Panorama of Dynamics and Geometry of Moduli Spaces and Applications

Lecture 12. Electron transport on Fermi-surfaces. Calabi Theorem. Classification of connected components of strata of Abelian differentials

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(after a joint work with M. Kontsevich,
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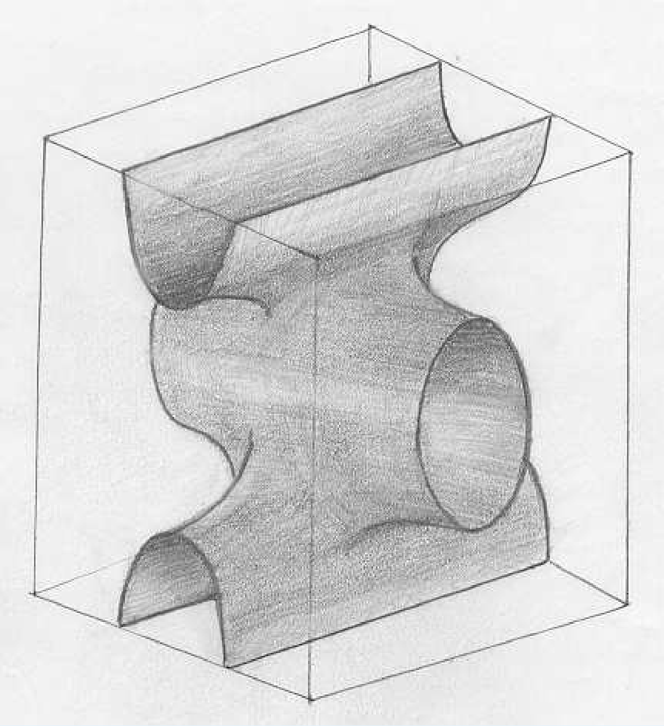
Measured foliations in the Nature

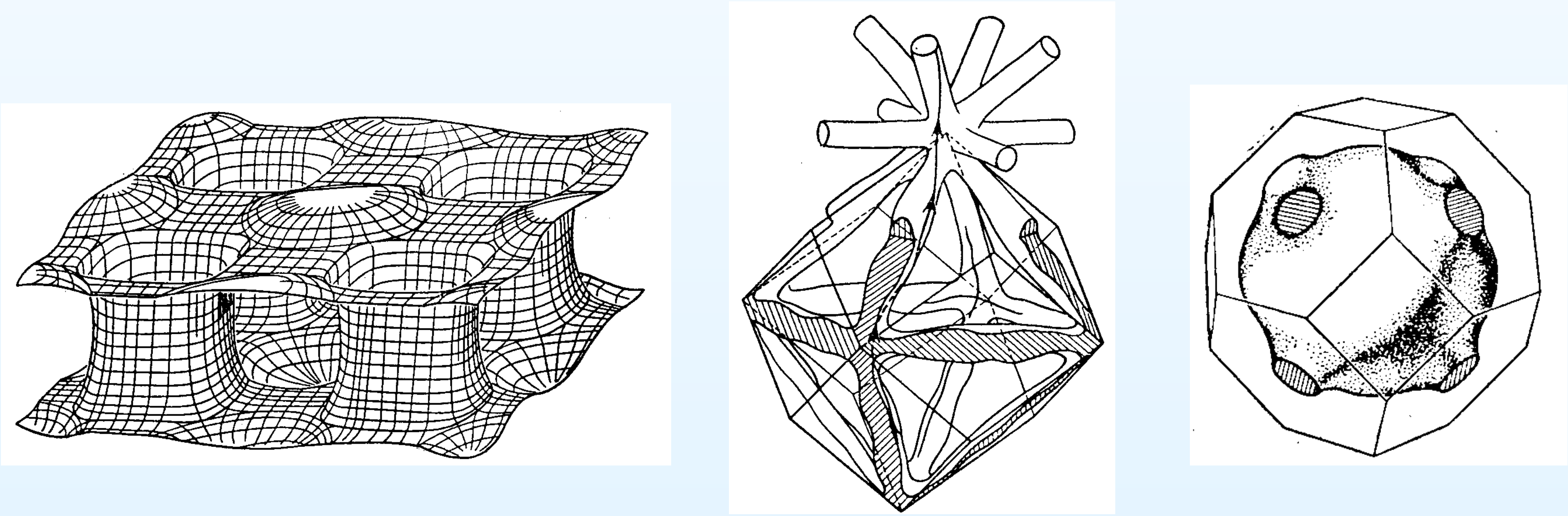
Plane sections of a periodic surface

Consider a $\mathbb{Z}^3$-periodic surface $\tilde{M}^2$ in $\mathbb{R}^3$ (a surface invariant under parallel translations preserving $\mathbb{Z}^3$). Such a surface can be constructed in a fundamental domain of a cubic lattice, as in the picture, and then reproduced repeatedly in the lattice. Choose now a family of parallel planes in $\mathbb{R}^3$ and consider intersection lines of the surface by the planes. These intersection lines might have some compact components and may also have some unbounded components. The question is *how do unbounded component propagate in* $\mathbb{R}^3$?



Electron transport in metals in homogeneous magnetic field

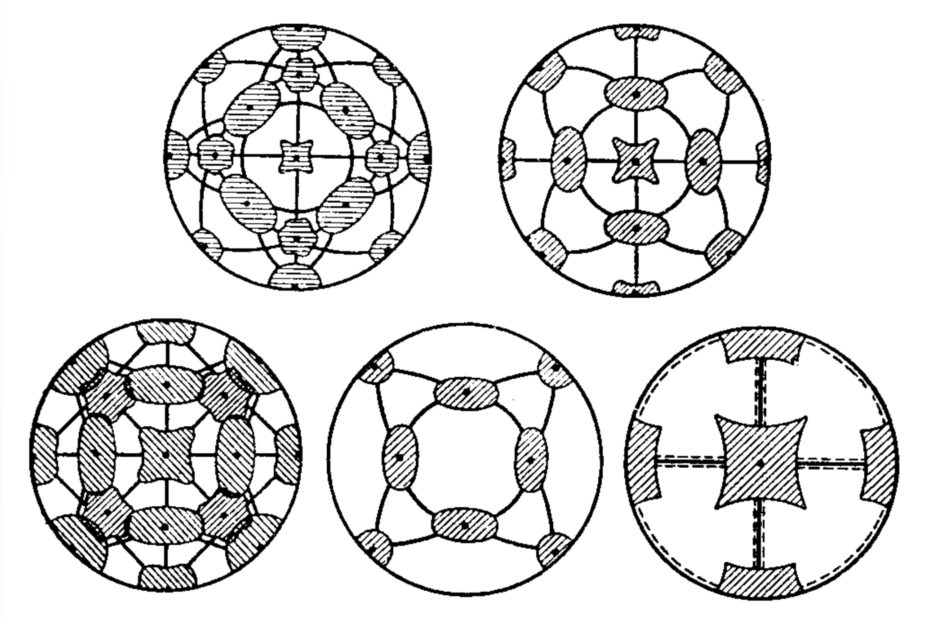
The study of this subject was suggested by S. P. Novikov about 1980 as a mathematical formulation of the corresponding problem concerning electron transport in metals. A periodic surface represents a *Fermi-surface* in the quasimomentum space, affine planes are orthogonal to a magnetic field, and the intersection lines are trajectories of an electron in the *inverse lattice*. The picture below shows that *Fermi-surfaces* might have sophisticated shape.



