers and cluster integrable systems"
Goncharov, Kenyon, 2011.

...

Integrable probability, Representation Theory

....

Calabi-Yau manifolds and dimers

- Limit shapes and mirror Calabi-Yau

  "Quantum Calabi-Yau and Classical Crystals", Okounkov, R., Vafa, 2003

  Toric CY, spectral curve of Γ_0 $\leftrightarrow$ limit shape

- Many interesting developments

  Nekrasov, Maulik, Okounkov, Pandharipande, Vafa, ...

- Dimers, dualities, toric diagrams, ...

- Dimers and Calabi-Yau algebras, ...

Happy Birthday Prof. Yau !!!