Lecture 9. Vassiliev's invariants. The chord diagram algebra

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МФТИ

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Contents

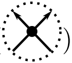
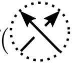
Contents

Each knot can be transformed to the unknot by switching some crossings. Any crossing switch can be thought of as performed in $\mathbb{R}^3$. Having a knot invariant f, one can consider its values on two knots that differ at only one crossing.

While switching the crossing continuously, the most interesting moment is the intersection moment: in this case we get what is called a singular knot. More precisely, a singular knot of degree n is an immersion of S^1 in $\mathbb{R}^3$ with only n simple transverse intersection points (i.e., points where two branches intersect transversely).

Singular knots are considered up to isotopy. The isotopy of singular knots is defined quite analogously to that for the case of classical knots. The set of singular knots of degree n (for $n = 0$ the set $\mathcal{X}_0$ consists of the classical knots) is denoted by $\mathcal{X}_n$. The set of all singular knots (including $\mathcal{X}_0$) is denoted by $\mathcal{X}$.
So, while switching a crossing of a classical knot, at some moment we get a singular knot of order one.

Then, we can define the derivative f' of the invariant f according to the following relation:

$f'($  $) = f($  $) - f($  $).$ (1)

This relation holds for all triples of diagrams that differ only outside a small domain (two of them represent classical knots and  represents the corresponding singular knot).

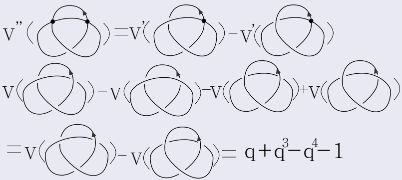
This relation is called the Vassiliev relation.

It is obvious that the invariant f' is a well-defined invariant of singular knots because with each singular knot and each vertex of it, we can associate the positive and the negative resolutions of it in $\mathbb{R}^3$. If we isotope the singular knot, the resolutions are "isotoped" together with it.

Having a knot invariant $f : \mathcal{X}_0 \to A$,[1] one can define all its derivatives of higher orders. To do this, one should take the same formula for two singular knots of order n and one singular knot of order $n+1$ (n singular vertices of each of them lie outside of the "visible" part of the diagram) and then apply the Vassiliev relation (1).
Thus, we define some invariant on the set $\mathcal{X}$. This invariant is called the extension of f for singular knots. Notation: $f^{(n)}$.

Example 1.1

Let us calculate the extension of the Jones polynomial evaluated on the simplest singular knot of order two. After applying the Vassiliev relation twice, we have:



[1] A can be a ring or a field; we shall usually deal with the cases of $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$

Definition 1.2

An invariant $f : \mathcal{X}_0 \to A$ is said to be a (Vassiliev) invariant of order $\leq n$ if its extension for the set of all $(n+1)$–singular knots equals zero identically.
Denote by $\mathcal{V}_n$ the space of all Vassiliev knot invariants of order less than or equal to n.

Definition 1.3

A Vassiliev invariant of order (type) $\leq n$ is said to have order n if it is not an invariant of order less than or equal to $n-1$.

Contents

The definition of the Vassiliev knot invariant shows us that if an invariant has degree zero then it has the same value on any two knots having diagrams with the same shadow that differ at precisely one crossing. Thus, it has the same value on all knots having the same shadow. Let K be a knot diagram, and S be the shadow of K. There is an unknot diagram with shadow S. So, the value of our invariant on K equals that evaluated on the unknot.

Thus, such an invariant is constant.

It turns out that the first order gives no new invariants (in comparison with 0–type invariants, which are constants).

Indeed, consider the simplest singular knot U shown in Fig. 1.

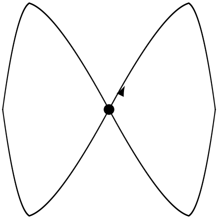


Figure 1: The simplest singular knot

Let S be a shadow of a knot with a fixed vertex which is a singular point. We will left the following statement as an exercise.

Corollary 2.1

Prove that one can arrange all other crossing types for S to get a singular knot isotopic to U.

It is easy to see that for each Vassiliev knot invariant I such that $\mathrm{I}'' = 0$ we have $\mathrm{I}'(\mathrm{U}) = 0$. Indeed, $\mathrm{I}'(\mathrm{U}) = \mathrm{I}($$) - \mathrm{I}($$) = 0$.

