Recent developments in Seiberg-Witten theory

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(~~Lecture 1 : Basics of SW theory~~) DONE!

(~~Lecture 2 : ① Finite-dim approx. ② Diff vs. Homeo in 4D~~) DONE!

Lecture 3 : SW Floer homotopy theory

Today.

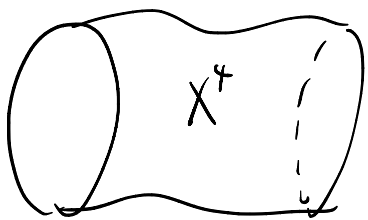
§1. Introduction to Floer theory

§2. SW Floer homotopy type SWF(Y)

§3. Sketch of the construction of SWF(Y)

§1. Introduction to Floer theory

Floer homology theory (in low dim. topology)



Y_0^3 $\quad$ Y_1^3

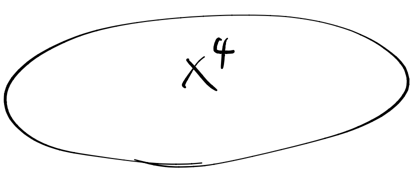
$$\longmapsto \quad HF(Y_0) \xrightarrow{HF(X)} HF(Y_1)$$

group hom.

Floer homology group

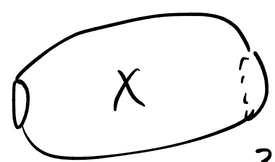
abelian groups

so that $HF(S^3) = \mathbb{Z}$ and



closed

$\doteq$



S^3 $\quad$ S^3

$$\longmapsto \quad \begin{array}{c} HF(S^3) \\ \| \\ \mathbb{Z} \end{array} \xrightarrow{\boxed{\begin{array}{c}\mathbb{Z} \\ \cup \\ HF(X)\end{array}}} \begin{array}{c} HF(S^3) \\ \| \\ \mathbb{Z} \end{array}$$

$\| \;$ Donaldson or SW invariant

① instanton Floer homology $\longleftrightarrow$ Donaldson theory

② monopole ——————— $\longleftrightarrow$ SW theory

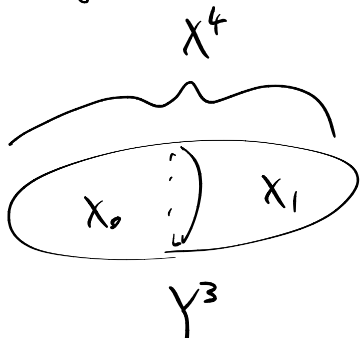
$\|$

(cf. <u>Heegaard Floer homology</u>)

$\uparrow$ combinatorial

can calculate Donaldson/SW invariant as in "Mayer-Vietoris"!

<u>Gluing axiom</u>:



$$\longmapsto \quad \begin{tikzcd} \mathbb{Z} = HF(S^3) \arrow[r, "HF(X_0)"'] \arrow[rr, bend left, "HF(X)"] \arrow[r, phantom, bend left, "\circlearrowright"] & HF(Y) \arrow[r, "HF(X_1)"'] & HF(S^3) = \mathbb{Z} \end{tikzcd}$$

Another major application of Floer homology

... Generalize Donaldson's diagonalization to 4-manifold with $\underline{\underline{\partial}}$.

which 3-manifold bound
which top / smooth 4-manifold ?

(3+1)-dim. problem.

• How to construct Floer homology

... ∞-dim. Morse homology!

In the case of monopole (SW) Floer homology:

$$(Y^3, \mathfrak{s}, g) \longmapsto \mathcal{C}(Y) : \text{a functional space}$$

$\mathfrak{s}$: spin^c str.

