1 Convergence of FK-Ising model with Dubrushin boundary conditions

1.1 Preliminaries and Setup

FK-Ising model

Let Ω_δ be a discrete simply connected domain on $\mathbb{L}_\delta := \sqrt{2}\delta e^{\frac{i\pi}{4}}\mathbb{Z}^2$, let Ω_δ^* be the corresponding dual domain and let $\Omega_\delta^\diamond$ be the graph such that $V(\Omega_\delta^\diamond) = \{\text{midpoints of edges of } \Omega_\delta \cup \Omega_\delta^*\}$ and $E(\Omega_\delta^\diamond) = \{\text{straight lines connecting nearest vertices}\}$. We orient $E(\Omega_\delta^\diamond)$ such that edges surrounding $V(\Omega_\delta^*)$ are clockwise. Define $\partial\Omega_\delta^\diamond := V(\Omega_\delta^\diamond) \cap (\partial\Omega_\delta \cup \partial\Omega_\delta^*)$. Fix two boundary points $a_\delta^\diamond$ and $b_\delta^\diamond$ on $\partial\Omega_\delta^\diamond$. Recall that the FK-Ising model on a graph Ω_δ is defined as follows: for every edge configuration ω, we have

$$\mathbb{P}[\omega] = \frac{1}{Z_{FK}}\left(\frac{p}{1-p}\right)^{|o(\omega)|} 2^{C(\omega)},$$

where $o(\omega)$ is the number of open edges and $C(\omega)$ is the number of connected components in ω. From now on, we always assume that p equals the critical value, that is

$$p = p_c = \frac{\sqrt{2}}{\sqrt{2}+1}.$$

Suppose that the boundary conditions are the Dubrushin boundary conditions: edges intersecting $(a_\delta^\diamond b_\delta^\diamond)$ are free and edges intersecting $(b_\delta^\diamond a_\delta^\diamond)$ are wired. When $p = p_c$, the dual edge configuration on Ω_δ^* is also the critical FK-Ising model with the Dubrushin boundary conditions: edges intersecting $(a_\delta^\diamond b_\delta^\diamond)$ are wired and edges intersecting $(b_\delta^\diamond a_\delta^\diamond)$ are free. The interface γ_δ is the unique path on $\Omega_\delta^\diamond$ from $a_\delta^\diamond$ to $b_\delta^\diamond$ which does not cross open edges or dual open edges.

Convergence of discrete domains

We will always assume that $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ converges to a simply connected domian $(\Omega; a, b)$ in the Carathéodory sense

- The boundary point $a_\delta^\diamond$(resp. $b_\delta^\diamond$) converges to a(resp. b).
- There exist conformal maps ψ_δ(resp. ψ) : $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$(resp. $(\Omega; a, b)$) $\to (\mathbb{U}; -1, 1)$ such that ψ_δ^{-1} converges to ψ^{-1} locally uniformly.

Space of curves

A path is defined by a continuous map from $[0,1]$ to $\mathbb{C}$. Let $\mathcal{C}$ be the space of unparameterized paths in $\mathbb{C}$. Define the metric on $\mathcal{C}$ as follows:

$$d(\gamma_1, \gamma_2) := \inf \sup_{t\in[0,1]} |\hat{\gamma}_1(t) - \hat{\gamma}_2(t)|, \tag{1.1}$$

where the infimum is taken over all the choices of parameterizations $\hat{\gamma}_1$ and $\hat{\gamma}_2$ of γ_1 and γ_2. The metric space $(\mathcal{C}, d)$ is complete and separable.

Let $\mathcal{P}$ be a family of probability measures on $\mathcal{C}$. We say $\mathcal{P}$ is tight if for any $\epsilon > 0$, there exists a compact set K_ϵ such that $\mathbb{P}[K_\epsilon] \geq 1-\epsilon$ for any $\mathbb{P} \in \mathcal{P}$. We say $\mathcal{P}$ is relatively compact if every sequence of elements in $\mathcal{P}$ has a weakly convergent subsequence. As the metric space is complete and separable, relative compactness is equivalent to tightness.