Now, consider an invariant I of degree less than or equal to one. Let K be an oriented knot diagram. By switching some crossing types, the knot diagram K can be transformed to some unknot diagram. Thus, $\mathrm{I}(\mathrm{K}) = \mathrm{I}(\bigcirc) + \sum \pm \mathrm{I}'(\mathrm{K_i})$ where $\mathrm{K_i}$ are singular knots with one singular point. But, each $\mathrm{K_i}$ can be transformed to some diagram U by switching some crossing types. Thus, $\mathrm{I}'(\mathrm{K_i}) = \mathrm{I}'(\mathrm{U}) + \sum \pm \mathrm{I}''(\mathrm{K_{ij}})$, where $\mathrm{K_{ij}}$ are singular knots of second order. By definition, $\mathrm{I}'' \equiv 0$, thus $\mathrm{I}'(\mathrm{K_i}) = 0$ and, consequently, $\mathrm{I}(\mathrm{K}) = \mathrm{I}(\bigcirc)$. Thus, the invariant function I is a constant. So, there are no invariants of order one.

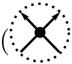
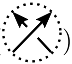
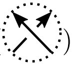
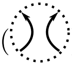
Contents

Consider the Conway polynomial C and its coefficients c_n.

Theorem 3.1

For each natural n, the function c_n is a knot invariant of degree less than or equal to n.

Proof.
Indeed, we just have to compare the Vassiliev relation and the Conway skein relation:

$c'_n($$) = c_n($$) - c_n($$) = x \cdot c_n($$).$

Thus we see that the first derivative of C is divisible by x; analogously, the n–th derivative of C is divisible by x^n. Thus, after $n+1$ differentiations, c_n vanishes. $\square$

This gives us the first non-trivial example. The second coefficient c_2 of the Conway polynomial is a second-order invariant (one can easily check that it is not constant; namely, its value on the trefoil equals one).

However, this invariant does not distinguish the two trefoils because the Conway polynomial itself does not. Later, we shall show how an invariant of degree three can distinguish the two trefoils.
As will be shown in the future, all even coefficients of the Conway polynomial give us finite–order invariants of corresponding orders.

Contents

As we have shown, each coefficient c_n of the Conway polynomial has order less than or equal to n.

Let v be a Vassiliev knot invariant of order n. By definition, $v^{(n+1)} = 0$. This means that if we take two singular knots K_1, K_2 of n–th order whose diagrams differ at only one crossing (one of them has the overcrossing and the other one has the undercrossing), then $v^{(n)}(K_1) = v^{(n)}(K_2)$. Thus, for singular knots of n–th order one can switch crossing types without changing the value of $v^{(n)}$. Hence, the value of $v^{(n)}$ does not depend on knottedness "that is generated" by classical crossings. It depends only on the order of passing singular points.

Definition 4.1

The function $v^{(n)}$ is called the symbol of v.

Now, it is clear that the space $\mathcal{V}_n/\mathcal{V}_{n-1}$ is just the space of symbols that can be considered in the diagrammatic language.
We shall show that for even n, the coefficient c_n of the Conway polynomial has order precisely n. Moreover, we shall calculate its symbols, according to [7].

Definition 4.2

By a chord diagram we mean a finite cubic graph consisting of one oriented cycle (circle) and unoriented chords (edges connecting different points on this cycle). The order of a chord diagram is the number of its chords.

Remark 4.3

Chord diagrams are considered up to natural graph isomorphism taking chords to chords, circle to the circle and preserving the orientation of the circle.

Remark 4.4

We shall never indicate the orientation of the circle on a chord diagram, always assuming that it is oriented counterclockwise.

The previous statements concerning singular knots can be put in formal diagrammatic language. Namely, with each singular knot one can associate a chord diagram that is obtained as follows. We think of a knot as the image of the standard oriented Euclidian S^1 in $\mathbb{R}^3$ and connect by chords the preimages of the same point in $\mathbb{R}^3$.

So, each invariant of order n generates a function on the set of chord diagrams with n chords. We can consider the formal linear space of chord diagrams with coefficients, say, in $\mathbb{Q}$, and then consider linear functions on this space generated by symbols of n–th order Vassiliev invariants (together with the constant zero function that has order zero).

Now, the main question is: Which functions on chord diagrams can play the role of symbols?

One–term relation

The simplest observation leads to the following fact. If we have a chord diagram C =  with a small solitary chord, then each symbol evaluated at this diagram equals zero. We have already discussed this in the language of singular knots.

This relation is called the 1T–relation (or one–term relation).

One can easily prove the generalised 1T–relation where we can take a diagram C =  with a chord that does not intersect any other chord. Then, each symbol of a Vassiliev knot invariant evaluated at the diagram C equals zero. The proof is left as an exercise.

Four–term relation

There exists another relation, consisting of four terms, the so-called 4T–relation. In fact, let us prove the following theorem.

Theorem 4.5 (The four–term relation)

For each symbol v^n of an invariant v of order n the following relation holds:

$v^n($  $) - v^n($  $) - v^n($  $) + v^n($  $) = 0.$

This relation means that for any four diagrams having n chords, where $(n-2)$ chords (not shown in the Figure) are the same for all diagrams and the other two look as shown above, the above equality takes place.

