Reference

- Cerf, "La stratification naturelle des espaces de fonctions différentiables réelles et la théorème de la pseudo-isotopie", 1970

<u>S1</u> $\eta_*: \pi_{i+1} Map(D^n;\partial, \frac{Top(n)}{O(n)}) \longrightarrow \pi_{i+1} Map(M, \frac{Top(n)}{O(n)})$

is injective provided that M is parallelizable and $i \geq n$.

<u>pf S1</u>:

$$\pi_{i+1} Map(D^n;\partial, \frac{Top(n)}{O(n)}) = \pi_0 \Omega^{i+1+n} \frac{Top(n)}{O(n)}$$

$$= [S^{i+n+1};*, \frac{Top(n)}{O(n)};*] \longrightarrow$$ the set of basept-preserving homotopy classes.

$$\pi_{i+1} Map(M, \frac{Top(n)}{O(n)}) = \pi_0 \Omega^{i+1} Map(M, \frac{Top(n)}{O(n)})$$

$$= \pi_0 Map(D^{i+1};\partial, Map(M, \frac{Top(n)}{O(n)}))$$

$$= \pi_0 Map(M\times D^{i+1}; M\times\partial D^{i+1}, \frac{Top(n)}{O(n)})$$

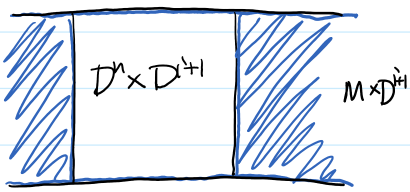
$$= [M\times D^{i+1}/\partial ;*, \frac{Top(n)}{O(n)};*]$$

Let $\hat{\eta}: M^n\times D^{i+1}/\partial \longrightarrow S^{i+n+1}$ be the map collapsing

$$M\times\partial D^{i+1} \cup (M\times D^{i+1}\setminus Int(D^n\times D^{i+1}))$$

to a point. where $M\times D^{i+1}\setminus Int(D^n\times D^{i+1})$ denotes the complement of the embedding

$$\nu\times id: D^n\times D^{i+1} \hookrightarrow M\times D^{i+1}$$



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Then $\eta_*: \pi_{i+1} Map(D^n;\partial, \frac{Top(n)}{O(n)}) \to \pi_{i+1} Map(M, \frac{Top(n)}{O(n)})$

can be identified with

can be identified with

$$\hat{\eta}^*: [S^{i+n+1};*, \frac{Top(n)}{O(n)};*] \longrightarrow [{}^{M\times D^{i+1}}/_{\partial};*, \frac{Top(n)}{O(n)};*]$$

Now construct a map $\varrho: S^{i+n+1} \longrightarrow {}^{M\times D^{i+1}}/_{\partial}$.

Since $i \geq n$, $\exists$ embedding

$$M \overset{j}{\hookrightarrow} S^{i+n+1}$$

$$\rightsquigarrow \underset{\text{tubular neighborhood}}{N(M)} \hookrightarrow S^{i+n+1}$$

Since $M \hookrightarrow S^{i+n+1}\setminus * = \mathbb{R}^{i+n+1} \Longrightarrow$

trivial bundle of rank $i+n+1$.

normal bundle $\leftarrow \nu_M \oplus \tau_M \cong \varepsilon^{i+n+1}$ $\rightarrow$ tangent bundle

$\tau_M \cong \varepsilon^n$ $\rightarrow$ M is parallel.

$$\nu_M \oplus \varepsilon^n$$

$$\underset{\text{Lemma 3.1 (see below)}}{\xRightarrow{\overset{i+1}{\overset{\shortparallel}{\dim \nu_M}} > \dim M = n}} \nu_M \cong \varepsilon^{i+1}$$

$$\Longrightarrow M\times D^{i+1} \cong N(M) \hookrightarrow S^{i+n+1}$$

$$\rightsquigarrow$$

Let $S^{i+n+1} \xrightarrow{\varrho} {}^{M\times D^{i+1}}/_{\partial}$ be the cond. map collapsing the complement of $M\times D^{i+1} \hookrightarrow S^{i+n+1}$ to a pt. $\Longrightarrow$

$$S^{i+n+1} \xrightarrow{\varrho} {}^{M\times D^{i+1}}/_{\partial} \xrightarrow{\hat{\eta}} S^{i+n+1}$$

is of degree 1 $\Longrightarrow \hat{\eta}\circ\varrho \simeq id$

(basepoint-preserving since $\pi_1 S^{i+n+1} = 1$)

$$\Longrightarrow \underset{\underset{id}{\shortparallel}}{\varrho^*\circ\hat{\eta}^*}: [S^{i+n+1};*, \frac{Top(n)}{O(n)};*] \circlearrowleft$$

$\Rightarrow \eta_* = \hat{\eta}^*$ is injective.

<u>Lemma 3.11</u> Let M be a compact mfd and let ν^k be a k-dim real vector bundle over M. If $k>n$ and $\nu^k \oplus \varepsilon^1 \cong \varepsilon^{k+1}$, then $\nu^k \cong \varepsilon^k$.