Fermi surfaces of tin, iron, and gold.

Open trajectories

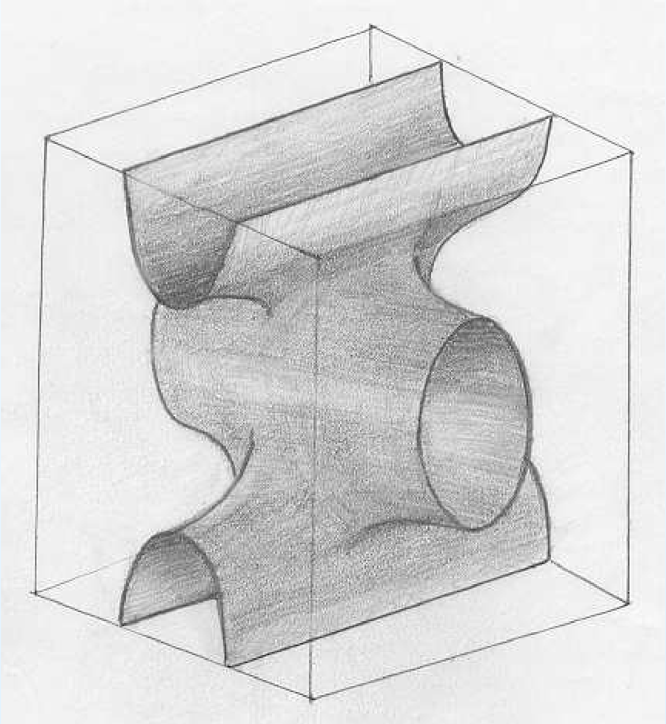
Physicists knew from experiments that open (unbounded) trajectories might exist, but they did not know whether they behave regularly, chaotically, whether they admit scattering etc.
The picture below, taken from the classical textbook on solid state physics by Azbel–Lifshitz–Kaganov, shows stereographic projection of the magnetic field directions (shaded regions and continuous curves) which give rise to open trajectories for some Fermi-surfaces (experimental results).



Mathematical reformulation

We are interested in plane sections of the initial periodic surface $\tilde{M}^2$. This plane sections can be viewed as level curves of a linear function $f(x, y, z) = ax + by + cz$ restricted to $\tilde{M}^2$.

Passing to a quotient $\mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3$ we get a closed orientable surface $M^2 \subset \mathbb{T}^3$ from the initial periodic surface $\tilde{M}^2$. Say, identifying the opposite sides of a unit cube in the picture, we get a closed surface M^2 of genus $g = 3$.



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The linear function $f(x, y, z) = ax + by + cz$ does not descend to M^2, but the differential 1-form $df = a\,dx + b\,dy + c\,dz$ in $\mathbb{R}^3$ does. A closed 1-form defines a codimension-one foliation on a manifold: locally one can represent any closed 1-form as a differential of a function; the foliation is defined by the levels of this function. After passing to a quotient over the lattice $\mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3$ the plane sections of $\tilde{M}^2$ project to leaves of the foliation defined by the restriction of the closed 1-form $\omega = a\,dx + b\,dy + c\,dz$ in $\mathbb{T}^3$ to the surface M^2. Thus, to study electron trajectories on Fermi-surfaces we have to study a measured foliation defined by $\omega|_{M^2}$.

Calabi Theorem

Foliation defined by a closed 1-form

Consider a closed smooth 1-form ω_1 on a closed orientable surface M^2. Since ω_1 is a closed form, locally it is a differential of a function f defined up to a constant: $\omega_1 = df$. The level curves of f define a foliation on M^2. This foliation is endowed with a transverse measure and with a natural orientation (defined by $\mathrm{grad}(f)$ and by the orientation of the surface).

Problem *Is it possible to introduce a “very flat structure” on M^2 such that the leaves of ω_1 would become straight lines in the flat metric?*

or, in other words

Problem *Is it possible to introduce a complex structure on M^2 and find a holomorphic form ω in this complex structure such that $\omega_1 = \mathrm{Im}(\omega)$?*

Obstruction for straightening

Obvious obstruction. If $\omega_1 = \mathrm{Im}(\omega)$, then the horizontal foliation (the one defined by ω_1) does not have closed leaves homologous to zero.

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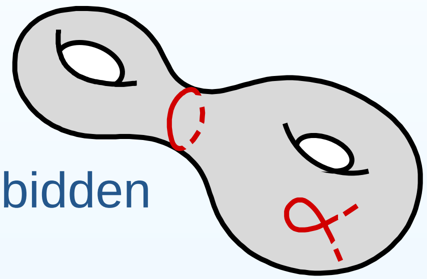

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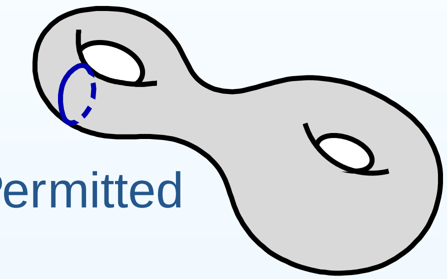

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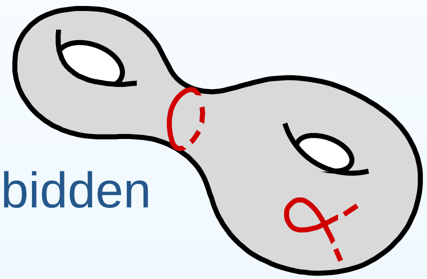