Proof of Theorem 4.5

Consider four singular knots S_1, S_2, S_3, S_4 of the order n, whose diagrams coincide outside some small circle, and their fragments s_1, s_2, s_3, s_4 inside this circle look like this:

$s_1 =$ , $s_2 =$ , $s_3 =$ , $s_4 =$ .

Consider an invariant v of order n and the values of its symbol on these four knots. Vassiliev's relation implies the relations shown in Fig. 2.

Proof of Theorem 4.5, continued

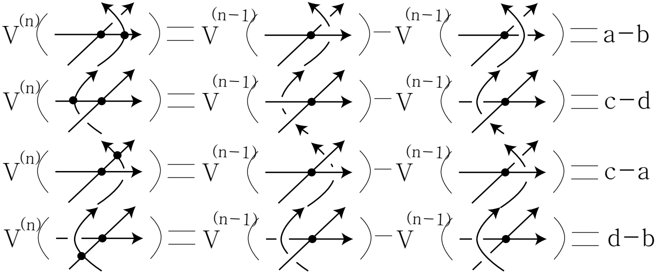


Figure 2: The same letters express $v^{(n-1)}$ for isotopic long knots

Obviously,

$$(a - b) - (c - d) + (c - a) - (d - b) = 0.$$

In order to get singular knots, one should close the fragments s_1, s_2, s_3, s_4.

Proof of Theorem 4.5, continued

Thus, the diagrams S_1, S_2, S_3, S_4 satisfy the relation

$$v^{(n)}(S_1) - v^{(n)}(S_2) + v^{(n)}(S_3) - v^{(n)}(S_4) = 0. \tag{2}$$

Each of the chord diagrams corresponding to S_1, S_2, S_3, S_4 has n chords; $(n-2)$ chords are the same for all diagrams, and only two chords are different for these diagrams.

Since the order of v equals n, the symbol of v is correctly defined on chord diagrams of order n. Thus, the value of $v^{(n)}$ on diagrams corresponding to singular knots S_1, S_2, S_3, S_4 equals the value on the singular knots themselves.

Taking into account the formulae obtained above, and the arbitrariness of the remaining $(n-2)$ singular vertices of the diagrams S_1, S_2, S_3, S_4, we obtain the statement of the theorem.

Both 1T– and 4T–relations can be considered for chord diagrams and on the dual space of linear functions on chord diagrams (since these two dual spaces can obviously be identified). For the sake of simplicity, we shall apply the terms 1T– and 4T–relation to both cases.

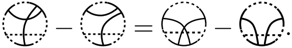
Definition 4.6

Each linear function on chord diagrams of order n, satisfying these relations, is said to be a weight system (of order n).

Notation: Denote the space of all weight systems of order n, by $\mathcal{A}_{\mathrm{n}}$ or by Δ_{n}.

In the last chapter, we considered invariants of orders less than or equal to two. The situation there is quite clear: there exists the unique non–trivial (modulo 1T–relation) chord diagram that gives the invariant of order two. As for dimension three, there are two diagrams:  and . It turns out that they are linearly dependent.

Namely, let us write the following 4T–relation (here the fixed chord is represented by the dotted line):



This means that  $= 2$ .
So, if there exists an invariant of order three,[2] then its symbol is uniquely defined by a value on . Suppose we have such an invariant v and $v'''($  $) = 1$.

[2]We showed that the second Vassiliev invariant is the second Conway coefficient up to addition of a constant.

Let us show that this invariant distinguishes the two trefoils; see Fig. 3.[3]

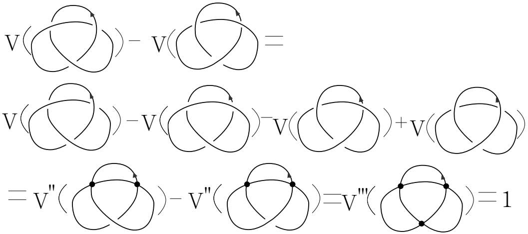


Figure 3: Vassiliev invariant of order 3 distinguishes trefoils

The existence of this invariant will be proved later.

[3]Let us recall that the second Vassiliev invariant detects the non-triviality of the trefoil, but does not separate the two trefoils.

It turns out that the chord diagrams factorised by the 4T–relation (with or without the 1T–relation) form an algebra. Namely, having two chord diagrams C_1 and C_2, one can break them at points $c_1 \in C_1$ and $c_2 \in C_2$ (which are not ends of chords) and then attach the broken diagrams together according to the orientation. Thus we get a chord diagram. The obtained diagram can be considered as the product $C_1 \cdot C_2$. Obviously, this way of defining the product depends on the choice of the base points c_1 and c_2; thus, different choices might generate different elements of $\mathcal{A}^c$. However, this is not the case since we have the 4T–relation.