1.2 Convergence of FK-Ising interfaces

We will prove the following theorem in this subsection:

Theorem 1.1. *Suppose $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ converges to $(\Omega; a, b)$ in the Carathéodory sense. Then, the random curve $\psi_\delta(\gamma_\delta)$ converges to $\mathrm{SLE}_{16/3}$ from -1 to 1 as curves in law.*

Note that we will prove the convergence of $\psi_\delta(\gamma_\delta)$ rather than γ_δ because we do not give any assumption on regularity of $\partial\Omega$ or the convergence of $\partial\Omega_\delta$. If $\partial\Omega_\delta$ has a long fjord near b_δ or $\partial\Omega$ is very irregular near b, we can not expect the convergence result.

The proof of Theorem 2.3 can be divided into three steps. First, we prove the tightness of $\{\phi_\delta(\gamma_\delta)\}_{\delta>0}$. Second, we construct a discrete observable and prove that the discrete observable will converge to a conformal invariant function. Third, we derive the law of any sublimit by using the observable. The uniqueness of the sublimit implies that the discrete curves converge in law.

For the first step, we need to check that $\{\phi_\delta(\gamma_\delta)\}_{\delta>0}$ satisfies C2 condition, which is defined as follows.

Definition 1.2. *We say $\{\gamma_\delta\}_{\delta>0}$ satisfies C2 condition, if there exists $M > 0$, such that for every $\delta > 0$, for any stopping time $0 \leq \tau_\delta \leq 1$ for γ_δ and for any avoidable quadrilateral Q of $\Omega_\delta \setminus \gamma_\delta[0, \tau_\delta]$, such that the modulus $m(Q)$ is larger than M,*

$$\mathbb{P}[\gamma_\delta[\tau_\delta, 1] \text{ crosses } Q | \gamma_\delta[0, \tau_\delta]] < \frac{1}{2}.$$

The constant $\frac{1}{2}$ is not important, we can replace $\frac{1}{2}$ by any constant smaller than 1. Note that if $\{\gamma_\delta\}_{\delta>0}$ satisfies C2 condition, then $\{\phi_\delta(\gamma_\delta)\}_{\delta>0}$ also satisfies C2 condition. The fact that $\{\gamma_\delta\}_{\delta>0}$ satisfies C2 condition can be proved by the following Theorem.

Theorem 1.3. *[CDH16, Theorem 1.1] For each $L > 0$, there exists $\eta = \eta(L) > 0$ such that the following holds: for any topological rectangle $(Q; x, y, z, w)$ such that the extremal distance between (xy) and (zw) is larger than L and for any boundary conditions ξ,*

$$\mathbb{P}^\xi[(xy) \text{ connects to } (zw) \text{ by open edges }] \leq 1 - \eta,$$

where $\mathbb{P}^\xi$ denotes the critical FK-Ising model on(the discrete approximation of) $(Q; x, y, z, w)$ with boundary conditions given by ξ.

Note that Theorem 1.3 and domain Markov property of FK-Ising model imply C2 condition immediately. Once we have proved that $\{\gamma_\delta\}_{\delta>0}$ satisfies C2 condition, we can use [KS17, Theorem 1.5] to get the tightness of $\{\phi_\delta(\gamma_\delta)\}_{\delta>0}$.

For the second step, we have constructed the following two discrete observables: For each edge $e \in E(\Omega_\delta^\diamond)$, the edge FK-fermionic observable is defined as

$$F_\delta(e) := \mathbb{E}\left[\mathbb{1}_{\{e\in\gamma_\delta\}} e^{\frac{i}{2}W_{\gamma_\delta}(e,b_\delta^\diamond)}\right],$$

where $W_{\gamma_\delta}(e, b_\delta^\diamond)$ is the total rotation in radians of the interface γ_δ from e to $b_\delta^\diamond$ and we always assume the edge connecting to $b_\delta^\diamond$ is horizontal(otherwise, we add an edge at $b_\delta^\diamond$). For each $v \in V(\Omega_\delta^\diamond)$, the vertex FK-fermionic observable is defined as