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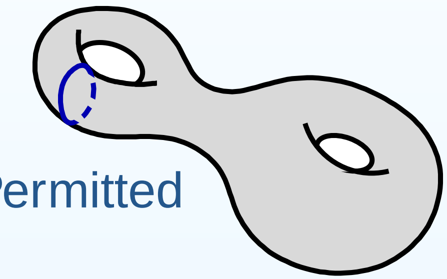

Proof. On the one hand, the integral of $\omega_0 = \mathrm{Re}(\omega)$ grows monotonously when we integrate it along a horizontal leaf. Hence, it is strictly positive along any *closed* horizontal leaf. On the other hand, since ω_0 is a closed form, its integral over any trivial cycle is zero. Thus, those closed leaves of ω_1 which represent trivial homology cycles, provide obvious obstructions for $\omega_1 = \mathrm{Im}(\omega)$. In particular,

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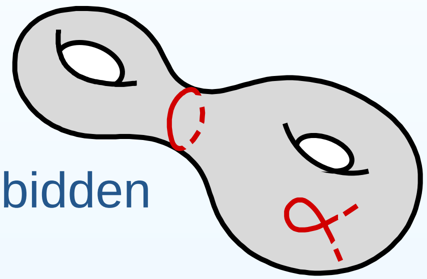

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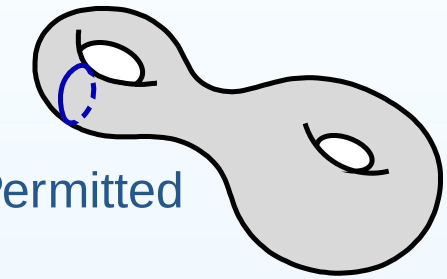

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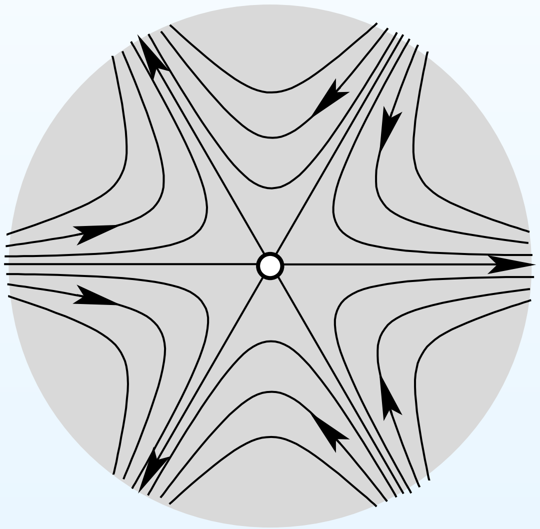
Corollary. If $\omega_1 = \mathrm{Im}(\omega)$, then it does not have minima or maxima, but only a finite number of isolated saddle points.

Saddles and saddle connections

Suppose now that this necessary condition is satisfied and that the closed 1-form has only isolated critical points and all of them are "saddles". Say, a form defined in local coordinates as df, where $f(x,y) = x^3 + y^3$ has a six-prongs saddle point at the orgine:



Recall that the singular leaves of the foliation landing at and going out of a saddle point are called *separatrices* .

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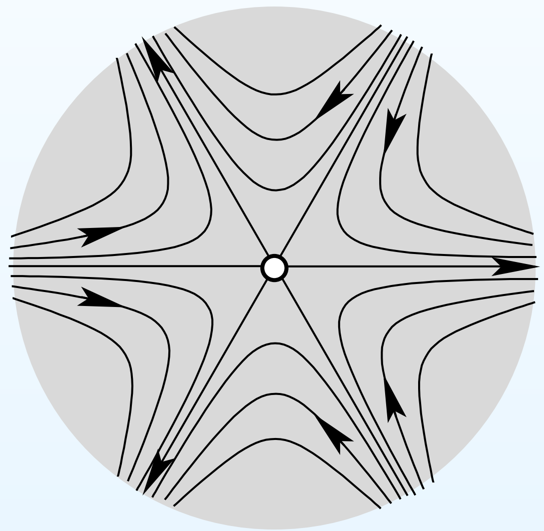

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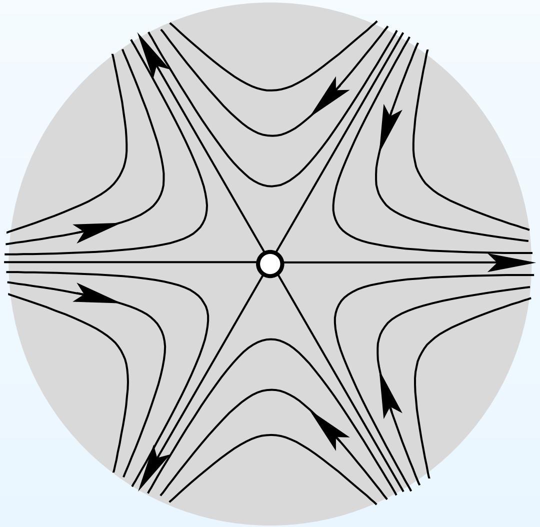

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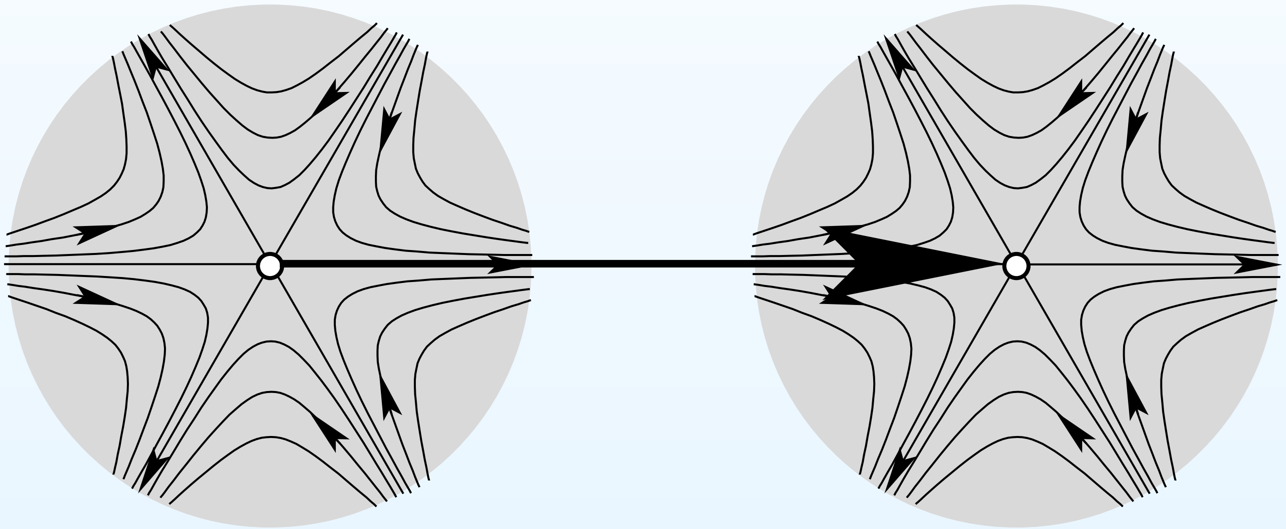

