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§2. Variational structure.

By definition, E, J, E_γ, H, M can be extended to $\mathcal{E}^1$.

On $\mathcal{H}$, we have the variational formula along a path $u_t = u + tv \in \mathcal{H}$.

$$\frac{d}{dt}\Big|_{t=0} E(u_t) = V^{-1}\int_X v\,\omega_u^n$$

$$\frac{d}{dt}\Big|_{t=0} M(u_t) = \int_M v(\bar{S} - S(u))\,\omega_u^n.$$

The convexity of K-energy along geodesic is a very crucial question to understand this functional and find its critical pts.

Berman-Berndtsson (17): M is convex along $C^{1,1}$-geodesic.

Berman-Darvas-Lu (17): M is convex along psh geodesic.

Rmk: · M is not continuous in $\mathcal{E}^1$ w.r.t. strong topology, just l.s.c..

· In general, $\mathcal{E}^1$ has no differential structure. so we can not do variation on $\mathcal{E}^1$ for those functional.

But for E, we have

$$\frac{d}{dt}\Big|_{t=0} E(P(u+tv)) = V^{-1}\int_X v\,\omega_u^n, \qquad v \in C^0,\ u \in \mathcal{E}^1.$$

where $P(u) := \sup\{\psi \mid \psi \in PSH, \psi \le u\}$

Thm. (Chen-Cheng 17)

(X, ω_0) admits a cscK metric iff. M is coercive. i.e. $\exists\, \delta, C > 0$.

s.t. $M \geq \delta J - C$. on $\mathcal{H}$.

Thm (Berman-Darvas-Lu 17, Chen-Cheng 17)

TFAE:

(i) M is not coercive on $\mathcal{H}$.

(ii) $\forall u \in \mathcal{E}^1$ with $M(u) < +\infty$, there is a non-trivial psh geodesic ray starting at u, s.t. M is non-increasing along the ray.

correct the def of K-energy (previous note).

$$M(u) = H(u) + \bar{s} E(u) - E_{Ric(\omega_0)}(u).$$

§3. Non-Archimedean approach.

From now on, fix a polarised smooth variety (X, L) with $c_1(L) = [\omega_0]$.

Recall: test.configuration $(\mathcal{X}, \mathcal{L}) \xrightarrow{\pi} \mathbb{C}$

- π proj. flat. ($\mathcal{X}$: normal)
- $(\mathcal{X}, \mathcal{L})$ admits a $\mathbb{C}^*$-action equivariantly
- $(\mathcal{X}_t, \mathcal{L}_t) = (\pi^{-1}(t), \mathcal{L}|_{\pi^{-1}(t)}) \cong (X, L)$. $\forall t \neq 0$.
- $\mathcal{L}$ is π-semiample.

if $\mathcal{L}$ is not π-semiample, $(\mathcal{X}, \mathcal{L})$ is called a model of (X, L)

Two model $(\mathcal{X}_1, \mathcal{L}_1) \sim (\mathcal{X}_2, \mathcal{L}_2)$ if $\exists$ a model $(\mathcal{X}_3, \mathcal{L}_3)$ with two $\mathbb{C}^*$-equiv. birational morphism $\mu_i : \mathcal{X}_3 \to \mathcal{X}_i$, s.t. $\mu_1^* \mathcal{L}_1 = \mu_2^* \mathcal{L}_2$

if there is a $\mathbb{C}^*$-equivariant birational morphism $r_{\mathcal{X}_1, \mathcal{X}_2} : \mathcal{X}_1 \to \mathcal{X}_2$ of two model, then we say $\mathcal{X}_1$ dominates $\mathcal{X}_2$ and write $\mathcal{X}_1 \geq \mathcal{X}_2$. $\mathcal{X}$ is dominating if $\mathcal{X} \geq X_{\mathbb{C}}$

· Berkovich analytification

$k (= \mathbb{C})$ equipped with the *trivial* absolute value.
The Berkovich analytification X^{an} of X consists of all pairs $x = (\xi, |\cdot|)$, where $\xi \in X$ be a point and $|\cdot| = |\cdot|_x$ is a multiplicative norm on the residue field $\kappa(\xi)$ extending the trivial norm on k.
We denoted by $\mathcal{H}(x)$ the completion of $\kappa(\xi)$ w.r.t. this norm. The surjective map $\ker : X^{\mathrm{an}} \to X$ sending $(\xi, |\cdot|)$ to the ξ is called the *kernel* map.

$$X^{\mathrm{val}} = \{x = (\xi, |\cdot|) : \xi \text{ is the generic point of } X\}.$$

The point in X^{val} are the valuations of the function field of X that are trivial on k.
The *Berkovich topology*: for any open affine $U = \mathrm{Spec}(A) \subset X$ and any $f \in A$, the function $\ker^{-1}(U) \ni x \mapsto |f(x)| \in \mathbb{R}_+$ is continuous, where $f(x)$ denote the image of f in $\kappa(\xi) \subset \mathcal{H}(x)$, so that $|f(x)| = |f|_x$.
The subset $X^{\mathrm{val}} \subset X^{\mathrm{an}}$ is dense. X^{an} is Hausdorff, locally compact, and locally path connected; it is compact iff X is proper.