$$F_\delta(v) = \frac{1}{2}\sum_{v\sim e} F_\delta(e).$$

Theorem 1.4. *Suppose $(\Omega_\delta^\diamond; a_\delta^\diamond, b_\delta^\diamond)$ converges to $(\Omega; a, b)$ in the Carathéodory sense. Then, for the vertex FK-fermionic observable F_δ, we have*

$$\frac{1}{\sqrt{2\delta}}F_\delta \to \sqrt{\phi'} \quad \textit{locally uniformly},$$

where ϕ is any conformal map from $(\Omega; a, b)$ onto $(\mathbb{R} \times (0, 1); -\infty, +\infty)$.

Now, we come to the third step: deriving the law of driving function of any sublimit from the observable. For any stopping time τ_δ for γ_δ, denote by $F_\delta^{\tau_\delta}$ the vertex fermionic observable on $(\Omega_\delta^\diamond \setminus \gamma_\delta[0, \tau_\delta]; \gamma_\delta(\tau_\delta), b_\delta^\diamond)$.

Lemma 1.5. *For every $v \in V(\Omega_\delta^\diamond)$, the process $\{F_\delta^n\}$ is a martingale before the hitting time of v.*

Proof. By domain Markov property of FK-Ising model, it suffices to check

$$\mathbb{E}[F_\delta^{\tau_\delta}(v)] = F_\delta(v)$$

for every stopping time τ_δ before the hitting time of v. By definition of vertex FK-fermionic observable, it suffices to check

$$\mathbb{E}[F_\delta^{\tau_\delta}(e)] = F_\delta(e)$$

for every $e \in E(\Omega_\delta^\diamond)$ adjacent to v. This follows from domain Markov property of FK-Ising model and the fact that $W_{\gamma_\delta}(e, b_\delta^\diamond)$ does not depend on $\gamma_\delta[0, \tau_\delta]$. This completes the proof. □

Proof of Theorem 1.1. Suppose $\psi(\gamma)$ is any sublimit. We may assume that $\psi_\delta(\gamma_\delta) \to \psi(\gamma)$ in law under the metric (1.1). By Theorem 1.4 and Lemma 1.5, we know that for every

$z \in \Omega$, we have $\{\sqrt{\phi_t'(z)}\}$ is a martingale before the hitting time of z, where ϕ_t is the conformal map from $(\Omega \setminus \gamma[0,t]; \gamma(t), b)$ onto $(\mathbb{R} \times (0,1); -\infty, +\infty)$. Define $M_t := \sqrt{\phi_t'(z)}$.

To derive the law of driving function of γ, we may assume $(\Omega; a, b)$ is $(\mathbb{H}; 0, \infty)$. Denote by $\{g_t : t \geq 0\}$ the corresponding conformal maps of γ and by W the driving function. Then, we have

$$\phi_t(z) = \frac{1}{\pi} \log(g_t(z) - W_t).$$

This implies that

$$W_t = g_t(z) - \frac{1}{\pi} \frac{g_t'(z)}{M_t^2}.$$

Thus, W_t is a semimartingale. Suppose $W_t = N_t + L_t$, where $\{N_t\}_{t \geq 0}$ is a matingale and $\{L_t\}_{t \geq 0}$ is a bounded variation process. Then, we have

$$dM_t(z) = \frac{3}{8\sqrt{\pi}} \sqrt{\frac{g_t'(z)}{(g_t(z) - W_t)^5}} \left(d\langle W \rangle_t - \frac{16}{3} dt \right) + \frac{1}{2\sqrt{\pi}} \sqrt{\frac{g_t'(z)}{(g_t(z) - W_t)^3}} dL_t.$$

As the drift term of $M_t(z)$ vanishes, we have

$$dL_t = 0, \quad d\langle W \rangle_t = \frac{16}{3} dt.$$

This implies that W_t has the same law as $\sqrt{\frac{16}{3}} B_t$, where B has the law of a standard Brownian motion. This completes the proof. $\square$