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Sometimes a critical leaf emitted from one saddle might can run into another (or into the same) saddle point. In this case we say that the foliation has a *saddle connection*.

Criterion of straightening

Theorem *Consider a nontrivial closed smooth 1-form ω_1 on a closed orientable surface M^2. There exist a complex structure on M^2 and a holomorphic form ω in this complex structure such that $\omega_1 = \mathrm{Im}(\omega)$ if and only if the following two conditions on ω_1 are valid:*

- *The form ω_1 has only isolated critical points, which are all saddles (it has no centers = no local minima or maxima).*
- *Any cycle obtained as a union of closed paths following in the positive direction a sequence of saddle connections is not homologous to zero.*

In particular, if there are no saddle connections at all, the foliation defined by ω_1 can always be “straightened” (provided it has no minima and maxima).

In different terms a similar statement was proved earlier by Calabi, Katok, and by Hubbard—Masur.

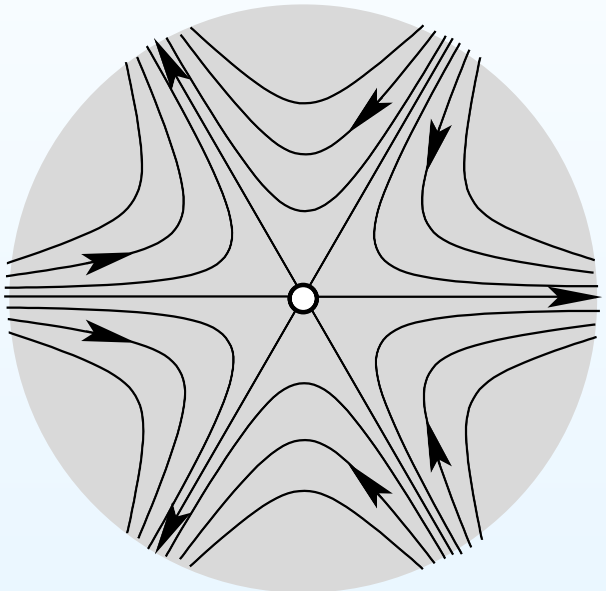
Calabi Theorem: original statement

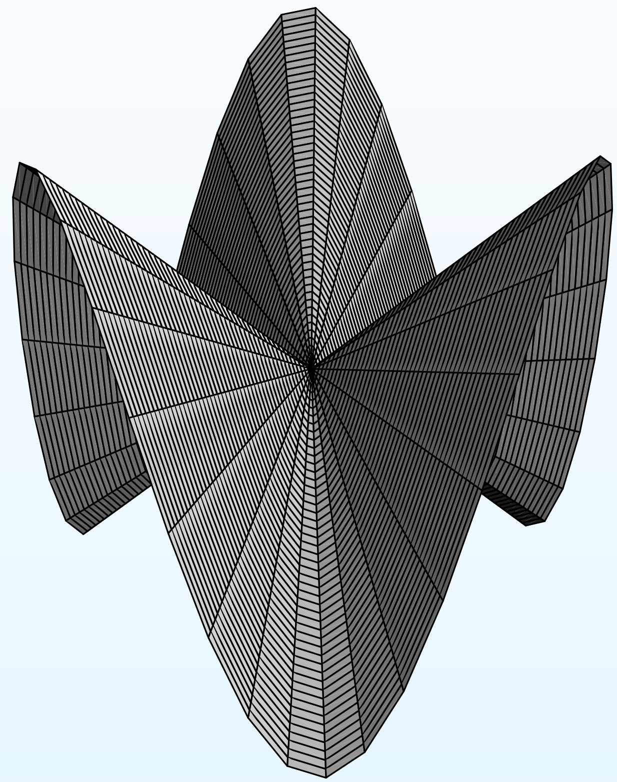
Theorem (E. Calabi, around 1969). *Let ω be a closed real 1-form on a smooth orientable compact manifold M^n. Suppose that ω has only Morse-type singularities. There exists a Riemannian metric on M^n for which ω is a harmonic 1-form if and only if for any point P of M^n which is not a zero of ω there exists a smooth closed path γ through P such that ω restricted to γ is everywhere nonzero on γ.*

Remark. Currently Maxim Kontsevich strongly advertises the structure (M^n, ω), where ω is not a closed 2-form, but a closed complex 1-form with moderate singularities.

Saddle points as conical points

Note that saddle points of the closed 1-form $\omega_1 = \mathrm{Im}(\omega)$ correspond to conical points of the resulting flat metric:

