Lecture No 7 March 29, 2022 (Tue) (Lect-6 p.10∼)

9.5 Doob's inequality

[Theorem 9.6] (X_n): submartingale. Then, for $^\forall a > 0$,

$$(1) \quad aP\left(\max_{1\le k\le n} X_k \ge a\right) \le E\left[X_n, \max_{1\le k\le n} X_k \ge a\right] \le E[X_n^+]$$

$$(2) \quad aP\left(\min_{1\le k\le n} X_k \le -a\right) \le E[X_n - X_1] - E\left[X_n, \min_{1\le k\le n} X_k \le -a\right] \le E[X_n^+] - E[X_1],$$

where X_n^+ stands for the positive part of X_n. □

[Remark] When $n = 1$, by Chebyshev's inequality,

$$aP(X_1 \ge a) \le E[X_1^+], ^\forall a > 0$$

∵ (LHS)$= E[a1_{\{X_1 \ge a\}}] \le E[X_1^+]$ □

[Proof]

proof of (1): Using a Markov time $\sigma = \min\{k; X_k \geq a\}$, we can write $\{\max_{1\leq k\leq n} X_k \geq a\} = \{\sigma \leq n\}$ so that

$$P\left(\max_{1\leq k\leq n} X_k \geq a\right) = P(\sigma \leq n) \leq \frac{1}{a} E\left[X_\sigma, \sigma \leq n\right].$$

($\sigma = \infty$ may happen, but on the event $\{\sigma \leq n\}$, X_σ is well-defined.)

However, the expectation in RHS is estimated as

$$\begin{aligned} &= \sum_{k=1}^{n} E\left[X_\sigma, \sigma = k\right] = \sum_{k=1}^{n} E\left[X_k, \sigma = k\right] \\ &\leq \sum_{k=1}^{n} E\left[X_n, \sigma = k\right] = E\left[X_n, \sigma \leq n\right]. \end{aligned}$$

Here the inequality in the 2nd line is shown by using the submartingale property of (X_n): $X_k \leq E[X_n|\mathcal{F}_k]$ and noting that σ is Markov time which implies $\{\sigma = k\} \in \mathcal{F}_k$. $\therefore$ 1st inequality in (1) is shown. 2nd inequality in (1) is easy from

$$E\left[X_n, \sigma \leq n\right] \leq E\left[X_n^+, \sigma \leq n\right] \leq E\left[X_n^+\right].$$

proof of (2): Set $\sigma = \min\{k; X_k \le -a\}$. Then, since $\sigma \wedge n$ ($= \min\{\sigma, n\}$) is a bounded Markov time, we have

$$E[X_1] \underset{\text{Theorem 9.4}}{\le} E[X_{\sigma\wedge n}] = E[X_{\sigma\wedge n}, \sigma \le n] + E[X_{\sigma\wedge n}, \sigma > n].$$

Noting that $X_{\sigma\wedge n} = X_\sigma \le -a$ for the 1st term when $\sigma \le n$ and $X_{\sigma\wedge n} = X_n$ for the 2nd term when $\sigma > n$, the above is further bounded by

$$\begin{aligned} &\le -aP(\sigma \le n) + E[X_n, \sigma > n] \\ &= -aP(\sigma \le n) + E[X_n] - E[X_n, \sigma \le n] \end{aligned}$$

Thus, we obtain

$$aP(\sigma \le n) \le E[X_n - X_1] - E[X_n, \sigma \le n]$$

This is the 1st inequality in (2). The 2nd inequality in (2) follows from

$$E[X_n] - E[X_n, \sigma \le n] = E[X_n, \sigma > n] \le E[X_n^+, \sigma > n] \le E[X_n^+].$$

□

[Corollary] (M_n): p-th power integrable (i.e., $E[|M_n|^p] < \infty$, $p \geq 1$) martingale. Then,

$$P\left(\max_{1\leq k\leq n} |M_k| \geq a\right) \leq \frac{1}{a^p} E\left[|M_n|^p\right], \quad \forall a > 0. \qquad \square$$

☺ We can apply Theorem 9.6-(1) for $(|M_n|^p)$, which is submartingale by Jensen's inequality. Take a^p instead of a. $\square$

• We used Kolmogorov's inequality to show the strong law of large numbers:

[Lemma 7.2] $\{X_n\}_{n=1}^{\infty}$: $\perp\!\!\!\perp$, $E[X_n] = 0$, $V_n = \mathrm{Var}(X_n) < \infty$

$$\Longrightarrow P\left(\max_{1\leq k\leq n} \left|\sum_{i=1}^{k} X_i\right| \geq a\right) \leq \frac{1}{a^2} \sum_{i=1}^{n} V_i, \quad {}^{\forall} a > 0 \qquad \square$$

However, this is a special case of the above Corollary taking $p = 2$ and $M_k = \sum_{i=1}^{k} X_i$.

9.6 Submartingale convergence theorem

As before, a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_n)_{n=1,2,\dots}$ are given.

① Martingale transform

For

$H = (H_n)_{n=2,3,\dots}$: predictable, i.e., H_n is $\mathcal{F}_{n-1}$–measurable,
$X = (X_n)_{n=1,2,\dots}$: $(\mathcal{F}_n)$–adapted,

we define martingale transform $H \cdot X = ((H \cdot X)_n)_{n=1,2,\dots}$ as

$$(H \cdot X)_n = \begin{cases} \sum_{k=2}^{n} H_k(X_k - X_{k-1}), & n \geq 2 \\ 0 \qquad\qquad\qquad , & n = 1 \end{cases}$$

[Remark] This is a prototype of a stochastic integral.

[Theorem 9.7] Assume each H_n is bounded: $\sup_\omega |H_n(\omega)| < \infty$.
(1) $X = (X_n)$: martingale $\Longrightarrow$ $H \cdot X$ is also martingale.
(2) $X = (X_n)$: submartingale and $H_n \geq 0$
$\Longrightarrow$ $H \cdot X$ is also submartingale. □

⊙ We easily see that $(H \cdot X)_n$ is $\mathcal{F}_n$–measurable and integrable (note the boundedness of H). Moreover, for $n \geq 1$,

$$\begin{aligned} &E[(H \cdot X)_{n+1} - (H \cdot X)_n | \mathcal{F}_n] \\ &\quad = E[H_{n+1}(X_{n+1} - X_n) | \mathcal{F}_n] \\ &\quad = H_{n+1} E[X_{n+1} - X_n | \mathcal{F}_n]. \end{aligned}$$

This shows (1), (2). □