g: metric

$$\mathcal{C}(Y) = \left(i\Omega^1(Y) \oplus \Gamma(S) \right)$$

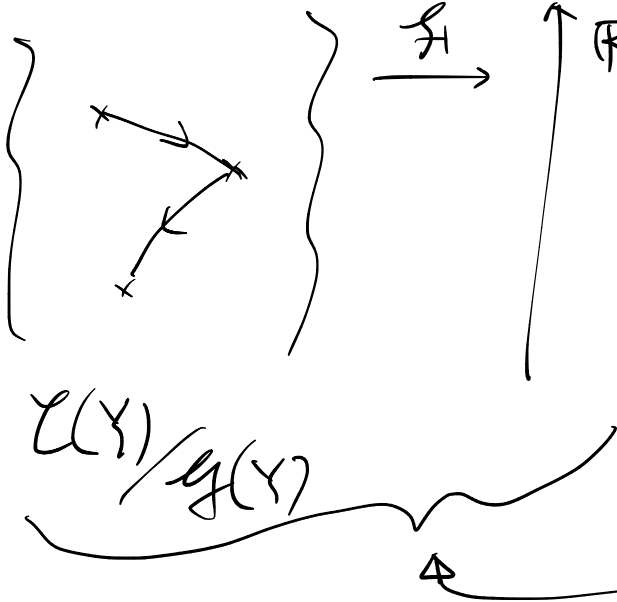
S: spinor bundle, 3-dim

$f_1 : \mathcal{C}(Y) \longrightarrow \mathbb{R}$: certain functional

s.t. $\operatorname{grad} f_1 \longleftrightarrow$ SW equations on $Y \times \mathbb{R}$.

$\circlearrowleft$

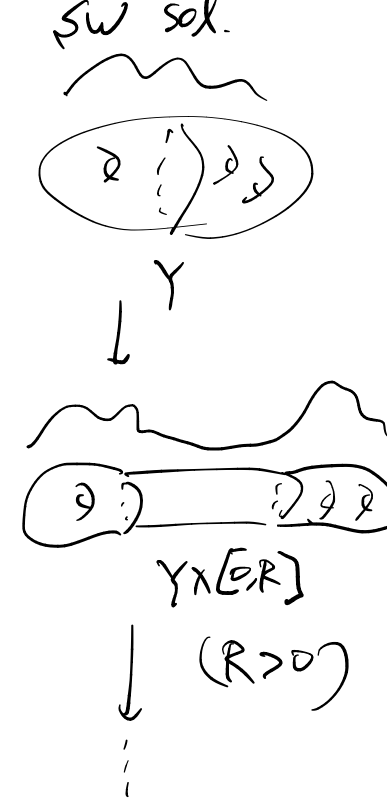
$$\mathcal{G}(Y) := \operatorname{Map}(Y, U(1))$$

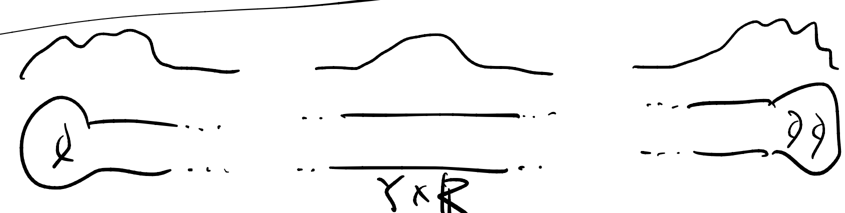


Construct the "Morse homology" for this

$\rightsquigarrow$ Monopole Floer homology $HM(Y)$

(Kronheimer-Mrowka (2007), 800 pages book!)





§2. SW Floer homotopy theory

A refinement of monopole Floer homology

... SW Floer (stable) homotopy type.

(Manolescu, 2003 ~)

closed 4-mfds	3-mfds (or 4-mfds with ∂)
SW invariant $\longleftrightarrow$	monopole Floer homology
$\downarrow$ refinement	$\downarrow$ refinement
Bauer-Furuta inv. $\longleftrightarrow$	SWF htpy type.

Mandescu (2003)

$\left[\begin{cases} Y : \mathbb{Q}HS^3 & \text{(i.e. ori. closed 3-mfd with } H^*(Y;\mathbb{Q}) \cong H^*(S^3;\mathbb{Q})\text{.)} \\ \mathfrak{s} : \text{a spin}^c \text{ str. on } Y \end{cases} \right.$

$\Updownarrow$

$b_1(Y) = 0$

$G_0 \hookrightarrow G \to S^1$

$\longmapsto SWF(Y,\mathfrak{s})$: homotopy type of a (pointed, "stable") S^1-space

called the SWF (stable) homotopy type.

Lidman-Manolescu (2018)

$\left[\quad H^*_{S^1}(SWF(Y,\mathfrak{s})) \cong HM(Y,\mathfrak{s}) \right.$ (monopole Floer homology)

Advantages of SWF homotopy type:

① Can apply any generalized cohomology theory

e.g. $FK(Y) := K(SWF(Y))$: Floer K-theory

$\rightsquigarrow$ Generalize Furuta's 10/8-ineq. to 4-mfds with ∂.

② No transversality problem!