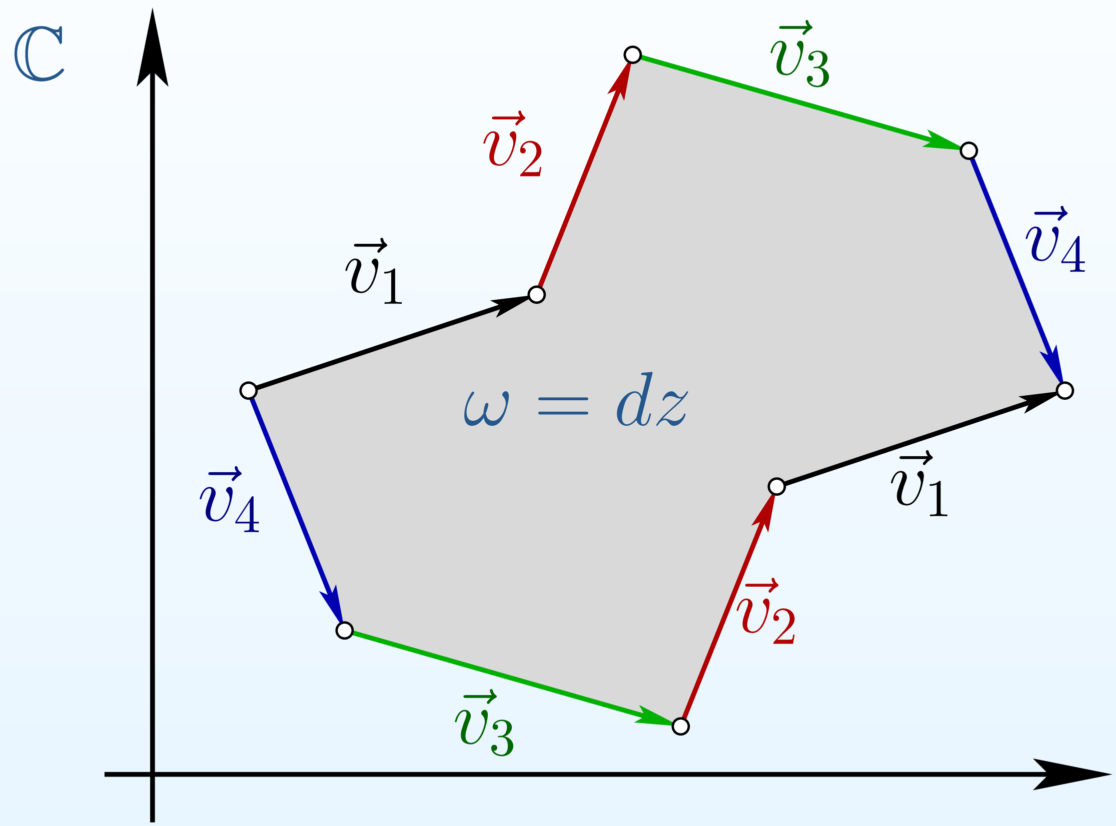




A neighborhood of a conical point with a cone angle 6π can be glued from six metric half discs. At this conical point we have 3 distinct directions to the East.

Holomorphic 1-forms versus very flat surfaces

Holomorphic 1-form associated to a flat structure



The form $\omega = dz$ has zeroes exactly at those points of S where the flat structure has conical singularities.

Flat structure defined by a holomorphic 1-form

Reciprocally a pair (Riemann surface X, holomorphic 1-form ω) uniquely defines a flat structure. Namely, in a simple-connected neighborhood $U \subset X$ one can choose a coordinate z such that $\omega = dz$. This is the flat coordinate. For a pair of such overlapping neighborhoods with coordinates z and z' one has $z' = z + const$ on the overlaps, so the change of coordinates is a parallel translation.

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In a neighborhood of zero a holomorphic 1-form can be represented as $w^d\, dw$, where d is the **degree** of zero. The form ω has a zero of degree d at a conical point with cone angle $2\pi(d+1)$.

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The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \dots + d_n = 2g - 2$.

Any holomorphic 1-forms corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly n zeroes of degrees $d_1, \dots, d_n$.

flat structure (including a choice of the vertical direction)	complex structure and a choice of a holomorphic 1-form ω
conical point with a cone angle $2\pi(d+1)$	zero of degree d of the holomorphic 1-form ω (in local coordinates $\omega = w^d\, dw$)
side $\vec{v}_j$ of a polygon	relative period $\int_{P_j}^{P_{j+1}} \omega = \int_{\vec{v}_j} dz$ of the 1-form ω
family of flat surfaces sharing the same cone angles $2\pi(d_1+1), \dots, 2\pi(d_n+1)$	stratum $\mathcal{H}(d_1, \dots, d_n)$ in the moduli space of holomorphic 1-forms
coordinates in the family: vectors $\vec{v}_i$ defining the polygon	coordinates in $\mathcal{H}(d_1, \dots, d_n)$: relative periods of ω in $H^1(S, \{P_1, \dots, P_n\}; \mathbb{C})$

Volume element

Vectors $\vec{v}_1, \ldots, \vec{v}_n$ defining the broken line serve as coordinates in the corresponding *space of flat surfaces*. The volume element $d\nu = d\vec{v}_1 \ldots d\vec{v}_n$ does not depend on a coordinate chart.

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It can be interpreted as a linear volume element in the vector space $H^1(S, \{zeroes\}\,;\, \mathbb{C})$ normalized by means of the integer lattice $H^1(S, \{zeroes\}\,;\, \mathbb{Z} \oplus \sqrt{-1}\,\mathbb{Z})$.

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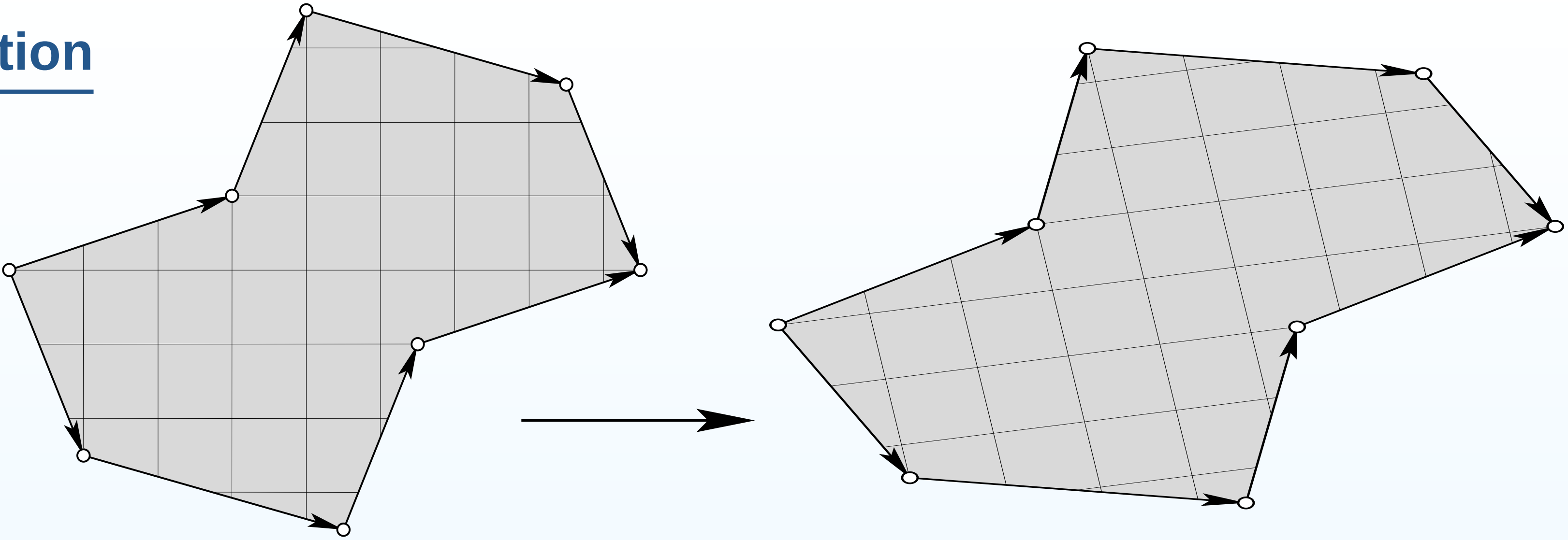
Consider now the real hypersurface $\mathcal{H}_1(d_1, \ldots, d_n) \subset \mathcal{H}(d_1, \ldots, d_n)$ defined by the equation $area(S) = 1$. The volume element $d\nu$ can be naturally restricted to the hypersurface defining the volume element $d\nu_1$ on $\mathcal{H}_1(d_1, \ldots, d_n)$.