$\|\cdot\| : K^* \to \mathbb{R}_+$ valuation $\longrightarrow$ norm.

$v(f) \longmapsto e^{-v(f)} =: \|f\|$.

<u>seminorm:</u> $\cdot\ \|0\| = 0$

(non-Archi.) $\cdot\ \|f+g\| \le \max\{\|f\|, \|g\|\}$

$\cdot\ \|f\cdot g\| \le \|f\|\cdot\|g\|$. (multiplicative)

<u>Valuation:</u> $K = k(X) \supseteq k$.

a (real) valuation v is a <u>gp homomorphism</u> : $v : K^* \to (\mathbb{R}, +)$. s.t.

$v(f\cdot g) = v(f) + v(g)$

$\cdot\ v|_{k^*} = 0$

$\cdot\ v(f+g) \ge \min\{v(f), v(g)\}$

$\cdot$ Trivial valuation: $v_{triv}(f) := 0$. $\forall f \in K^*$

$X^{div}_{\mathbb{Q}}$:= the set of rational divisorial valuations on X.

i.e. $v \in X^{div}_{\mathbb{Q}} : k(X)^* \to \mathbb{Q}$, $\exists$. prime div. $E \subset Y \xrightarrow[bir]{\pi} X$, s.t.

$v = c\,\mathrm{ord}_E$. for some $c \in \mathbb{Q}_{>0}$

In particular, $X^{div}_{\mathbb{Q}} \subseteq X^{an}$ is dense.

$P_1 : X_{\mathbb{C}} := X \times \mathbb{C} \to X \rightsquigarrow (X_{\mathbb{C}})^{an} \longrightarrow X^{an}$ there exists a canonical cont. section.

σ. Gauss extension.

$\forall v \in X^{div}_{\mathbb{Q}}$. $\sigma(v)\left(\sum_i f_i \tau^i\right) = \min_i \{v(f_i) + i\}$ for finite $f_0, \cdots, f_r \in \mathbb{C}(X)$.

$\mathrm{Im}(\sigma) := \{ w \in (X_{\mathbb{C}})^{div}_{\mathbb{Q}} \mid \mathbb{C}^*\text{-inv. } w(\tau) = 1 \}$

where. $\mathbb{C}^*$-action: $\forall f \in K(t)$. $(a\cdot f)(t) = f(a^{-1}t)$. for $a \in k^*$

$w(a\cdot f) = w(f)$

Any dominating t.f. $p_{\mathcal{X}}: (\mathcal{X}, \mathcal{L}) \longrightarrow X_{\mathbb{C}}$ defines a positive non-Archimedean metric $\phi_{(\mathcal{X},\mathcal{L})}$ on (X^{an}, L^{an}), represented as a function on X^{an} as follows: for any $v \in X^{an}$

$$\phi_{(\mathcal{X},\mathcal{L})}(v) = G(v)(\mathcal{L} - p_{\mathcal{X}}^{*} L_{\mathbb{C}})$$

In fact, by definition of test.conf., $\mathcal{L} - p_{\mathcal{X}}^{*} L_{\mathbb{C}} = D$ for a unique $\mathbb{Q}$-Cartier div D supported on the central fiber.

$\mathcal{H}^{NA} := \mathcal{H}^{NA}(L)$ the set of (sm. positive) non-Archimedean metrics defined by test configuration.

Def: A psh metric on L^{an} is a function $\phi: X^{an} \longrightarrow \mathbb{R} \cup \{-\infty\}$, $(\not\equiv -\infty)$ that can be written as the limit of a decreasing sequence in $\mathcal{H}^{NA}$. Denoted by PSH^{NA}.

Definition

For any $\phi \in \mathrm{PSH}^{\mathrm{NA}}$, define

$$\mathbf{E}^{\mathrm{NA}}(\phi) = \inf\{\mathbf{E}^{\mathrm{NA}}(\psi);\ \psi \in \mathcal{H}^{\mathrm{NA}} \text{ and } \psi \geq \phi\}.$$

$$\mathcal{E}^{1,\mathrm{NA}} := \mathcal{E}^{1,\mathrm{NA}}(L) = \{\phi \in \mathrm{PSH}^{\mathrm{NA}}(L); \mathbf{E}^{\mathrm{NA}}(\phi) > -\infty\}.$$

where $\mathbf{E}^{\mathrm{NA}}(\psi) := \frac{1}{n+1}(\bar{\mathcal{L}}^{n+1})$ and $\psi = (\mathcal{X}, \mathcal{L})$.

A sequence $\{\phi_m\}$ in $\mathcal{E}^{1,\mathrm{NA}}$ converges strongly to $\phi \in \mathcal{E}^{1,\mathrm{NA}}$ if $\lim_{m\to+\infty}(\phi_m - \phi) = 0$ on X^{qm} (set of quasi-monomial points in X^{NA}) and $\lim_{m\to+\infty} \mathrm{E}^{\mathrm{NA}}(\phi_m) = \mathrm{E}^{\mathrm{NA}}(\phi)$.