$\rightsquigarrow$ Easy to take into account group actions

e.g. Manolescu (2016): triangulation conjecture.

using $H^*_{Pin(2)}(SWF(Y))$

$\mathbb{H} \supset Pin(2) = S^1 \cup jS^1$

$\downarrow \mathbb{Z}/2$

$O(2)$

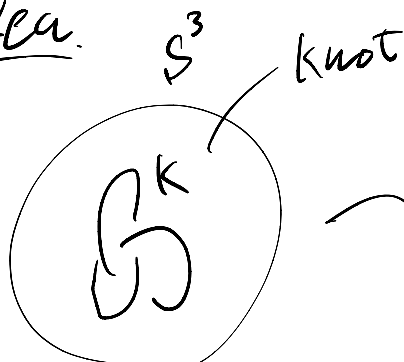
A recent topic (K.-Miyazawa-Taniguchi (2021)):

Using advantages ①, ②

$\rightsquigarrow$ Floer K-theory for knots $\underset{\text{K-theory}}{\underbrace{FK}}(\overset{\text{knot}}{K})$

(can be used to study knot invariants).

Idea.



$\rightsquigarrow$

$$\begin{array}{c} \Sigma(K) \circlearrowleft \iota_K \\ \downarrow \\ S^3 \end{array}$$

involution

double cover branched along K

Construct a variant of SWF htpy type

$SWF(Y, \iota)$ for $Y^3 \circlearrowleft \iota$ (following Y. Kato's idea)

involution

and $FK(K) := \kappa(SWF(\Sigma(K), \iota_K))$

§3. Sketch of the construction of $SWF(Y)$.

(Y, s) : a $\mathbb{Q}HS^3$ with a spin^c str.

(Fix a metric g.)

$$\rightsquigarrow \quad \begin{array}{c} S \\ \downarrow \mathbb{C}^2 \\ Y \\ \text{spinor bundle} \end{array} \qquad \begin{array}{c} L \\ \downarrow \mathbb{C} \\ Y \\ \text{det line bdle} \end{array} \qquad \longleftrightarrow \quad Spin^c(3) = \frac{SU(2) \times U(1)}{\pm 1}$$

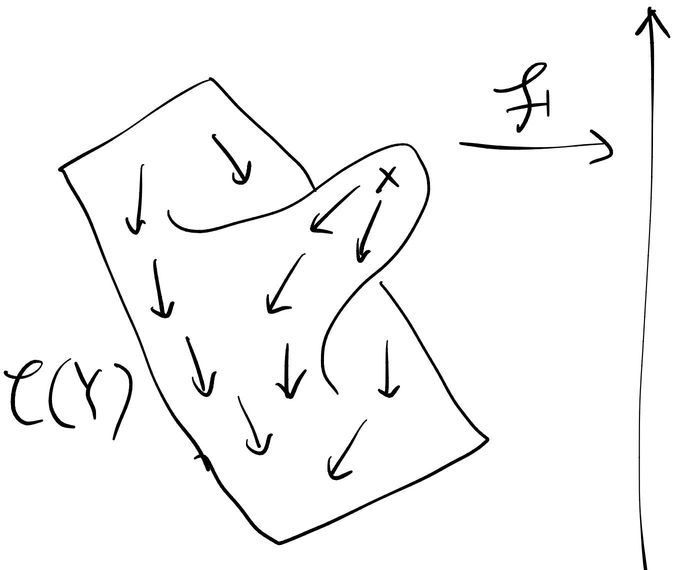
$$\begin{array}{ccc} [A, \lambda] & \longmapsto & A \cdot \lambda \\ \cap & & \cap \\ \frac{SU(2)\times U(1)}{\pm 1} & \xrightarrow{\Delta} & U(2) \\ & \xrightarrow{\det} & U(1) \\ & & \psi \\ & \longrightarrow & \lambda^2 \end{array}$$

$$\rightsquigarrow \underset{\substack{\circlearrowleft \\ S^1}}{\mathcal{C}(Y)} = \mathcal{C}(Y, s, g) = \underbrace{\operatorname{Ker}(d^* : \Omega^1 \to \Omega^0)}_{\substack{\circlearrowleft \text{ trivial} \\ S^1}} \oplus \underbrace{\Gamma(S)}_{\substack{\circlearrowleft \text{ scalar} \\ S^1 \text{ multi.}}}$$