Theorem (H. Masur; W. A. Veech) *Volume* $\mathrm{Vol}(\mathcal{H}_1(d_1, \ldots, d_n))$ *of every stratum is finite.*

Group action

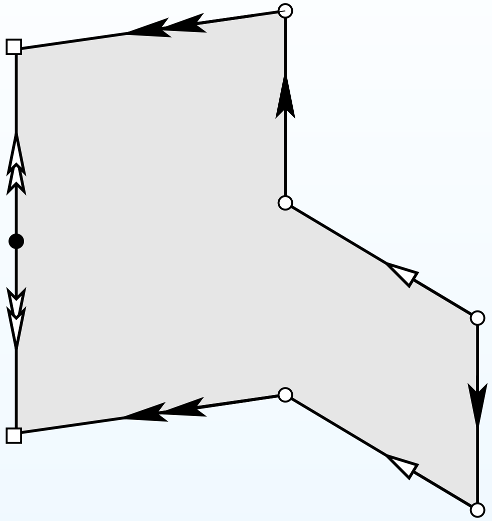


The subgroup $\mathrm{SL}(2,\mathbb{R})$ of area preserving linear transformations acts on the "unit hyperboloid" $\mathcal{H}_1(d_1,\dots,d_n)$. In cohomological coordinates $H^1(\dots,\mathbb{C}) = H^1(\dots,\mathbb{R}) \otimes \mathbb{C} \simeq H^1(\dots,\mathbb{R}) \otimes \mathbb{R}^2$ this action can be interpreted as the linear action of $\mathrm{SL}(2,\mathbb{R})$ on the factor $\mathbb{R}^2$.

Key Theorem (H. Masur; W. A. Veech) *The action of the groups* $\mathrm{SL}(2,\mathbb{R})$ *and* $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ *preserves the measure* $d\nu_1$. *Both actions are ergodic with respect to this measure on each connected component of every stratum* $\mathcal{H}_1(d_1,\dots,d_n)$.

Hyperelliptic connected components

Flat surfaces and quadratic differentials



Identifying pairs of sides of this polygon by isometries we obtain a surface of genus $g = 1$. Now the flat metric has holonomy group $\mathbb{Z}/2\mathbb{Z}$. The cone angles are multiples of π.

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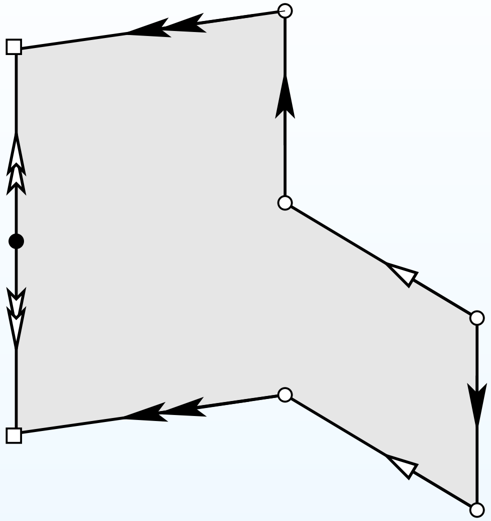

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Flat surfaces of this type correspond to quadratic differentials $q(z)(dz)^2$. For example, the quadratic differential representing the surface from the picture belongs to the stratum $\mathcal{Q}(2, -1, -1)$: it has a single zero of order two and two simple poles.

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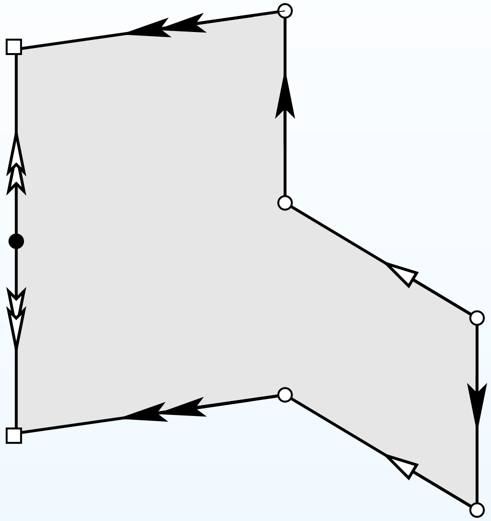