Theorem 4.7

The product of chord diagrams in $\mathcal{A}^c$ is well defined; i.e., it does not depend on the choice of initial points.

To prove this theorem, we should consider arc diagrams rather than chord diagrams.

Definition 4.8

By an arc diagram we mean a diagram consisting of one straight oriented line and several arcs connecting points of it in such a way that each arc connects two different points and each point on the line is incident to no more than one arc.

These diagrams are considered up to the natural equivalence; i.e., a mapping of the diagram, taking the line to the line (preserving the orientation of the line) and taking all arcs to arcs.
Obviously, by breaking one and the same chord diagram at different points, we obtain different arc diagrams.

Now, we can consider the 4T–relation for the case of the arc diagrams, namely the relation obtained from a 4T–relation by breaking all four circles at the same point (which is not a chord end). The point is that the two arc diagrams A_1 and A_2 obtained from the same chord diagram D by breaking this diagram at different points are equivalent modulo 4T–relation. This will be sufficient for proving Theorem 4.7. Obviously, one can obtain A_2 from A_1 by "moving a chord end through infinity". Thus, it suffices to prove the following lemma.

Lemma 4.9

Let A_1, A_2 be two arc diagrams that differ only at the chord: namely, the rightmost position of a chord end of A_2 corresponds to the leftmost position of the corresponding chord end of A_1; the other chord ends of A_1 and A_2 are on the same places. Then A_1 and A_2 are equivalent modulo the four–term relation.

Proof of Lemma 4.9

Suppose that each of the diagrams A_1 and A_2 have n arcs. Denote the common arc ends A_1 and A_2 by $X_1, X_2, \ldots, X_{2n-1}$ enumerated from the left to the right. They divide the line into 2n intervals $I_1, \ldots, I_{2n}$ (from the left to the right). Denote by D_j the arc diagram having the same "fixed" arc ends as A_1 and A_2 and one "mobile" arc end at I_j. Thus, $A_1 = D_1, A_2 = D_{2n}$. Suppose that the second end of the "mobile" arc is X_k. Then, obviously, $D_k = D_{k+1}$. See Fig. 4 below.

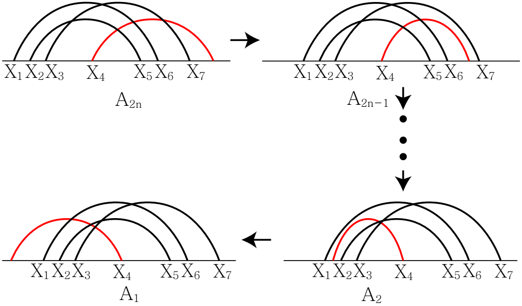


Figure 4: Sequence $A_{2n} \to A_{2n-1} \to \cdots \to A_2 \to A_1$

Proof of Lemma 4.9, continued

Now, consider the following expression

$$\begin{aligned} A_{2n} - A_1 = A_{2n} - A_{2n-1} + A_{2n-1} - A_{2n-2} + \dots \\ \dots + A_{k+2} - A_{k+1} + A_k - A_{k-1} + \dots A_2 - A_1. \end{aligned}$$

Here we have $4n - 4$ summands. It is easy to see that they can be divided into $n - 1$ groups, each of which forms the 4T–relation concerning one immobile chord and the mobile chord. Thus, $A_{2n} = A_1$. This completes the proof of the theorem.

Contents

Coproduct of chord diagrams

The chord diagram algebra $\mathcal{A}^c$ has, however, very sophisticated structures. It is indeed a bialgebra. The coalgebra structure of $\mathcal{A}^c$ can be introduced as follows.

Let C be a chord diagram with n chords. Denote the set of all chords of the diagram C by $\mathcal{X}$. Let $\Delta(\mathrm{C})$ be

$$\sum_{\mathrm{s}\in 2^{\mathcal{X}}} \mathrm{C}_{\mathrm{s}} \otimes \mathrm{C}_{\mathcal{X}\setminus \mathrm{s}},$$

where the sum is taken over all subsets s of $\mathcal{X}$, and C_{y} denotes the chord diagram consisting of all chords of C belonging to the set y. Now, let us extend the coproduct Δ linearly.

Now we should check that this operation is well defined. Namely, for each four diagrams $A =$ , $B =$ , $C =$ , $D =$  such that $A - B + C - D = 0$ is the 4T–relation, one must check that $\Delta(A) - \Delta(B) + \Delta(C) - \Delta(D) = 0$.
Actually, let A, B, C, D be four such diagrams (A differs from B only by a crossing of two chords, and D differs from C in the same way). Let us consider the comultiplication Δ. We see that when the two "principal" chords are in different parts of $\mathcal{X}$, then we have no difference between A, B as well as between C, D. Thus, such subsets of $\mathcal{X}$ give no impact. And when we take both chords into the same part for all A, B, C, D, we obtain just the 4T–relation in one part and the same diagram at the other part. Thus, we have proved that Δ is well–defined.