2 Convergence of spin-Ising model with positive-negative-free boundary conditions

2.1 Preliminaries and Setup

Let Ω_δ be a discrete simply connected domain on $\mathbb{L}_\delta$ and let $\Omega_\delta^\diamond$ be the subgraph of $\delta\mathbb{Z}^2$ which contains all the faces intersecting with Ω_δ. Fix three boundary points $a_\delta, b_\delta, c_\delta$(we allow that b_δ equals c_δ) on $\partial\Omega_\delta$. Denote by $a_\delta^\diamond$(resp. $b_\delta^\diamond, c_\delta^\diamond$) $\in V(\Omega_\delta^\diamond)$ the vertices nearst to a_δ(resp. b_δ, c_δ). Suppose the boundary conditions are positive-negative-free: σ equal $+1$ on faces which are along the outside of $(a_\delta b_\delta)$ and equal -1 on faces which are along the outside of $(c_\delta a_\delta)$. Recall that the spin-Ising model on Ω_δ is defined as follows: for every spin configuration σ, we have

$$\mathbb{P}[\sigma] = \frac{1}{Z_{sp}} e^{\beta \sum_{x \sim y} \sigma(x)\sigma(y)},$$

where the sum is taken over the set of pairs of adjacent faces separated by $E(\Omega_\delta)$, except for those edges that belong to the free arc $(b_\delta c_\delta)$. The interface γ_δ is the unique path on Ω_δ from a_δ to b_δ such that spins on its left are positive and spins on its right are negative(turning left when there is ambiguity). From now on, we always assume β equals the critical value, that is

$$\beta = \beta_c = \frac{1}{2}\log(\sqrt{2}+1).$$

We denote by ψ_δ the conformal map from $(\Omega; a, b, c)$ onto $(\mathbb{U}; -1, 1, i)$. We will consider two cases: the case that $b_\delta = c_\delta$ and the case that $b_\delta \neq c_\delta$.

2.2 Convergence of interfaces when $b_\delta = c_\delta$

We will prove the following theorem in this subsection:

Theorem 2.1. *Suppose $(\Omega_\delta; a_\delta, b_\delta)$ converges to $(\Omega; a, b)$ in the Carathéodory sense. Then, the random curve $\psi_\delta(\gamma_\delta)$ converges to SLE_3 from -1 to 1 as curves.*

Theorem 2.1 can be proved in the same way as Theorem 1.1: First, we prove the tightness of $\{\psi_\delta(\gamma_\delta)\}_{\delta>0}$. Second, we construct a discrete fermionic observable and prove the convergence. Third, we derive the law of any sublimit by using the observable.

For the first step, we still need to check C2 condition. In this case, we need the following result.

Corollary 2.2. *[CDH16, Corollary 1.7] For each $L > 0$, there exists $\eta = \eta(L) > 0$ such that the following holds: for any topological rectangle $(Q; x, y, z, w)$ such that the extremal distance between (xy) and (zw) is smaller than L,*

$$\mathbb{P}[\text{ there exists a crossing of } -1 \text{ spins connecting } (xy) \text{ and } (zw)] \geq \eta,$$

where $\mathbb{P}$ denotes the critical spin-Ising model on(the discrete approximation of) $(Q; x, y, z, w)$ with free boundary conditions on $(xy) \cup (zw)$ and $+1$ boundary conditions on $(yz) \cup (wx)$.

Proof of C2 condition. By domain Markov property of spin-Ising model, we only need to consider the case that $\tau_\delta = 0$. For any avoidable quadrilateral $Q = (Q; x, y, z, w)$, we assume that spins on $(yz) \cup (wx)$ all equal -1. The other case can be dealt similarly. Denote by $\mathbb{P}^Q$ the spin-Ising model on Q such that spins on the outside of $(xy) \cup (zw)$ equal $+1$ and spins on the outside of $(yz) \cup (wx)$ equal -1. Then, we have

$$\begin{aligned}\mathbb{P}[\gamma_\delta \text{ crosses } Q] &\leq \mathbb{P}^Q[\text{ there exists a crossing of } +1 \text{ spins connecting } (xy) \text{ and } (zw)] \\ &\qquad\qquad \text{(by monotonicity)} \\ &= 1 - \mathbb{P}^Q[\text{ there exists a crossing of } -1 \text{ spins connecting } (yz) \text{ and } (wx)] \\ &\qquad\qquad \text{(by duality)} \\ &\leq 1 - \eta. \qquad \text{(by monotonicity and Corollary 2.2)}\end{aligned}$$

