Minimal surfaces, WZW and multiple zeta values

GeometrieWerkstatt

Current Developments in Mathematics and Physics
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Sebastian Heller
BIMSA

Motivating questions

What is the 'best' realization of a given surface in space?

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Minimal surfaces: critical points for the area functional

- $H = 0$
- CMC surfaces ($H = const.$): with fixed enclosed volume
- compact embedded examples only in $\mathbb{S}^3$

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- space of embedded minimal (CMC) surfaces in $\mathbb{S}^3$

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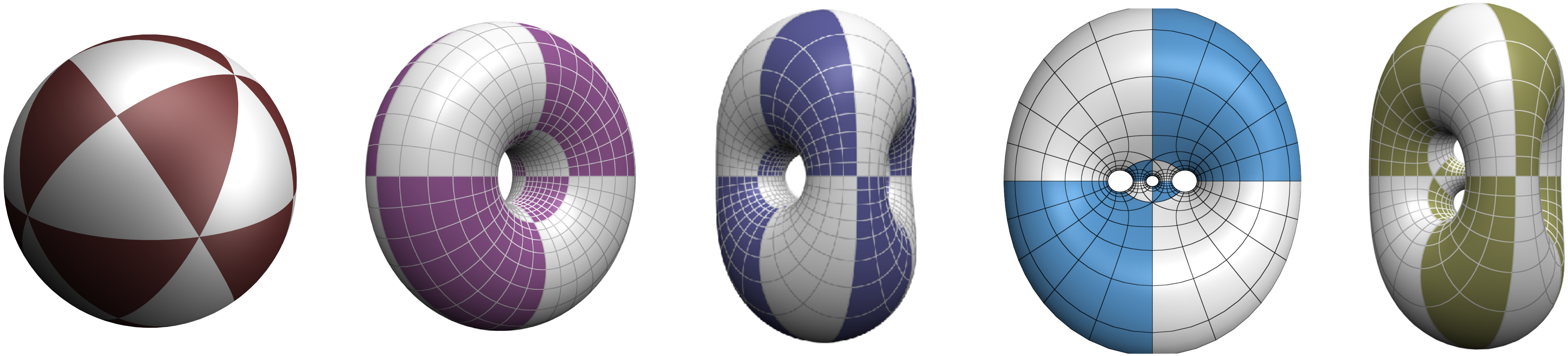
- Li-Yau: compact surfaces with $\mathscr{W} < 8\pi$ are embedded

Candidates

Lawson surfaces $\xi_{k,l}$ in $\mathbb{S}^3$



$\xi_{0,1}$ $\xi_{1,1}$ $\xi_{2,1}$ $\xi_{3,1}$ $\xi_{2,2}$

Images by N. Schmitt.

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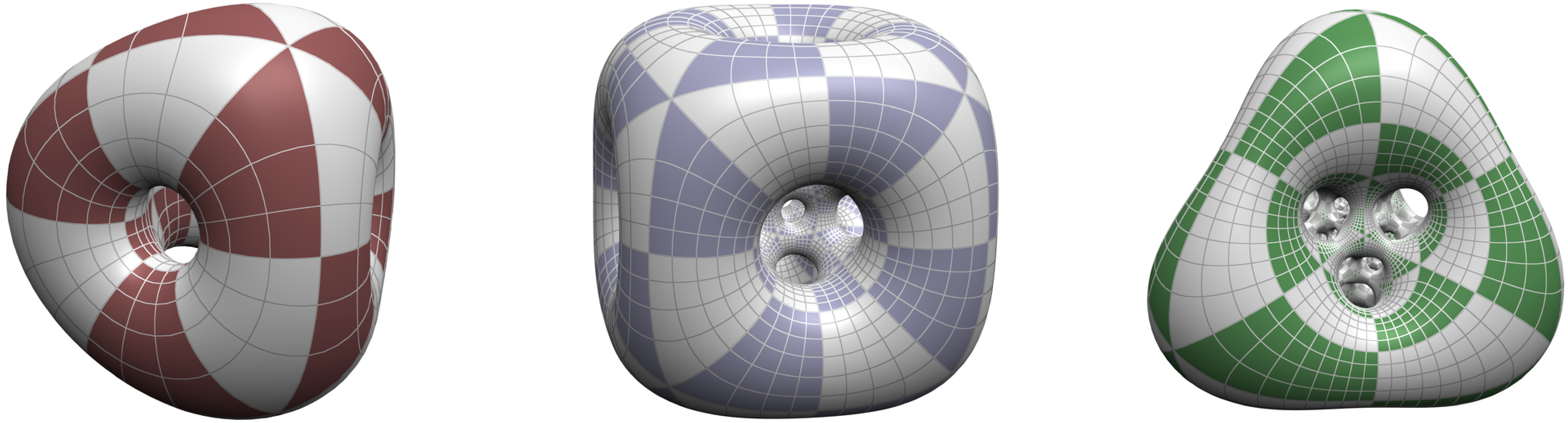
- gonality: $min(k+1, l+1)$?

  true for $l=1$, $k=l$ and $k \to \infty, l\ fixed$

Further examples

Karcher-Pinkall-Sterling



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Theorem [Charlton-H^2-Traizet]

The areas of the Lawson surfaces $\xi_{g,1}$ can be computed in terms of alternating multi zeta values and are strictly monotonic in the genus g.

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$$\zeta(\bar{n}) := \zeta(n; -1) = \sum_{0<k} \frac{(-1)^k}{k^n}$$, etc.