$\mathcal{F} : \mathcal{C}(Y) \to \mathbb{R}$: Chern-Simons-Dirac functional

$$(a,\phi) \mapsto -\int_Y a \wedge da + \int_Y \langle \phi, \underbrace{D}_{\text{Dirac}}{}_{\underbrace{A_0}_{\text{reference conn.}}+a}\, \phi \rangle \, dvol.$$

with a L^2-metric



compactness of SW

$\Downarrow$

$\{$critical points of $\mathcal{F}\}$

$\subset B(\mathcal{C}(Y), R)$

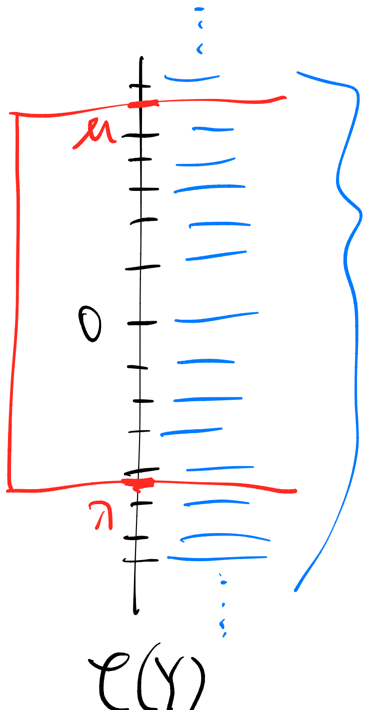
ball with radius $R \gg 0$

$$\operatorname{grad} \mathcal{F}_{(a,\phi)} = (\underbrace{L}_{\text{linear}} + \underbrace{Q}_{\text{non-linear operator.}})(a,\phi) \qquad ((a,\phi) \in \mathcal{C}(Y))$$

$L \circlearrowright \mathcal{C}(Y)$

$V = V_\lambda^\mu$

finite dim vector space



eigenvalues of L.

$$\mathcal{C}(Y) \xrightarrow{p_\lambda^\mu} V_\lambda^\mu$$

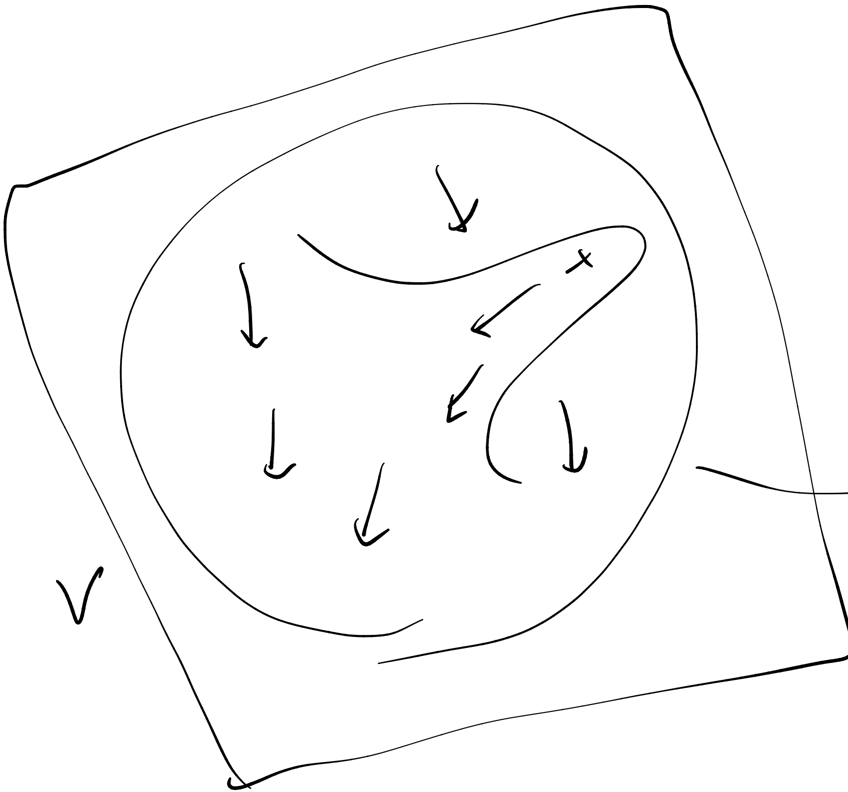
L^2-projection

Take finite dim. approx. :

$$V \longrightarrow TV \cong V \times V$$

$$v \longmapsto \big(v, (L + \rho_n^{\mu} \circ Q)(v)\big)$$

new vector field in V



V

$B(V, 3R)$

cut-off the vector field outside $B(V, 3R)$

$\mathbb{R} \curvearrowright V \Longleftarrow$ integrate

$$Y \longmapsto (\mathcal{A} : \mathcal{C}(Y) \longrightarrow \mathbb{R})$$

$$\longmapsto (\mathbb{R} \curvearrowright V) \quad : \text{a finite dim dynamical system}$$

$$\longmapsto SWF(Y)$$

Conley index

a "space-valued" invariant of finite dim dynamical systems

- Conley index

M : finite dim space
$\circlearrowleft$
$\mathbb{R}$

$A \subset M$, let
sub set

$$\mathrm{Inv}\,A := \{ x \in M \mid t\cdot x \in A \ (\forall t \in \mathbb{R}) \}$$

Def.