For the second inequality, note that the extremal distance between (yz) and (wx) equals $\frac{1}{m(Q)}$. This completes the proof. □

Once we have proved that $\{\gamma_\delta\}_{\delta>0}$ satisfies C2 condition, by [KS17, Theorem 1.5], we can get the tightness of $\{\psi_\delta(\gamma_\delta)\}$.

For the second step, we can construct the discrete fermionic observable in a similar way as before: For every $z_\delta^\diamond \in V(\Omega_\delta^\diamond)$, let $\mathcal{E}$ is the set of collections of contours drawn on Ω_δ composed of loops and one interface γ_δ from $a_\delta^\diamond$ to $z_\delta^\diamond$. Define

$$F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}(z_\delta^\diamond) := \frac{\sum_{\omega\in\mathcal{E}(a_\delta^\diamond,z_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(a_\delta^\diamond,z_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}{\sum_{\omega\in\mathcal{E}(a_\delta^\diamond,b_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(a_\delta^\diamond,b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}},$$

where W_{γ_δ} is the total rotation in radians. We still assume the edge connecting to $b_\delta^\diamond$ is horizontal. Let φ be any conformal map from Ω onto $\mathbb{H}$ such that $\varphi(a)=\infty$ and $\varphi(b)=0$.

Theorem 2.3. *Suppose that $(\Omega_\delta; a_\delta, b_\delta)$ converges to $(\Omega; a, b)$ in the Carathéodory sense and $\partial\Omega$ is smooth near b, then*

$$F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)} \to \sqrt{\frac{\varphi'}{\varphi'(b)}} \quad \textit{locally uniformly.}$$

We have given a sketch of the proof in lectures before. Now, we come to the third step: deriving the law of driving function from the observables. For any stopping time τ_δ of γ_δ, define

$$F_{(\Omega_\delta\setminus\gamma_\delta[0,\tau_\delta];\gamma_\delta(\tau_\delta),b_\delta^\diamond)}(z_\delta^\diamond) := \frac{\sum_{\omega\in\tilde{\mathcal{E}}(\gamma_\delta(\tau_\delta),z_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(\gamma_\delta(\tau_\delta),z_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}{\sum_{\omega\in\tilde{\mathcal{E}}(\gamma_\delta(\tau_\delta),b_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(\gamma_\delta(\tau_\delta),b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}},$$

where $\tilde{\mathcal{E}}$ is the corresponding set of contours on $\Omega_\delta \setminus \gamma_\delta[0,\tau_\delta]$.

Lemma 2.4. *For every $z_\delta^\diamond \in V(\Omega_\delta^\diamond)$, the fermionic observable $\{F_{(\Omega_\delta\setminus\gamma_\delta[0,n];\gamma_\delta(n),b_\delta^\diamond)}(z_\delta^\diamond)\}$ is a martingale before the hitting time of $z_\delta^\diamond$.*

Proof. By domain Markov property, it suffices to check

$$\mathbb{E}\left[F_{(\Omega_\delta\setminus\gamma_\delta[0,\tau_\delta];\gamma_\delta(\tau_\delta),b_\delta^\diamond)}(z_\delta^\diamond)\right] = F_{(\Omega_\delta;\gamma_\delta(n),b_\delta^\diamond)}(z_\delta^\diamond),$$

for every stopping time τ_δ before the hitting time of $z_\delta^\diamond$. Note that