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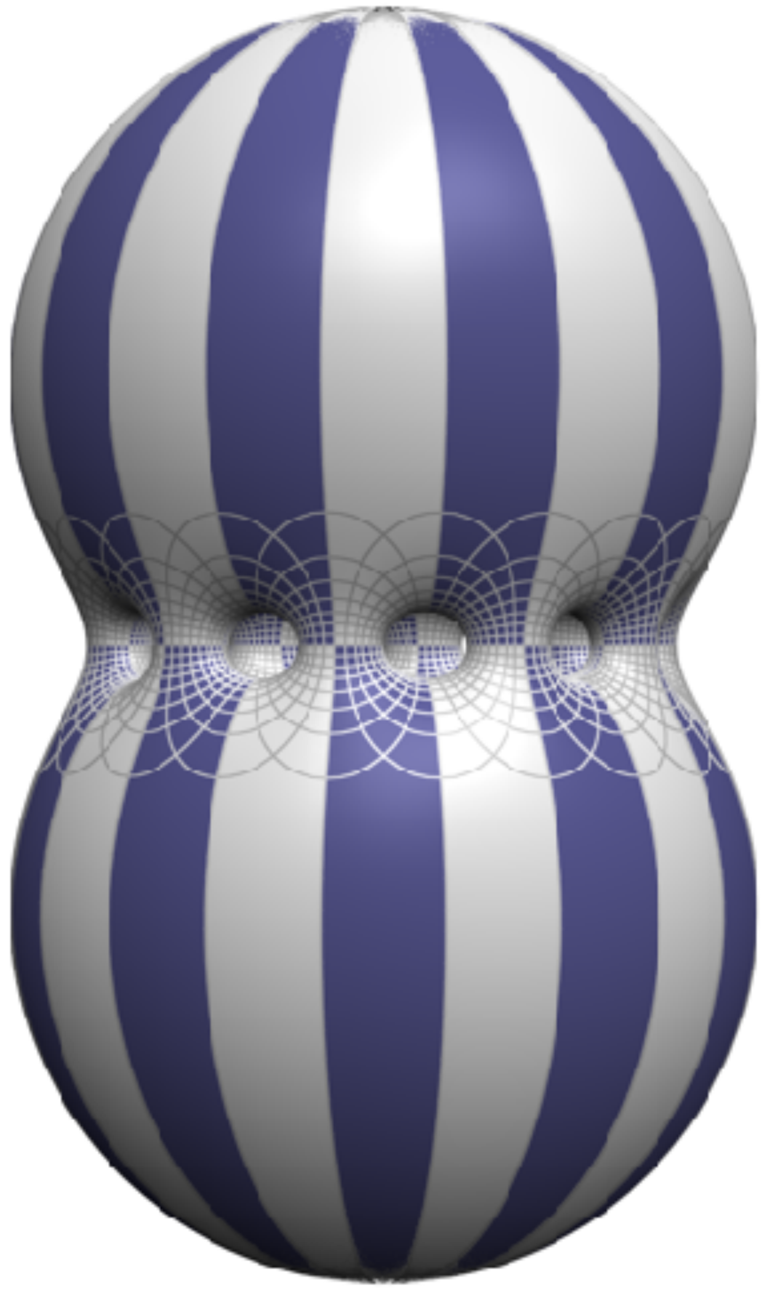
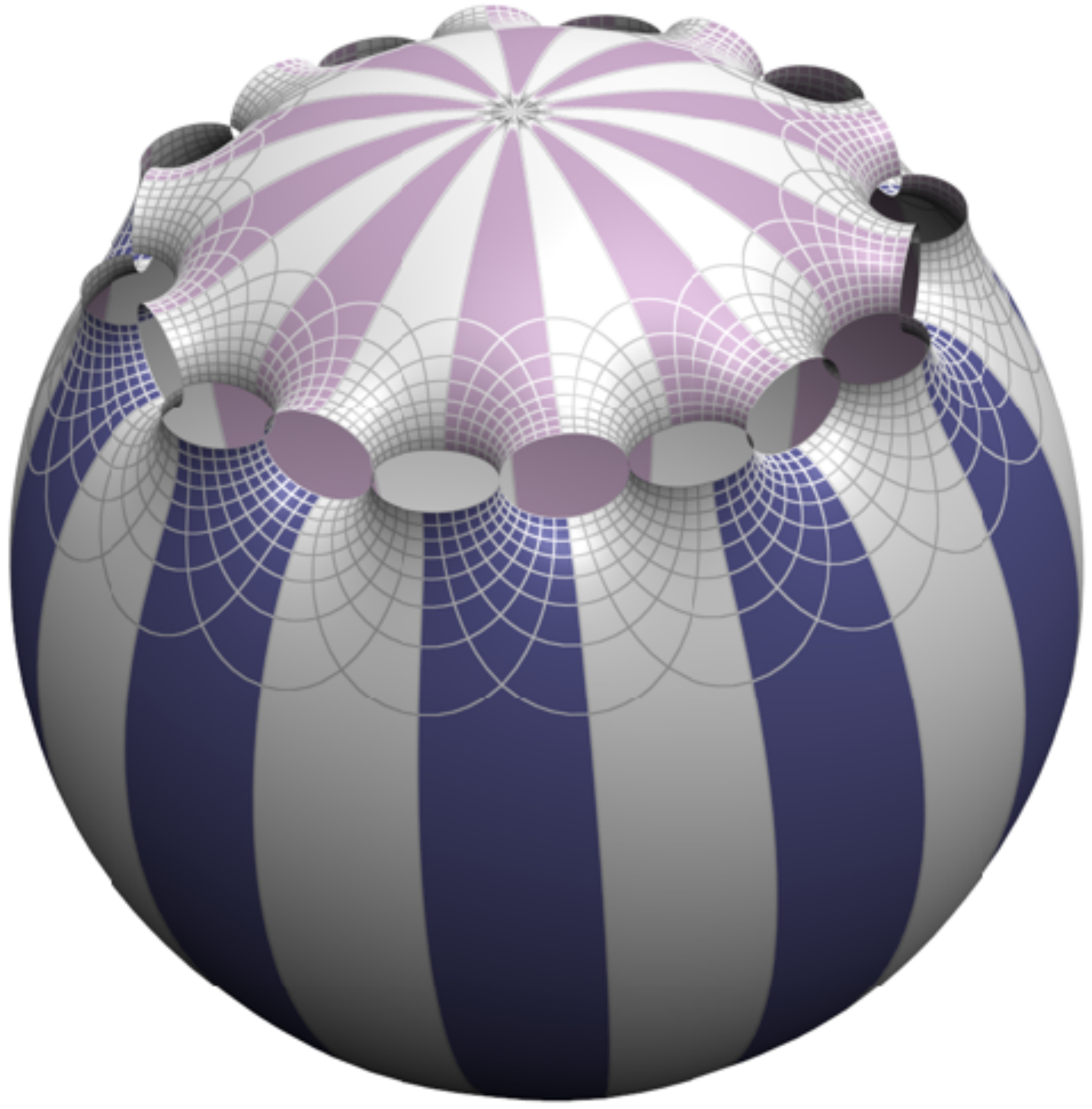
$\mathscr{A}(\xi_{\infty,1}) = 8\pi$