Now, let us give the formal definition of the bialgebra.[4].

Definition 4.10

An algebra A with algebraic operation μ and unit map e and with coalgebraic operation Δ and counit map ϵ is called a bialgebra if

1. e is an algebra homomorphism;
2. ϵ is an algebra homomorphism;
3. Δ is an algebra homomorphism.

Definition 4.11

An element x of a bialgebra B is called primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

[4] In [13] this is also called a Hopf algebra. One usually requires more constructions for the algebra to be a Hopf algebra, see e.g. [6, 11]. However, the bialgebras of chord and Feynman diagrams that we are going to consider are indeed Hopf algebras: the antipode map is defined by induction on the number of chords. We shall not use the antipode and its properties.

Obviously, for the case of $\mathcal{A}^c$ with natural e, ϵ and endowed with the product and coproduct Δ, e and ϵ are homomorphic. The map Δ is monomorphic: it has the empty kernel because for each $\mathrm{x} \neq 0$, $\Delta(\mathrm{x})$ contains the summand $\mathrm{x} \otimes 1$. Thus, $\mathcal{A}^c$ is a bialgebra.

Consider a chord diagram D of order n. Let us "double" each chord and erase small arcs between the ends of parallel chords. The constructed object (oriented circle without 2n small arcs but with n pairs of parallel chords) admits a way of walking along itself. Indeed, starting from an arbitrary point of the circle, we reach the beginning of some chord (after which we can see a "deleted small arc"), then we turn to the chord and move along it. After the end of the chord we again move to the arc (that we have not deleted), and so on. Obviously, we shall finally return to the initial points. Here we have two possibilities.

In the first case we pass all the object completely; in the second case we pass only a part of the object.

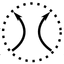
By performing a small perturbation in $\mathbb{R}^3$ we can make all chords non-intersecting. In this case our object becomes a manifold m(D). The first possibility described above corresponds to a connected manifold and the second one corresponds to a disconnected manifold.

Proposition 4.12 ([7])

The value of the n–th derivative of c_n on D equals one if m(D) has only one connected component and zero, otherwise.

Proof.

Let L be a singular knot with chord diagram D. Let us resolve vertices of D by using the skein relation for the Conway polynomial and the Vassiliev relation:

C′() = x · C().

Applying this relation n times, we see that the value of the n–th derivative of the invariant C on L (on D) equals the value of C on the diagram obtained from D by resolving all singular crossings, multiplied by x^n. Herewith, the coefficient c_n of the n–th derivative of the Conway polynomial for the case of the singular knot is equal to the coefficient c_0 evaluated at the “resolved” diagram.

This value does not depend on crossing types: it equals one on the unknot and zero on the unlink with more than one component. That completes the proof. □

It turns out that knots (as well as odd–component links) have only even–degree non-zero monomials of the Conway polynomial: $c_n \equiv 0$ for odd n.
This fact can be proved by using the previous proposition. Let D be a chord diagram of odd order n. Suppose that the curve m(D) corresponding to D has precisely one connected component. Let us attach a disc to this closed curve. Thus we obtain an orientable (prove it!) 2–manifold with disc cut. Thus, the Euler characteristic of this manifold should be odd. On the other hand, the Euler characteristic equals $V - E + S = 2n - 3n + 1 = -n + 1$. Taking into account that n is odd, we obtain a contradiction that completes the proof.
Obviously, for even n, there exist chord diagrams, where c_n does not vanish.

Exercise 4.13

Show that for each even n the value of the n–th derivative of the invariant c_n evaluated on the diagram with all chords pairwise intersecting is equal to one.

This exercise shows the existence of Vassiliev invariants of arbitrary even orders.
Thus we have proved that the Conway polynomial is weaker than the Vassiliev knot invariants.
Thus, we can say the same about the Alexander polynomial that can be obtained from the Conway polynomial by a simple variable change.

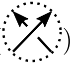
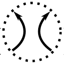
Contents

If we try to apply formal differentiation to the coefficients of other polynomials, we might fail. Thus, for example, coefficients of the Jones polynomial themselves are not Vassiliev invariants. The main reason is that the Jones polynomial evaluated at some links might have negative powers of the variable q in such a way that after differentiation we shall still have negative degrees.
In [9] the authors give a criterion to detect whether the derivatives of knot polynomials are Vassiliev invariants. They also show how to construct a polynomial invariant by a given Vassiliev invariant. Although other polynomials can not be obtained from the Conway (Alexander) polynomial by means of a variable change, Vassiliev invariants are stronger than any of those polynomial invariants of knots (possibly, except for the Khovanov polynomial). The results described here first arose in the work by Birman and Lin [4] (the preprint of this work appeared in 1991); see also [5, 10].