$$\begin{aligned}\mathbb{E}\left[F_{(\Omega_\delta\setminus\gamma_\delta[0,\tau_\delta];\gamma_\delta(\tau_\delta),b_\delta^\diamond)}(z_\delta^\diamond)\right]=&\sum_\eta \mathbb{P}[\gamma_\delta[0,\tau_\delta]=\eta]F_{(\Omega_\delta\setminus\eta;\tilde{\eta},b_\delta^\diamond)}(z_\delta^\diamond)\\ =&\sum_\eta \frac{(\sqrt{2}-1)^{|\eta|}\times\sum_{\omega\in\tilde{\mathcal{E}}(\tilde{\eta},b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}{\sum_{\omega\in\mathcal{E}(a_\delta^\diamond,b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}\\ &\times\frac{\sum_{\omega\in\tilde{\mathcal{E}}(\tilde{\eta},z_\delta^\diamond)}e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(\tilde{\eta},z_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}{\sum_{\omega\in\tilde{\mathcal{E}}(\tilde{\eta},b_\delta^\diamond)}e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(\tilde{\eta},b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega|}}\\ =&F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}(z_\delta^\diamond),\end{aligned}$$

where $\tilde{\eta}$ is the endpoint of η on $V(\Omega_\delta^\diamond)$. □

Proof of Theorem 2.3. Suppose $\psi(\gamma)$ is any sublimit. We may assume that $\psi_\delta(\gamma_\delta)\to\psi(\gamma)$ in law under the metric (1.1). By Theorem 2.3 and Lemma 2.4, for every $z\in\Omega$, we have that $\left\{\sqrt{\frac{\varphi_t'(z)}{\varphi_t'(b)}}\right\}$ is a martingale before the hitting time of z, where φ_t is any conformal map from $(\Omega\setminus\gamma[0,t];\gamma(t),b)$ onto $(\mathbb{H};\infty,0)$. Define $M_t(z):=\sqrt{\frac{\varphi_t'(z)}{\varphi_t'(b)}}$.

To derive the law of the driving function of γ, we may assume $(\Omega;a,b)$ to be $(\mathbb{H};0,\infty)$. Denote by $\{g_t : t\geq 0\}$ the corresponding conformal maps of γ and by W the driving function. Then,

$$M_t(z)=\left(\frac{g_t'(z)}{(g_t(z)-W_t)^2}\right)^{1/2}.$$

Thus, W_t is a semimartingale. Suppose $W_t=N_t+L_t$, where $\{N_t\}_{t\geq 0}$ is a matingale and $\{L_t\}_{t\geq 0}$ is a bounded variation process. Then, we have

$$dM_t(z)=\frac{\sqrt{g_t'(z)}}{(g_t(z)-W_t)^3}(-3dt+d\langle W\rangle_t)+\frac{\sqrt{g_t'(z)}}{(g_t(z)-W_t)^2}(dN_t+dL_t).$$

As the drift term of $M_t(z)$ vanishes, we have

$$dL_t=0,\quad \text{and } d\langle W\rangle_t=3dt.$$

This implies that W_t has the same law as $\sqrt{3}B_t$, where B has the law of a standard Brownian motion. This completes the proof. □

2.3 Convergence of interfaces when $b_\delta\neq c_\delta$

We will prove the following theorem in this subsection:

Theorem 2.5. *Suppose that* $(\Omega_\delta; a_\delta, b_\delta, c_\delta)$ *converges to* $(\Omega; a, b, c)$ *in the Carathéodory sense. Then, the random curve* $\psi_\delta(\gamma_\delta)$ *converges to* $\mathrm{SLE}_3\,(-3/2)$ *from* -1 *to* 1 *with marked point* i *as curves.*

$\mathrm{SLE}_3(-3/2)$ is a variant of SLE_3 with a marked point. We will give a concrete description in the proof. Theorem 2.5 can be proved in the same way as Theorem 2.1. The proof of tightness is same as the proof of the case that $b_\delta = c_\delta$. The construction of observable is different. In this case, the fermionic observable is defined as

$$F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}(z_\delta^\diamond) := \frac{\sum_{\omega\in\mathcal{E}(a_\delta^\diamond,z_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(a_\delta^\diamond,z_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega\setminus(b_\delta c_\delta)|}}{\sum_{\omega\in\mathcal{E}(a_\delta^\diamond,b_\delta^\diamond)} e^{-\frac{i}{2}W_{\gamma_\delta(\omega)}(a_\delta^\diamond,b_\delta^\diamond)}\left(\sqrt{2}-1\right)^{|\omega\setminus(b_\delta c_\delta)|}}.$$

For every $z \in \mathbb{H}$, define $f_{\mathbb{H},b}(z) := \frac{z-2b}{\sqrt{\pi z}\sqrt{b-z}}$ and define $f_{\Omega,b}(z) = \phi'(z)^{1/2} f_{\mathbb{H},\phi(b)}(\phi(z))$, where ϕ is any conformal map from Ω onto $\mathbb{H}$ such that $\phi(a) = 0$ and $\phi(c) = \infty$.