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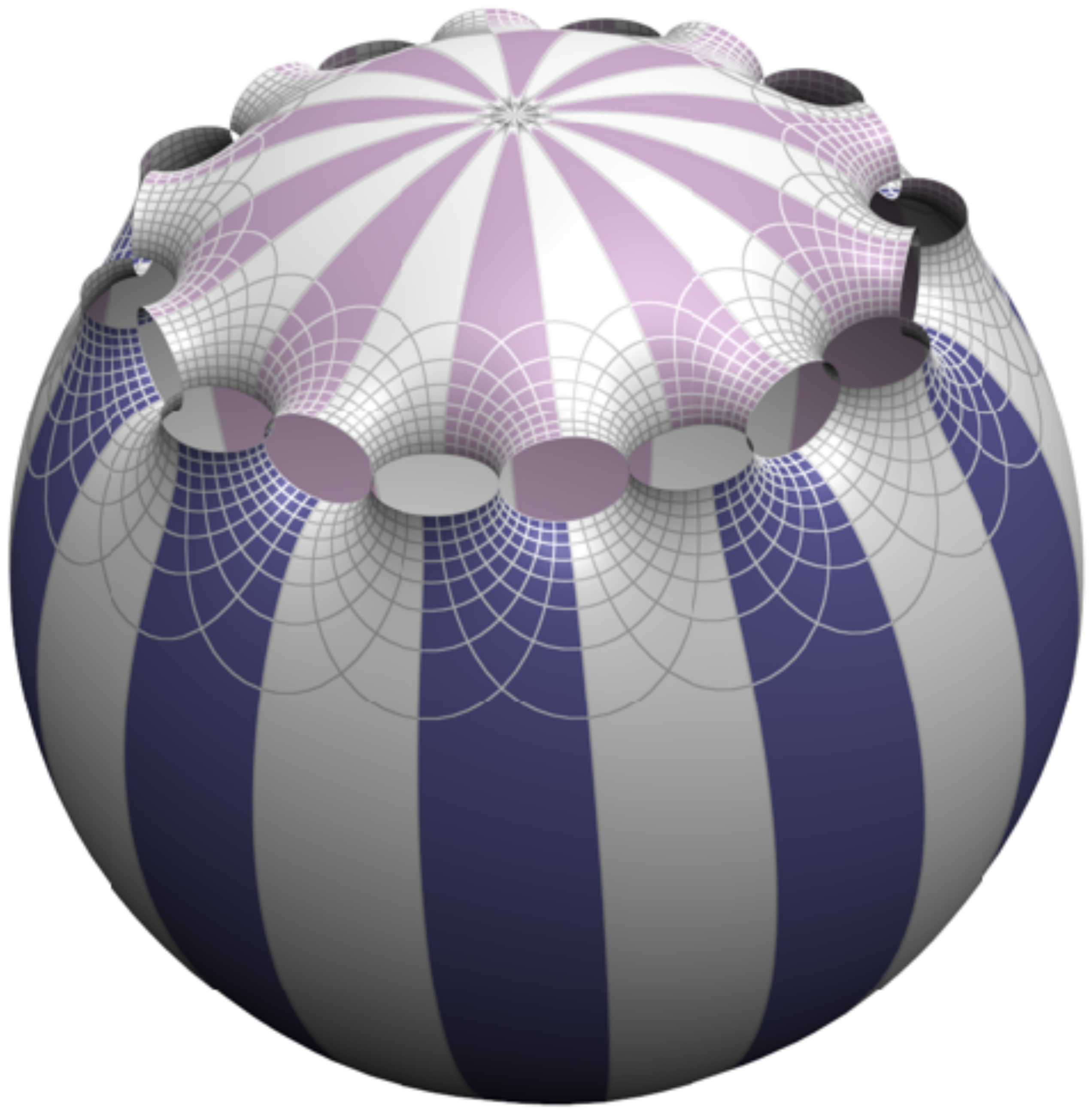
$$\mathscr{A}(\xi_{g,1}) = 8\pi\left(1 - \sum_k \frac{\alpha_{2k+1}}{(2g+2)^{2k+1}}\right)$$