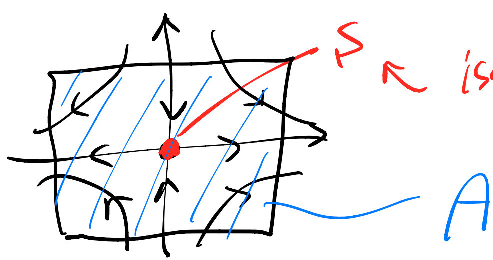
$S \subset M$ subset

S is an isolated invariant set if

$\exists$ cpt nbd A of S s.t. $S = \mathrm{Inv}\,A \subset \mathrm{Int}\,A$.

interior

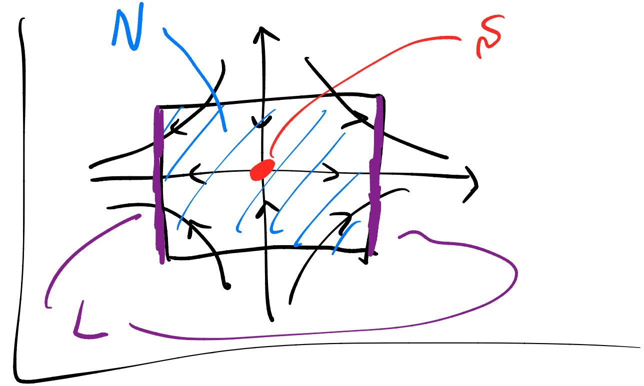
e.g. $M = \mathbb{R}^2$



Thm/Def

S : an isolated inv. set. Then

$\exists \underbrace{L}_{\text{exit set}} \subset^{\exists} N \subset M$ (cpt) with



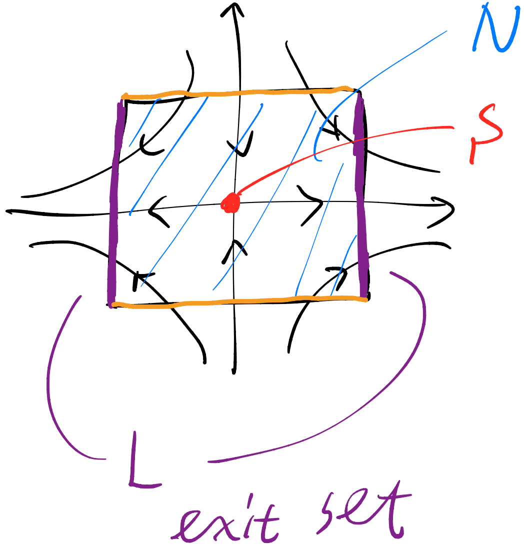
(1) $\mathrm{Inv}(N \setminus L) = S \subset \mathrm{Int}(N \setminus L)$

(2) $\forall x \in N$, if $^{\exists} t > 0$ s.t. $t \cdot x \notin N$, then $t' \cdot x \in L$ $(0 < {}^{\exists} t' < t)$.

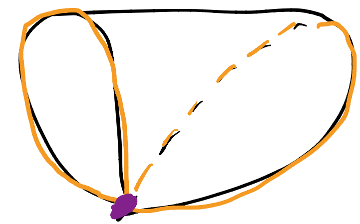
(3) $x \in L$, $t > 0$, $[0,t] \cdot x \subset N$

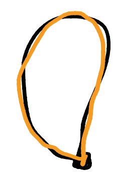
$\Rightarrow [0,t] \cdot x \subset L$.

$I(S) := N/L$: Conley index of S.



$I(S) = N/L$

$=$ 

$\simeq$ (htpy equiv.)  $\cong S^{\boxed{1}}$

$\| $ Morse index of S

Thm

$[I(S)]$: (based) htpy type

is indep. of (N, L) (depends only on S and $\mathbb{R} \curvearrowright M$.)

Come back to SW ...

compactness of SW

Thm (Manolescu)

$B(V,R)$ is an isolated invariant set in $V \circlearrowleft \mathbb{R}$

$$SWF(Y) := [I(B(V,R))] \text{ (up to suspension).}$$

- de Rham
- Dolbeault
- sign
- Dirac

- Yang-Mills ASD
- SW

G_2 $Spin(7)$

$* : \Lambda^2 \circlearrowleft$

$** = 1$