Theorem 2.6. *Suppose* $(\Omega_\delta; a_\delta, b_\delta, c_\delta)$ *converges* $(\Omega; a, b, c)$ *and suppose* (bc) *is smooth. Then,*

$$F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)} \to f_{\Omega,b} \quad \textit{locally uniformly.}$$

We only give a sketch of proof here. The complete proof can be found in [Izy15].

Lemma 2.7. $F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}$ *is s-holomorphic.*

Thus, we can define $H_\delta^\bullet$ and H_δ° as before. That is, for a pair of neighboring vertices $v \in V(\Omega_\delta)$ and $v^* \in V(\Omega_\delta^*)$, we put

$$H_\delta^\bullet(v) - H_\delta^\circ(v^*) = P_{l(e)}[F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}(x)]^2,$$

where $e = (x, y)$ is the edge crosses the edge (vv^*). Set $H_\delta^\circ(a_\delta^*) = 0$. Note that H° is also defined at faces of $\mathbb{L}_\delta \setminus \Omega_\delta$ adjacent to Ω_δ.

Lemma 2.8.

- *$H^\circ = 0$ at faces of $L_\delta \setminus \Omega_\delta$ adjacent to $(a_\delta b_\delta) \cup (c_\delta a_\delta)$ and $H^\bullet = 1$ at the vertices of $(b_\delta c_\delta)$.*
- *Set $H^\circ = 1$ at faces of $L_\delta \setminus \Omega_\delta$ adjacent to $(b_\delta c_\delta)$ and set $H^\bullet = 0$ on vertices of $L_\delta \setminus \Omega_\delta$ adjacent to $(a_\delta b_\delta) \cup (c_\delta a_\delta)$. Then, we have $\Delta H^\bullet \geq 0$ for every $v \neq a_\delta$ and $\Delta H^\circ \leq 0$ for any face in Ω_δ. In this case, the Laplacian is modified on the boundary: $\Delta H(z) = \sum_{w\sim z}(H(w) - H(z))$, where $c(z, w) = 1$ unless w is either a face of $L_\delta \setminus \Omega_\delta$ adjacent to $(b_\delta c_\delta)$, or a vertex of $L_\delta \setminus \Omega_\delta$ adjacent to $(a_\delta b_\delta) \cup (c_\delta a_\delta)$, in which case $c(z, w) = 2(\sqrt{2} - 1)$.*

Proof of Theorem 2.6(sketch). Though $H_\delta^\bullet$ is not subharmonic at a_δ, we may assume $H_\delta^\bullet$ is uniformly bounded on $\Omega_\delta \setminus B(a_\delta, r)$ for every fixed r. Then, the limit of $H_\delta^{\bullet,\circ}$, denote by

h has the following boundary conditions: it equals 0 on $(ab)\cup(ca)$ and it equals 1 on (bc). Thus, we have

$$h(z)=1-\frac{1}{\pi}\mathrm{Im}\left[\log(\phi(z)-\phi(b))+\frac{\alpha}{\phi(z)}\right]\quad\text{for some }\alpha\geq 0.$$