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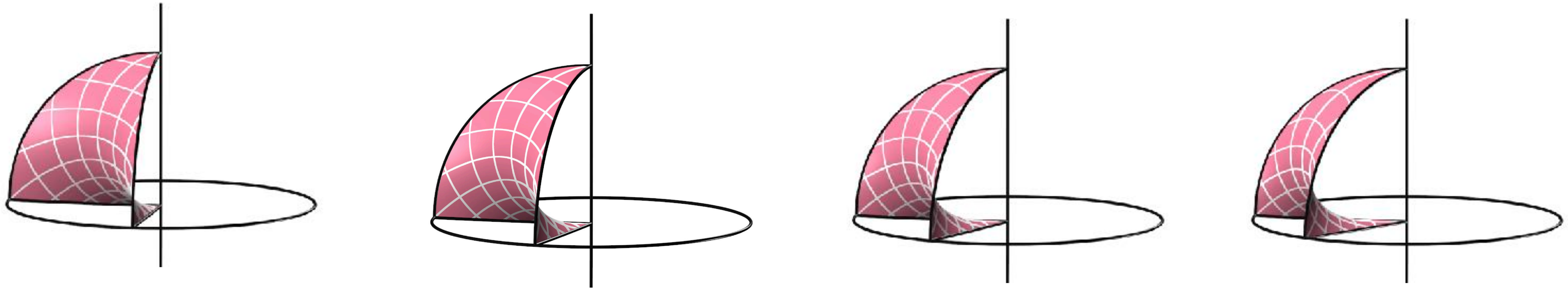
deformation via angle $\frac{\pi}{g+1} = 2\pi t$ of the geodesic polygon



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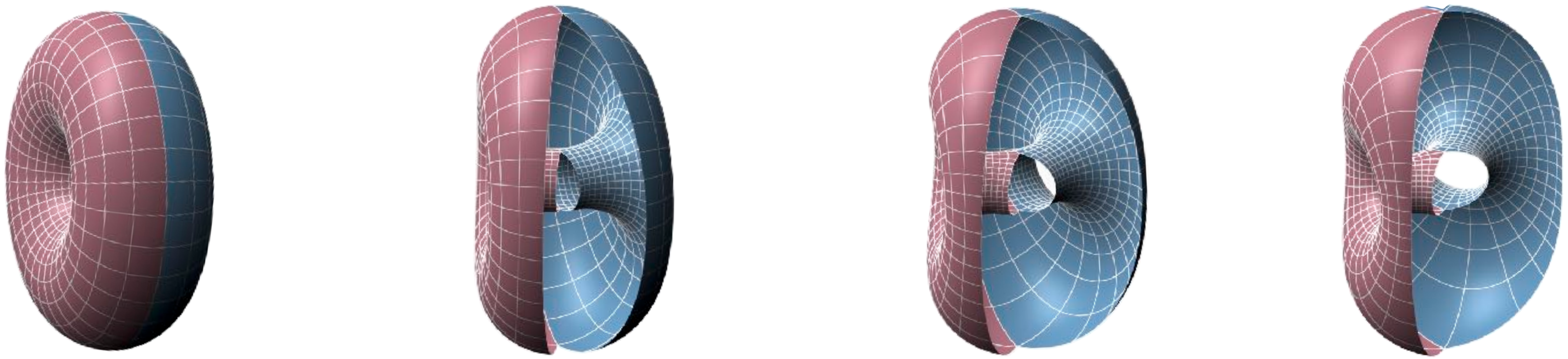

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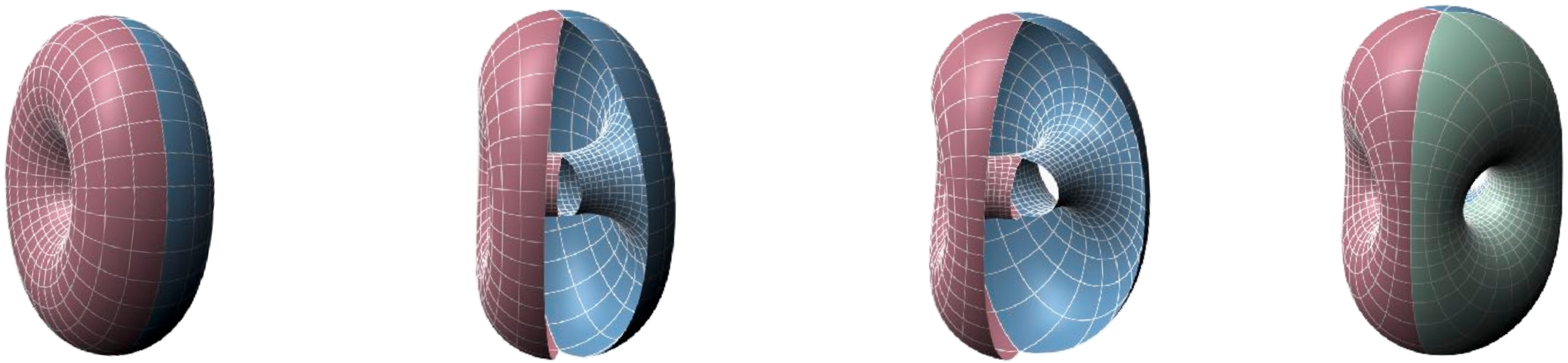

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- $\alpha_5 = -8\zeta(1,1,\bar{3}) + \frac{121}{16}\zeta(5) + \frac{2\pi^2}{3}\zeta(3) - 21\zeta(3)\zeta(\bar{1})^2$

with α_l of weight l

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- $\alpha_7 = -256\zeta(1,1,1,1,\bar{3}) + \frac{1392}{17}\zeta(1,1,\bar{5}) + \frac{720}{17}\zeta(1,3,\bar{3}) + 128\log^2(2)\zeta(1,1,\bar{3})$
  $+28\zeta(3)\zeta(1,\bar{3}) + \frac{296921}{1088}\zeta(7) - \frac{418\pi^2}{51}\zeta(5) - \frac{473\pi^4}{765}\zeta(3) - \frac{109}{2}\zeta(5)\log^2(2)$
  $+\frac{280}{3}\zeta(3)\log^4(2) - \frac{32\pi^2}{3}\zeta(3)\log^2(2) - 112\zeta(3)^2\log(2)$