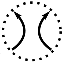
Contents

Outline

First, let us consider the Jones polynomial. Recall that the Jones polynomial satisfies the following skein relation:

$q^{-1}V($$) - qV($$) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V($$)$

Now, perform the variable change $q = e^{x}$. We get:

$e^{-x}V($$) - e^{x}V($$) = (e^{\frac{x}{2}} - e^{-\frac{x}{2}})V($$)$.

Now let us write down the formal Taylor series in x of the expression above and take all members divisible by x explicitly to the right part. In the right part we get a sum divisible by x and in the left part we obtain the derivative of the Jones polynomial plus something divisible by x:

$V($$) - V($$) = x\langle \text{some mess} \rangle$

.

Arguing as above, we see that after the second differentiation, only terms divisible by x^2 arise in the right part.
Consequently, after $(n+1)$ differentiations, the n–th term of the series expressing the Jones polynomial in x, becomes zero. Thus, all terms of this series, are Vassiliev invariants. So, we obtain the following theorem.

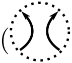
Theorem 5.1

The Jones polynomial in one variable is weaker than Vassiliev invariants.

Contents

Jones polynomial in two variables

One can do the same with the Jones polynomial (denoted by $\mathcal{X}$)) in two variables.
Let us write down the skein relation for it:

$\frac{1}{\sqrt{\lambda}\sqrt{q}}\mathcal{X}($  $) - \sqrt{\lambda}\sqrt{q}\mathcal{X}($  $) = \frac{q-1}{\sqrt{q}}\mathcal{X}($  $)$

and let us make the variable change $\sqrt{q} = e^x$, $\sqrt{\lambda} = e^y$ and write down the Taylor series in x and y.

In the right part we get something divisible by x and in the left part something divisible by xy plus the derivative of the Jones polynomial. Finally, we have

$\mathcal{X}($  $) - \mathcal{X}($  $) = x\langle\text{some mess}\rangle.$

Thus, after $(n+1)$ differentiations, all terms of degree $\leq n$ in x, vanish.

Consequently, we get the following theorem.

Theorem 5.2

The Jones polynomial in two variables is weaker than Vassiliev invariants.

Since the HOMFLY-PT polynomial is obtained from the Jones polynomial by a variable change, we see that the following theorem holds.

Theorem 5.3

The HOMFLY-PT polynomial is weaker than Vassiliev invariants.

Contents

Kauffman polynomial in two variables

The most difficult and interesting case is the Kauffman 2–variable polynomial because this polynomial does not satisfy any Conway relations. This polynomial can be expressed in the terms of functions z, a, and $\frac{a-a^{-1}}{z}$.

In order to represent the Kauffman polynomial as a series of Vassiliev invariants, we have to represent all these functions as series of positive powers of two variables.

We recall that the Kauffman polynomial in two variables is given by the formula[5]

$$\mathrm{Y}(\mathrm{L}) = \mathrm{a}^{-\mathrm{w}(\mathrm{L})}\mathrm{D}(\mathrm{L}),$$

where D is a function on the chord diagram that satisfies the following relations:

$$\mathrm{D}(\mathrm{L}) - \mathrm{D}(\mathrm{L}') = \mathrm{z}(\mathrm{D}(\mathrm{L}_\mathrm{A}) - \mathrm{D}(\mathrm{L}_\mathrm{B})); \tag{3}$$

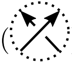
$$\mathrm{D}(\bigcirc) = \left(1 + \frac{\mathrm{a} - \mathrm{a}^{-1}}{\mathrm{z}}\right); \tag{4}$$

D(X# ) = aD(X), D(X# ) = a^{-1}D(X), (5)

where the diagrams L = , L′ = , L_A = , L_B =  coincide outside a small neighbourhood of some vertex.

[5] Here we denote the oriented and the unoriented diagrams by the same letter L.

Let us rewrite (3) for Y. We get:

$a^{-1}Y($$) - aY($$) = z(Y($$) - Y($$)) \cdot \langle \text{Power of } a \rangle.$ (6)

Let us perform the variable change: $p = \ln(\frac{a-1}{z})$. Then, in terms of z and p, one can express $z, a, \frac{a-a^{-1}}{z}$ by using only positive powers and series. Actually, we have:

$$z = z,$$

$$a = ze^{p} + 1 = z(1 + p + \dots) + 1,$$

$$a^{-1} = 1 - z(1 + p + \dots) + z^2(1 + p + \dots)^2 + \dots,$$

$$\frac{a - a^{-1}}{z} = a^{-1}(a + 1)e^{p}$$

Each of these right parts can evidently be represented as sequences of positive powers of p and z.