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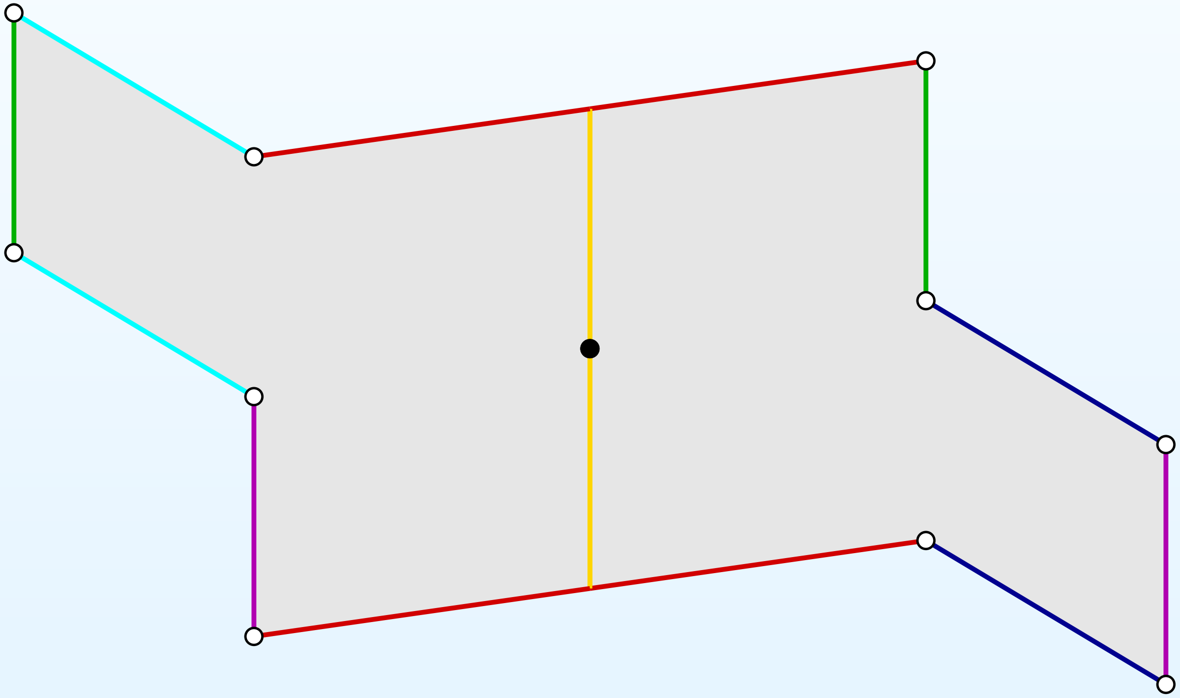
The cone angle at a simple pole is π; at a zero of degree d it is $(d+2)\pi$.

Canonical double cover: flat picture

For any flat surface with $\mathbb{Z}/2\mathbb{Z}$-holonomy there exists a canonical double cover such that the induced flat metric already has trivial holonomy. It is ramified at every conical singularity with odd cone angle $\pi(2k+1)$ and at no other points.



Canonical double cover: analytic picture

In other words, having a Riemann surface X and a meromorphic quadratic differential q with at most simple poles on it there exists a canonical double cover $p : \hat{X} \to X$ such that the induced quadratic differential p^*q is already a square of a holomorphic 1-form $\hat{\omega}$ globally defined on $\hat{X}$, i.e. $p^*q = \hat{\omega}^2$.

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- Every simple pole of q is a ramification point; it becomes a regular point of $\hat{\omega}$.
- Every zero $z^{2d}(dz)^2$ of even order of q gives rise to two zeroes of degree d of $\hat{\omega}$.
- Every zero $z^{2d+1}(dz)^2$ of odd order of q is a ramification point; it gives rise to a single zero of degree $d+1$ of $\hat{\omega}$.
- The induced holomorphic form $\hat{\omega}$ does not have other singularities.

Induced maps of the strata

For a meromorphic quadratic differential on a Riemann surface of genus g with singularities of orders $d_1, \dots d_n$ one has

$$d_1 + \cdots + d_n = 4g - 4.$$

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The induced holomorphic 1-form $\hat{\omega}$ on the ramified double cover $\hat{X} \to \mathbb{CP}^1$ has a single zero of degree $d + 1$ when d is odd and a pair of zeroes of degrees $\frac{d}{2}$ when d is even. We get two maps of the strata:

$$\mathcal{Q}(2g - 3, -1^{2g+1}) \to \mathcal{H}(2g - 2)$$
$$\mathcal{Q}(2g - 2, -1^{2d+2}) \to \mathcal{H}(g - 1, g - 1)$$

Dimension of a stratum

Each stratum is a complex-analytic orbifold of complex dimension

$$\dim_{\mathbb{C}} \mathcal{H}(d_1, \ldots, d_n) = \dim_{\mathbb{C}} H^1(S, \{P_1, \ldots, P_n\}; \mathbb{C}) = 2g + n - 1$$

The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is a $\mathbb{C}^g$-vector bundle over the moduli space $\mathcal{M}_g$ of complex structures.

Note that $\dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3$ while, say, $\dim_{\mathbb{C}} \mathcal{H}(2g-2) = 2g$. It is not surprising: for a generic Riemann surface one cannot construct a zero with such a hight multiplicity. Thus, individual strata **are not** fiber bundles over the moduli space $\mathcal{M}_g$.

Moreover, they might have several connected components!

Dimension count

We can place zeroes and poles of a quadratic differential on $\mathbb{CP}^1$ at any configuration of points. A quadratic differential is defined by its zeroes and poles up to a constant factor. We can send any triple of points on $\mathbb{CP}^1$ to $0, 1, \infty$ by a holomorphic map after which there remain no holomorphic automorphisms. Thus,

$$\dim_{\mathbb{C}} \mathcal{Q}(2g-3, -1^{2g+1}) = (2g+2) + 1 - 3 = 2g,$$

where $2g+2 = 1 + (2g+1)$ stands for the *number of singularities*; $+1$ for a multiplicative factor; and -3 stands for three points which should be sent to $0, 1, \infty$ and which should not be counted as parameters.

Dimension count

We can place zeroes and poles of a quadratic differential on $\mathbb{C}\mathrm{P}^1$ at any configuration of points. A quadratic differential is defined by its zeroes and poles up to a constant factor. We can send any triple of points on $\mathbb{C}\mathrm{P}^1$ to $0, 1, \infty$ by a holomorphic map after which there remain no holomorphic automorphisms. Thus,

$$\dim_{\mathbb{C}} \mathcal{Q}(2g-3, -1^{2g+1}) = (2g+2) + 1 - 3 = 2g,$$

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On the other hand, $\dim_{\mathbb{C}} \mathcal{H}(2g-2) = 2g$. Hence,

$$\dim_{\mathbb{C}} \mathcal{Q}(2g-3, -1^{2g+1}) = \dim_{\mathbb{C}} \mathcal{H}(2g-2).$$

Similarly

$$\dim_{\mathbb{C}} \mathcal{Q}(2g-2, -1^{2g+2}) = \dim_{\mathbb{C}} \mathcal{H}(g-1, g-1).$$

Hyperelliptic connected components

Every stratum $\mathcal{Q}(d_1, \ldots, d_n)$ on $\mathbb{C}\mathrm{P}^1$ is connected: up to a multiplicative factor a quadratic differential q in $\mathcal{Q}(d_1, \ldots, d_n)$ is parameterized by a configuration of n points on $\mathbb{C}\mathrm{P}^1$.