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convergence radius $t \geq 0.137$

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The areas of the Lawson surfaces $\xi_{g,1}$ can be computed in terms of alternating multi zeta values and are strictly monotonic in the genus g for all $g \geq 0$.

convergence radius $t \geq 0.137$

full control for genus $g \geq 2.65$

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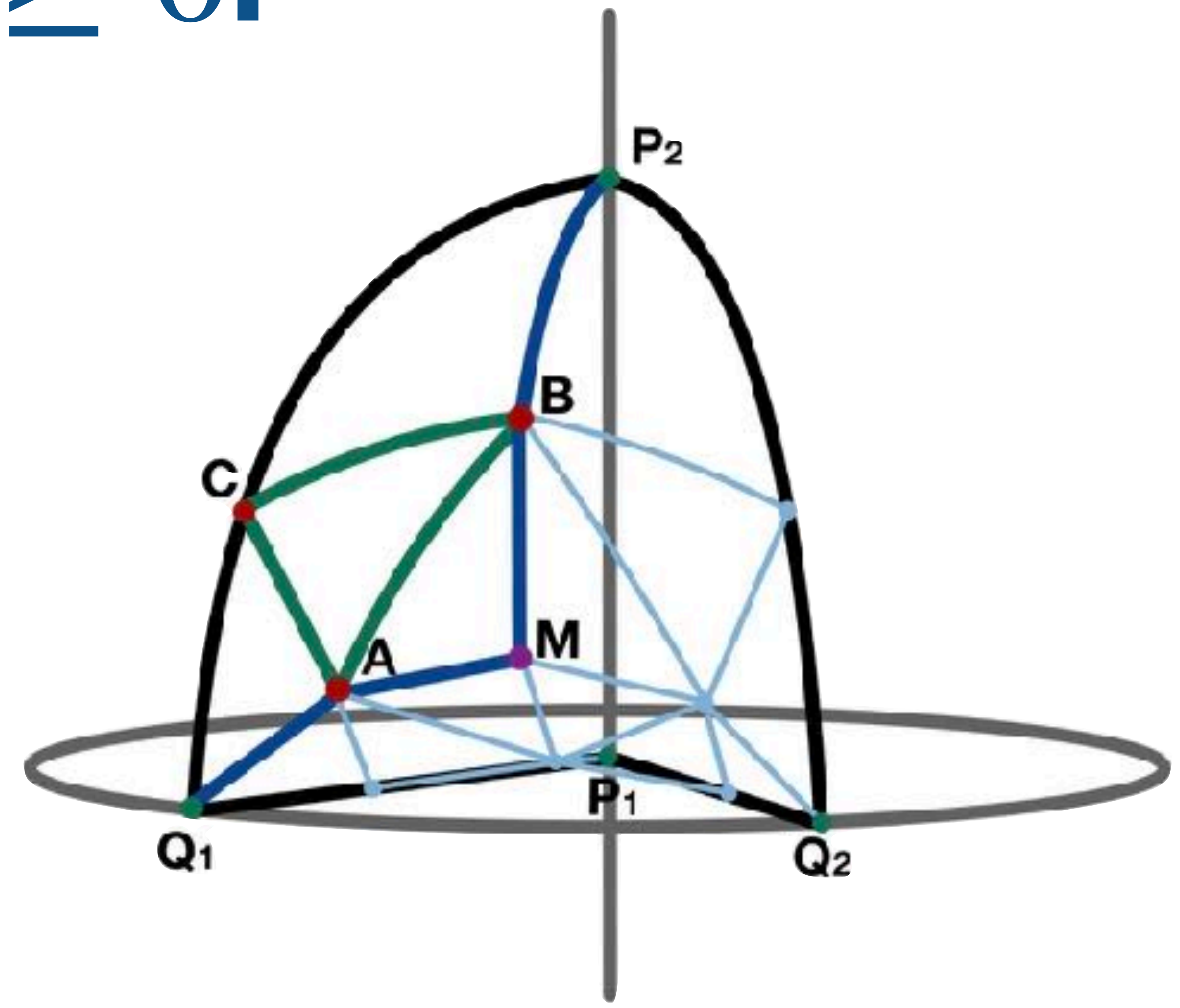
Using the resolution of the Willmore conjecture, only the area $g = 2$ needs to be estimated from above.