Thus, the value of the Kauffman polynomial in two variables on each knots is represented by positive powers of p and z. On the other hand, taking into account that $a = 1 + z\langle \text{some mess} \rangle_1$ and $a^{-1} = 1 + z\langle \text{some mess} \rangle_2$, we can deduce from (6) and (5) that

$$Y' = z\langle \text{some mess} \rangle.$$

Herewith, all terms of our double sequence having degree less than or equal to n in the variable z, vanish after the $(n+1)$–th differentiation. Thus, all these terms are Vassiliev invariants.
Thus, we have proved the following theorem.

Theorem 5.4

The Kauffman polynomial in two variables is weaker than Vassiliev invariants.

Contents

Let us show how to calculate the derivative of products of two functions.

For any two functions f and g defined on knot diagrams one can formally define the derivatives f' and g' on diagrams of first-order singular knots just as we define the derivatives of the invariants. Analogously, one can define higher-order derivatives.

Consider the function $f \cdot g$ and consider a singular knot diagram K of order n. By a splitting is meant a choice of a subset of i singular vertices of n singular vertices belonging to K. Choose a splitting s. Let K_{1s} be the diagram obtained from K by resolving $(n-i)$ unselected vertices of s negatively, and let K_{2s} be the knot diagram obtained by resolving i selected vertices positively.

Lemma 5.5

Let K be a chord diagram of degree n. Then the Leibniz formula holds:

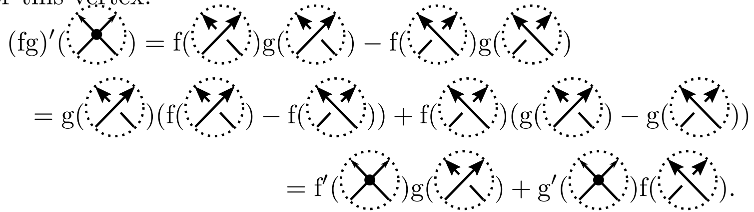
$$(fg)^{(n)}(K) = \sum_{i=0}^{n} \sum_{s} f^{(i)}(K_{1s}) g^{(n-i)}(K_{2s}).$$

Proof of Lemma 5.5

Proof.

We shall use induction on n.

First, let us establish the induction base (the case $n = 1$). Given a singular knot of order one, let us consider a diagram of it and the only singular vertex A of this diagram. Write down the Vassiliev relation for this vertex:



(7)

The equality (7) holds by definition of f' and g'. Thus, we have proved the claim of the theorem for $n = 1$. Note that we can apply the obtained formula for functions on singular (not ordinary) knots, when all singular points do not take part in the relation; i.e., lie outside the neighbourhood.

Proof of Lemma 5.5

Now, for any given singular knot K of order n, let us fix a singular vertex A of the knot diagram K. The value of $(fg)^{(n)}$ on K equals the difference of $(fg)^{(n-1)}$ evaluated on two singular knots K^1 and K^2; these two diagrams of singular knots of order $n-1$ are obtained by positive and negative resolution of A, respectively.
By the induction hypothesis, we have:

$$(fg)^{(n-1)}(K^i) = \sum_{i=0}^{n-1} \sum_{s} f^{(i)}(K^i_{1s}) g^{(n-1-i)}(K^i_{2s}), \tag{8}$$

where s runs over the set of all splittings of order $(n-1)$.

Proof of Lemma 5.5

We have:

$$
\begin{aligned}
(fg)^{(n)}(K) &= (fg)^{(n-1)}(K^1) - (fg)^{(n-1)}(K^2) \\
&= \sum_{i=0}^{n-1} \sum_s \left[f^{(i)}(K^1_{1s}) g^{(n-1-i)}(K^1_{2s}) - f^{(i)}(K^2_{1s}) g^{(n-1-i)}(K^2_{2s}) \right] \\
&= \sum_{i=0}^{n-1} \sum_s \Big[f^{(i)}(K^1_{1s}) g^{(n-1-i)}(K^1_{2s}) - f^{(i)}(K^2_{1s}) g^{(n-1-i)}(K^1_{2s}) \\
&\qquad + f^{(i)}(K^2_{1s}) g^{(n-1-i)}(K^1_{2s}) - f^{(i)}(K^2_{1s}) g^{(n-1-i)}(K^2_{2s}) \Big] \\
&= \sum_{i=0}^{n-1} \sum_s \left[f^{(i+1)}(K_{1s}) g^{(n-1-i)}(K^1_{2s}) - f^{(i)}(K^2_{1s}) g^{(n-i)}(K_{2s}) \right] \\
&= \sum_{i=0}^{n} \sum_s f^{(i)}(K_{1s}) g^{(n-i)}(K_{2s}).
\end{aligned}
$$

□

Corollary 5.6

Let f and g be two functions defined on the set of knot diagrams (not necessarily knot invariants) such that $f^{(n+1)} \equiv 0$, $g^{(k+1)} \equiv 0$. Then $(fg)^{(n+k+1)} \equiv 0$.
In particular, the product of Vassiliev invariants of orders n and k is a Vassiliev invariant of order less than or equal to $(n + k)$.