We still need to derive α and then we can derive the explict form of the limit of $F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}$. Note that

$$\partial_\omega\left(\log(\omega-\phi(b))+\frac{\alpha}{\omega}\right)=\frac{1}{\omega-\phi(b)}-\frac{\alpha}{w^2}\tag{2.1}$$

has two simple zeros on $(\phi(b),\infty)$ if $\alpha>4\phi(b)$ and a simple zero in $\mathbb{H}$ if $0<\alpha<4\phi(b)$. The former is impossible from the Harnack-type estimate that $\partial_n h$ is always positive on (bc), which implies that the derivative in (2.1) increases strictly. The later case is also impossible since this implies that the limit of $F_{(\Omega_\delta;a_\delta^\diamond,b_\delta^\diamond)}$ is not a single valued function. Thus, we only need to consider two cases: $\alpha=0$ or $\alpha=4\phi(b)$. The convergence result in positive-free boundary conditions case implies that the limit function should have singularity near a. This implies $\alpha=4\phi(b)$. This completes the proof. □

We can still define the process $\left\{F_{(\Omega_\delta\setminus\gamma_\delta[0,n];\gamma_\delta(n),b_\delta^\diamond)}(z_\delta^\diamond)\right\}$ as before.

Lemma 2.9. *For every $z_\delta^\diamond\in V(\Omega_\delta^\diamond)$, the fermionic observable $\left\{F_{(\Omega_\delta\setminus\gamma_\delta[0,n];\gamma_\delta(n),b_\delta^\diamond)}(z_\delta^\diamond)\right\}$ is a martingale before the hitting time of $z_\delta^\diamond$.*

Proof of Theorem 2.5. Suppose $\psi(\gamma)$ is any sublimit. We may assume that $\psi_\delta(\gamma_\delta)\to\psi(\gamma)$ in law under the metric (1.1). By Theorem 2.6 and Lemma 2.9, we know that for every $z\in\Omega$, we have $\left\{\phi_t'(z)^{1/2}f_{\mathbb{H},\phi_t(b)}(\phi_t(z))\right\}$ is a martingale before the hitting time of z, where ϕ_t is any conformal map from $(\Omega\setminus\gamma[0,t];\gamma(t),c)$ onto $(\mathbb{H};0,\infty)$. Define $M_t(z):=\phi_t'(z)^{1/2}f_{\mathbb{H},\phi_t(b)}(\phi_t(z))$.

To derive the law of the driving function of γ, we may assume $(\Omega;a,b)$ to be $(\mathbb{H};0,\infty)$. Denote by $\{g_t:t\geq 0\}$ the corresponding conformal maps of γ and by W the driving function. Then, $\phi_t(z)=\frac{g_t(z)-W_t}{g_t(z)-g_t(c)}$. Thus, we have

$$M_t=\frac{1}{\sqrt{\pi}}\left(\sqrt{\frac{g_t'(z)}{g_t(z)-g_t(c)}}-2\frac{\sqrt{(g_t(z)-g_t(c))g_t'(z)}}{g_t(z)-W_t}\right).$$

Thus, W_t is a semimartingale. Suppose $W_t=N_t+L_t$, where $\{N_t\}_{t\geq 0}$ is a matingale and $\{L_t\}_{t\geq 0}$ is a bounded variation process. By direct computation, we have

$$dM_t=\frac{2}{\sqrt{\pi}}\frac{\sqrt{g_t'(z)(g_t(z)-g_t(c))}}{(g_t(z)-W_t)^3}(3dt-d\langle W\rangle_t)-\frac{2}{\sqrt{\pi}}\frac{\sqrt{g_t'(z)(g_t(z)-g_t(c))}}{(g_t(z)-W_t)^2}\left(dN_t+dL_t-\frac{3}{2}\frac{1}{g_t(c)-W_t}\right).$$

As the drift term of $M_t(z)$ vanishes, we have

$$dL_t = \frac{3}{2}\frac{1}{g_t(c) - W_t}, \quad \text{and } d\langle W\rangle_t = 3dt.$$

This implies that W_t satisfies the following SDEs

$$\begin{cases} dW_t = \sqrt{3}dB_t - \frac{3}{2}\frac{1}{W_t - V_t}dt, \quad W_0 = 0; \\ dV_t = \frac{2}{V_t - W_t}dt, \quad V_0 = c. \end{cases}$$

This implies that γ has the same law as a $\mathrm{SLE}_3(-3/2)$ curve from 0 to ∞ with marked point c. This completes the proof. □

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