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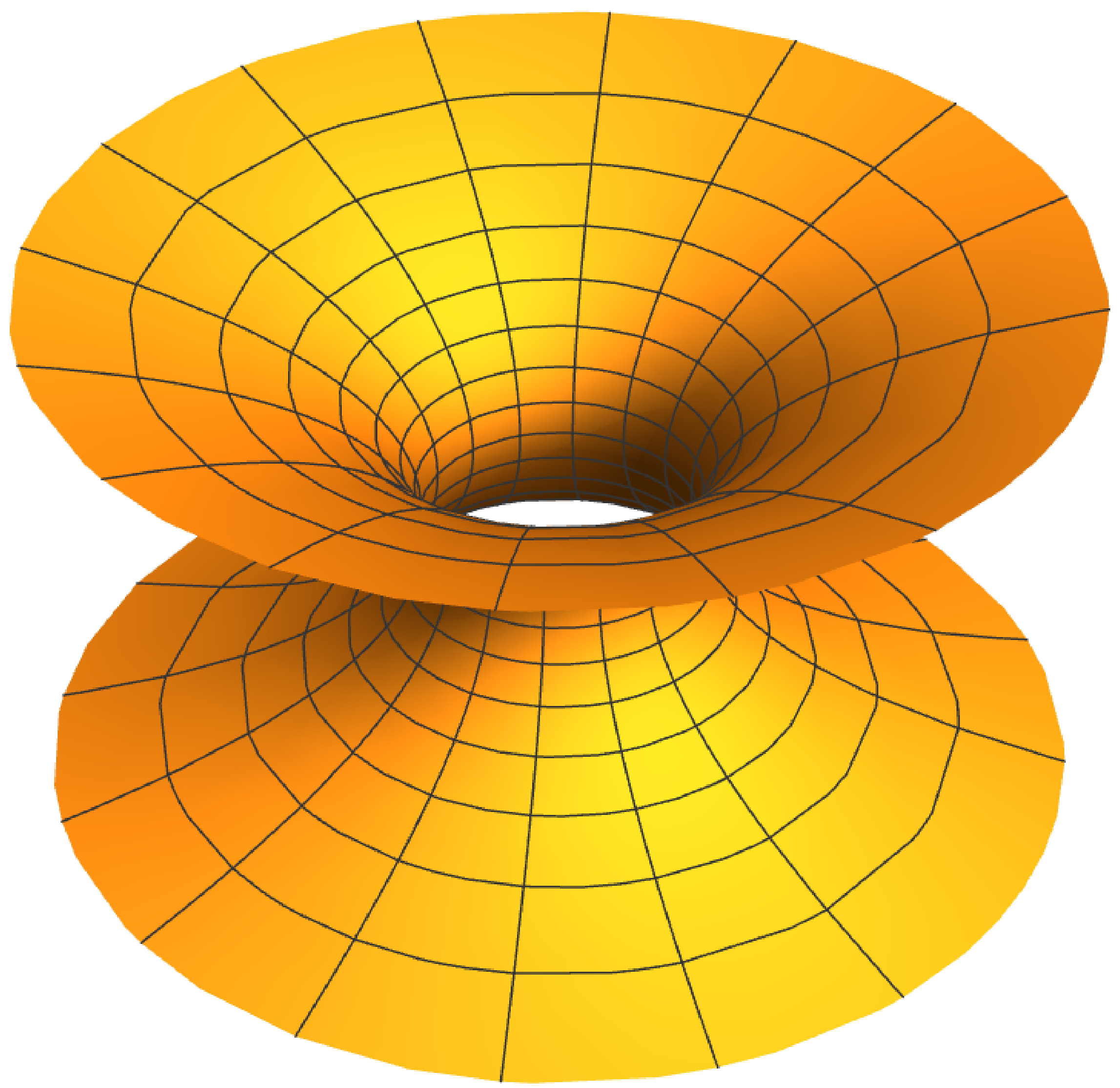

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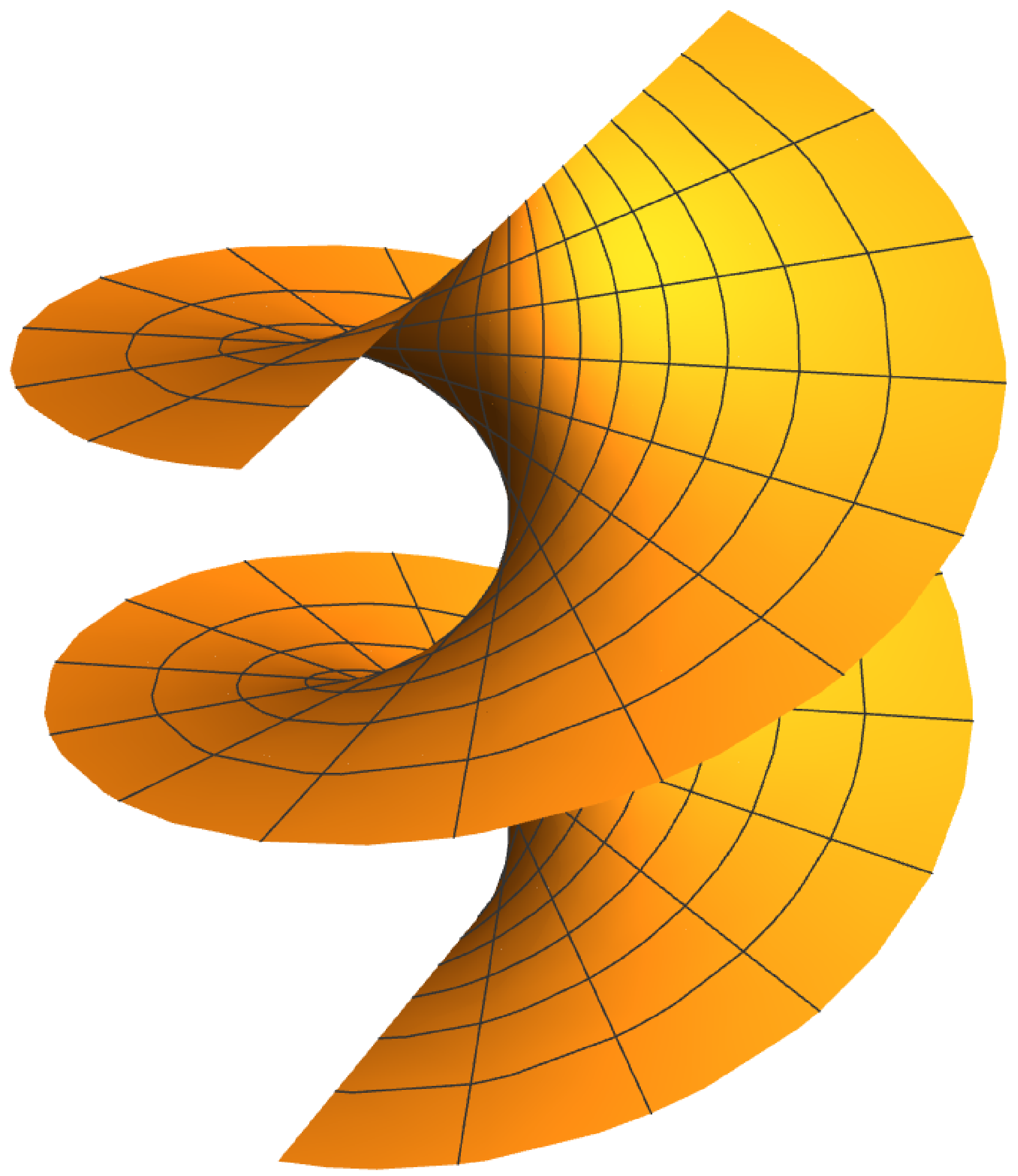

$\Rightarrow$ **(non-abelian) monodromy depending on** λ

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1	abelian	reducible	linear flow
>1	non-abelian	generically irreducible	?