We left the proof of this corollary as an exercise.

Remark 5.7

The converse of Corollary 5.6 follows from remarkable structures: the chord diagram algebra, weight systems, Hopf algebras and Milnor-Moore theorem. We will study in the next subsequence lectures.

Contents

Until now, we have dealt only with invariants either having finite-order or invariants that can be reduced to finite order invariants. We have not yet given any proof that some knot invariant has infinite order.

Here we give an example of a knot invariant that has infinite order, [4].

Definition 6.1

The unknotting number U(K) of an (oriented) link K is the minimal number $n \in \mathbb{Z}_+$ such that K can be transformed to the unlink by passing n times through singular links. In other words, n is the minimal number such that there exists a diagram of K that can be transformed to an unlink diagram by switching n crossings.

By definition, our invariant equals zero only for unlinks.

Theorem 6.2

The invariant U has infinite order.

Proof. Let us fix an arbitrary $i \in \mathbb{N}$. Now, we shall give an example of the singular knot for which $U^{(i)} \neq 0$. Fix an integer $m > 0$ and consider the knot K_{4m} with $4m$ singularity points which are shown in Fig. 5.

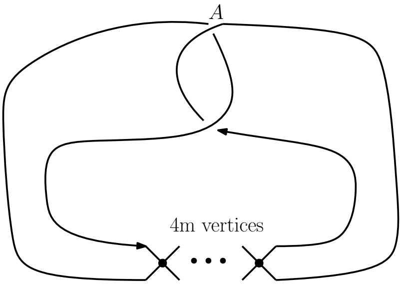


Figure 5: Singular knot, where $U^i \neq 0$.

Proof of Theorem 6.2

By definition of the derivative, the value of $U^{(4m)}$ on this knot is equal to the alternating sum of $2^{(4m)}$ summands; each of them is the value of U on a knot, obtained by somehow resolving all singular vertices of K_{4m}.

Note that for each such singular knot the value of U does not exceed one: by changing the crossing at the point A, we obtain the unknot. On the other hand, the knot obtained from K_{4m} by splitting all singular vertices is trivial if and only if the number of positive splittings equals the number of negative splittings (they are both equal to 2m).

Proof of Theorem 6.2

The case of q positive and $4m - q$ negative crossings generates the sign $(-1)^q$.
Thus we finally get that $U^{(4m)}(K_{4m})$ is equal to

$$U^{(4m)}(K_{4m}) = 2[C^0_{4m} - C^1_{4m} + \cdots - C^{2m-1}_{4m}].$$

This sum is, obviously, negative: $U^{(4m)}(K_{4m}) \neq 0$. So, for $m > \frac{i}{4}$, we get $U^{(i)} \not\equiv 0$. Thus, the invariant U is not a finite type invariant of order less than or equal to i. Since i was chosen arbitrarily, the invariant U is not a finite type invariant. □

Remark 6.3

We do not claim that U cannot be represented via finite type invariants.

Contents

Exercises

1. Show that every invariant of order ≤ 1 is constant.
2. Consider the simplest singular knot U shown in Fig. 6.

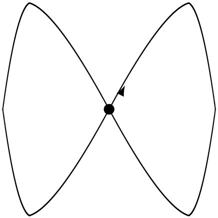


Figure 6: The simplest singular knot

   Let S be a shadow of a knot with a fixed vertex which is a singular point. Prove that one can arrange all other crossing types for S to get a singular knot isotopic to U.
3. Show that the Kauffman polynomial in two variables is weaker than Vassiliev invariants.
4. Express coefficients of Kauffman and Homflypt polynomials in terms of Vassiliev invariants.

Hard Exercises

1. Prove that the product of invariants f and g of degrees m and n, respectively, is an invariant of degree less than or equal to $m + n$. This can be split into two parts:
   a) Prove that it is of degree $(m + n)$. (relatively easy)
   b) Prove that it is exactly $(m + n)$. (very hard)

Research problem:

1. Construct the non-commutative Vassiliev invariant theory. (The idea is to make the relation $f(X^*) = f(X_+) - f(X_-)$ into $f(X_+) = f(X_-) \cdot f(X_+)$).
2. Construct Vassiliev's invariant for surface knots.
3. To construct a complexification of Vassiliev's invariant.
4. Do Vassiliev knot invariant detect unknot?
5. Do Vassiliev knot invariant detect knot invertibility? In other words: do there exist any K such that the knot K' with the inverse orientation K' can be separated by K by some Vassiliev invariant?

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