Pf: Consider fiber bundle

$$S^k \hookrightarrow BO(k) \xrightarrow{j} BO(k+1)$$

$\rightsquigarrow$ exact seq: $\langle \cdot, \cdot \rangle$ denote the set of basept-preserving homotopy classes

$$\langle M, S^k \rangle \longrightarrow \langle M, BO(k) \rangle \xrightarrow{j_*} \langle M, BO(k+1) \rangle$$

$$f \longmapsto [*]$$

the homotopy class of constant map

classify ν^k

since $\nu^k \oplus \varepsilon^1 \cong \varepsilon^{k+1}$

has only one homotopy class since $\dim M < k$.

$\Rightarrow j_*$ injective $\Rightarrow f \sim *$ $\square$

To show <u>Th 3.4</u>, it remains to prove

<u>Lemma 3.9</u> Let $i \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{>0}$

Suppose $\pi_{i+1} \mathrm{Diff}(D^{n-1}, \partial) \xrightarrow{\alpha_{n*}} \pi_i \mathrm{Diff}(D^n, \partial)$

$$\exists \cdot \longmapsto \bar{x}$$

Then $\exists r>0$ with the following property:

for any closed real hyp. mfd (N, g) with injectivity radius at least $3r$ at some point, $\exists \nu : D^n \hookrightarrow N$ s.t.

$$\begin{tikzcd}
 & \pi_i \mathrm{Diff}(D^n, \partial) \ni \bar{x} \arrow[d, "(\iota(\nu))_*"] \\
 & \pi_i \mathrm{Diff}_0(N) \arrow[d, equal] \\
\pi_{i+1} T^0(N) \arrow[r, "F_*"] & \pi_{i+1} T(N) \\
\exists \cdot \arrow[r, mapsto] & (\iota(\nu))_* \bar{x}
\end{tikzcd}$$

$$\begin{array}{ccc} \pi_{i+1}T(N) & \longrightarrow & \pi_{i+1}T(N) \\ \exists\, \bullet & \longmapsto & \iota(\nu)_*\bar{x} \end{array}$$

<u>Rk</u>: When $i=0$ and $n\geq 6$, Lemma 3.9 is restated as follows:

<u>Lemma 3.9'</u> Suppose $n\geq 6$ and $\bar{x}\in\pi_0\mathrm{Diff}(D^n,\partial)$, then $\exists\, r>0$ with the following property:
for any closed real hyp. mfd (N,g) with injectivity radius $\geq 3r$ at some point, $\exists\, \nu: D^n \hookrightarrow N$ s.t.

$$\iota(\nu)_*\bar{x}\in \mathrm{Im}(F_*: \pi_1 T^{<0}(N) \longrightarrow \underset{\pi_0\mathrm{Diff}_0(N)}{\overset{\pi_1 T(N)}{\parallel}})$$

Note that the condition

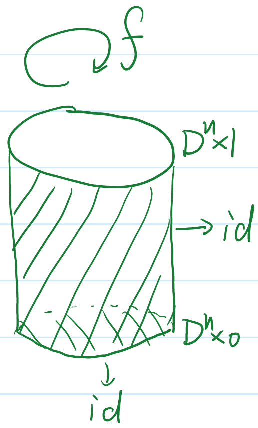
$$\text{"} = \pi_1\mathrm{Diff}(D^{n-1},\partial) \xrightarrow{\alpha_{n*}} \pi_0\mathrm{Diff}(D^n,\partial) \text{"}$$
$$\exists\, \bullet \longmapsto \bar{x}$$

is dropped here because α_* is always surjective for $n\geq 6$ (Reason:

Let

$$P(D^n) = \{ f\in\mathrm{Diff}(D^n\times I) \mid f|_{D^n\times 0\,\cup\,\partial D^n\times[0,1]} = id \}$$

It is a subgrp of $\mathrm{Diff}(D^n\times I)$, called <u>pseudoisotopy space</u> (or <u>concordance space</u>) of D^n.



Consider

$$\begin{array}{ccc} P(D^{n-1}) & \xrightarrow{\tau} & \mathrm{Diff}(D^{n-1},\partial) \\ f & \longmapsto & f|_{D^n\times 1} \end{array}$$

τ is a fibration with fiber $\mathrm{Diff}(D^n,\partial)$

Moreover, $\exists$ fibration seq

Gromoll's map

$$\Omega\mathrm{Diff}(D^{n-1},\partial) \xrightarrow{\alpha} \mathrm{Diff}(D^n,\partial) \longrightarrow P(D^{n-1}) \xrightarrow{\tau} \mathrm{Diff}(D^{n-1},\partial)$$

$\rightsquigarrow$ homotopy exact seq

$$\pi_1\mathrm{Diff}(D^{n-1},\partial) \xrightarrow{\alpha_*} \pi_0\mathrm{Diff}(D^n,\partial) \longrightarrow \pi_0 P(D^n)$$

homotopy exact seq

$$\begin{tikzcd}
\pi_1 \mathrm{Diff}(D^{n-1},\partial) \arrow[r, "\alpha_*"] & \pi_0 \mathrm{Diff}(D^n,\partial) \arrow[r] & \pi_0 P(D^n) \arrow[d, equal, "\text{for } n\geq 6"] \\
& & 0
\end{tikzcd}$$

cf. [C70]