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Consider 'compactified' moduli spaces of monodromies $\mathscr{M}$ for the associated families of ODEs (flat connections)

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monodromies satisfy reality conditions depending on the type of the iPDE, e.g., unitary for unimodular λ in the case of minimal surfaces

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The associated monodromy curve

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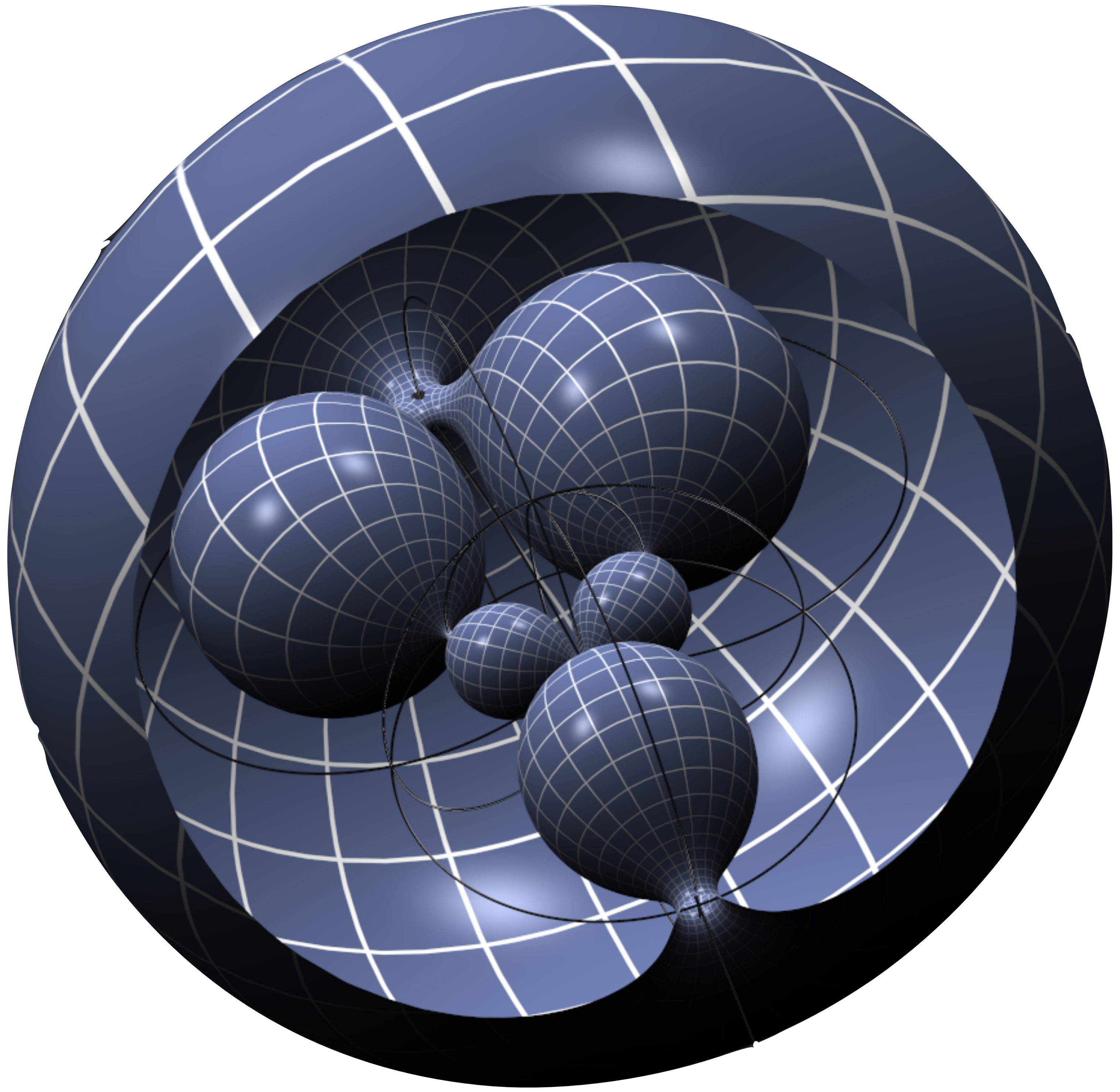

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- $A_i(t,\lambda) = \lambda^{-1}\sum A_i^k(\lambda)t^k$ with polynomial $A_i^k(\lambda)$
- expand monodromy via iterated integrals
- express MZVs and MPLs via iterated integrals

$$Li_{n_1,\dots,n_d}(z_1,\dots,z_d) = (-1)^d \int_L \frac{dw}{w-a_1}\{\frac{dw}{w}\}^{n_1-1}\dots\frac{dw}{w-a_d}\{\frac{dw}{w}\}^{n_d-1}$$

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- computable in terms of the residues A_i in the Fuchsian system representation

Conserved Quantities

Enclosed volume

- Chern-Simons line bundle $\mathscr{L} \to \mathscr{M}^u$ with unitary connection $\mathscr{D}$

$$(\nabla, \mu) \sim (\nabla . g, \Theta(\nabla, g)\mu) \text{ with}$$

$$\Theta(\nabla, g) = \exp(iCS(\nabla . g) - iCS(\nabla))$$

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- extension to CMC surface in $\mathbb{S}^3$ of genus $g \geq 2$

$$\mathrm{Hol}(\mathscr{D}, \gamma) = \exp\left(\frac{i}{\pi}\frac{c - \sin(c)}{4}\mathscr{W}(f) - \frac{i}{\pi}\mathscr{V}(f)\right) \qquad \text{for } c = 2\cot^{-1}(H)$$

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Isoperimetric Inequalities in $\mathbb{R}^3/\Gamma$

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- Example: hexagonal lattice

$$(a,b) = \frac{1}{2}(1,\sqrt{3}), \quad V = \frac{3}{4\pi}, \quad A_{conjecture} = A_{planes} = A_{cylinder} = \sqrt{3}$$

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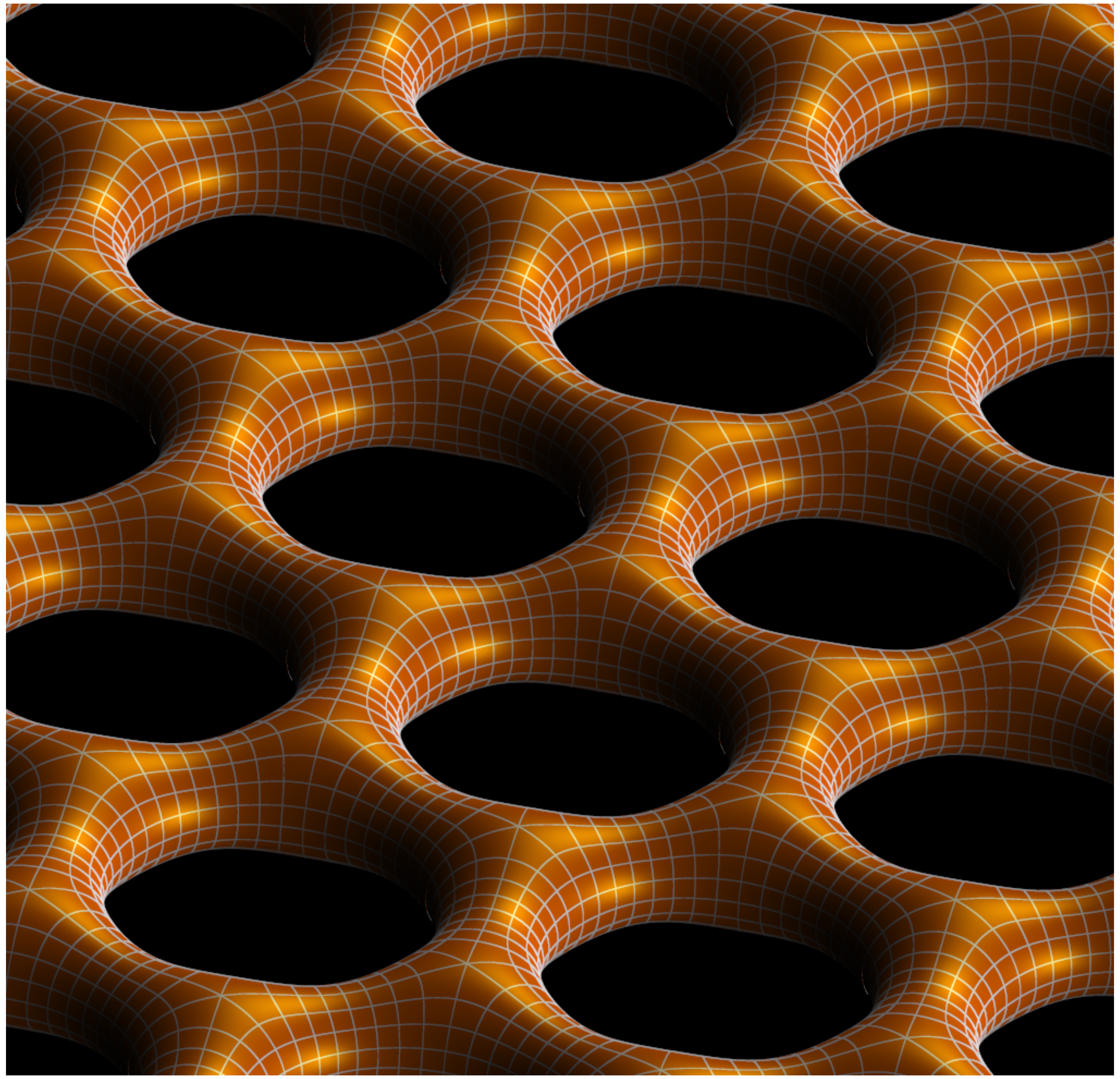

- Ros et. al.: a surface close to CMC cousin of Lawson $\xi_{2,2}$ with area less than $1.0003 \times \sqrt{3}$ is a possible competitor

Enclosed volume for CMC in $\mathbb{R}^3/\Gamma$

Theorem [Charlton-H^2-Traizet]

The enclosed volume can be computed in terms of the monodromy curve:

$$K = -\frac{i}{2\pi}\mathscr{A}(f) + \frac{3i}{2\pi}\mathscr{V}(f) - \frac{3i}{2\pi}\mathscr{V}(\Gamma_\Sigma)$$

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Theorem [Charlton-H[2]-Traizet]

The enclosed volume can be computed in terms of the monodromy curve:

$$K = -\frac{i}{2\pi}\mathscr{A}(f) + \frac{3i}{2\pi}\mathscr{V}(f) - \frac{3i}{2\pi}\mathscr{V}(\Gamma_\Sigma)$$

- next step: check on competitors!


Happy Birthday!