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Thus we get *hyperelliptic connected components* $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ in the corresponding strata of holomorphic 1-forms as images of the strata $\mathcal{Q}(2g-3, -1^{2g+1})$ and of $\mathcal{Q}(2g-2, -1^{2g+2})$.

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A holomorphic 1-form with two zeroes of order $g-1$ on a hyperelliptic surface X belongs to $\mathcal{H}^{hyp}(g-1, g-1)$ if and only if the zeroes are interchanged by the involution σ (and does not, if σ fixes them).

Exercises

- Show that in flat coordinates defined by any holomorphic 1-form on a hyperelliptic Riemann surface, the hyperelliptic involution σ has differential $D\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Show that a flat surface obtained by gluing the opposite sides of a regular $2n$-gon belongs to a hyperelliptic connected component.
- Determine the corresponding stratum in terms of n.
- Find a hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g+2$ such points.

Parity of the spin structure

Index of a smooth closed path on a flat surface

Consider a simple smooth closed path ρ on a flat surface avoiding conical singularities. At any point of the surfaces we know where is the "direction to the North". Hence, at any point $z = \rho(t)$ we can apply a compass and measure the direction of the tangent vector $\dot{z}$. Moving along ρ we make the tangent vector turn in the compass.

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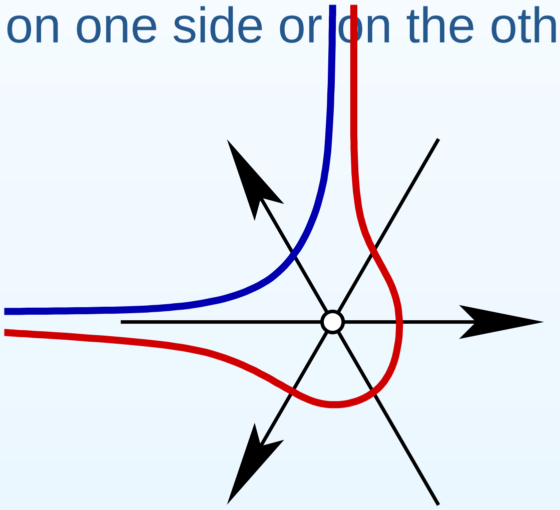
We define the *index* $ind(\rho)$ of the path ρ as a degree of the corresponding Gauss map (or, in other words as the algebraic number of turns of the tangent vector around the compass) taken modulo 2.

$$ind(\rho) = \deg G(\rho) \mod 2$$

Parity of the spin structure

It is easy to see that $ind(\rho)$ does not depend on parametrization. Moreover, it does not change under small deformations of the path.

Exercise. If a conical point P has a cone angle which is an odd multiple of 2π, then bypassing P on one side or on the other we get the same $ind(\rho)$.



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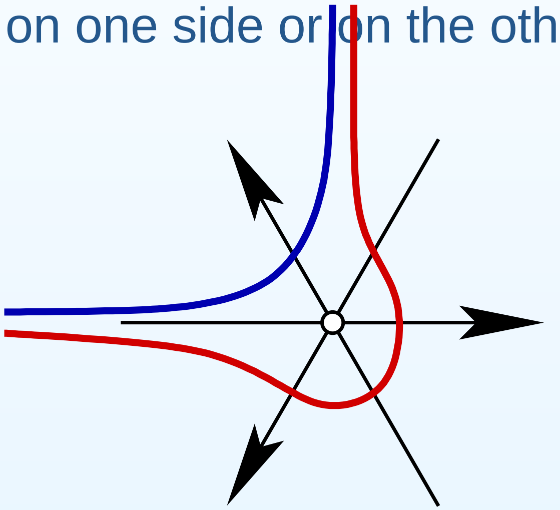


Consider a collection of simple closed smooth paths $a_1, b_1, \ldots, a_g, b_g$ representing a symplectic basis of homology $H_1(S, \mathbb{Z}/2\mathbb{Z})$. We define a *parity of the spin-structure* of a flat surface $S \in \mathcal{H}(2d_1, \ldots, 2d_n)$ as

$$\phi(S) = \sum_{i=1}^{g} (ind(a_i) + 1)(ind(b_i) + 1) \mod 2$$

Parity of the spin structure

Theorem *The value $\phi(S)$ does not depend neither on the choice of symplectic basis of cycles $\{a_i, b_i\}$, nor on its representation by simple closed paths. It does not change under continuous deformations of S in $\mathcal{H}(2d_1, \dots, 2d_n)$.*

Variants of this theorem are proved by M. Atiyah, J. Milnor, D. Mumford and D. Johnson in appropriate terms.

Theorem above shows that the parity of the spin structure is an invariant of connected components of strata of Abelian differentials $\mathcal{H}(2d_1, \dots, 2d_n)$ with zeroes of even degrees.

Exercise. Compute a parity of the spin structure for a flat torus.

Classification Theorem: genus four and higher

Theorem (M. Kontsevich, A.Z.) *General case: $g \geq 4$.*

• *The stratum $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2g-2)$, and two other components: $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures.*

• *The stratum $\mathcal{H}(2l,2l)$, $l \geq 2$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2l,2l)$, and two other components: $\mathcal{H}^{even}(2l,2l)$ and $\mathcal{H}^{odd}(2l,2l)$.*

• *All the other strata of the form $\mathcal{H}(2l_1,\dots,2l_n)$, where all $l_i \geq 1$, have two connected components: $\mathcal{H}^{even}(2l_1,\dots,2l_n)$ and $\mathcal{H}^{odd}(2l_1,\dots,2l_n)$, corresponding to even and odd spin structures.*

• *The strata $\mathcal{H}(2l-1,2l-1)$, $l \geq 2$, have two connected components; one of them, $\mathcal{H}^{hyp}(2l-1,2l-1)$, is hyperelliptic; the other one, $\mathcal{H}^{nonhyp}(2l-1,2l-1)$, is not.*

• *All other strata of Abelian differentials on complex curves of genera $g \geq 4$ are nonempty and connected.*

Classification Theorem: low genera

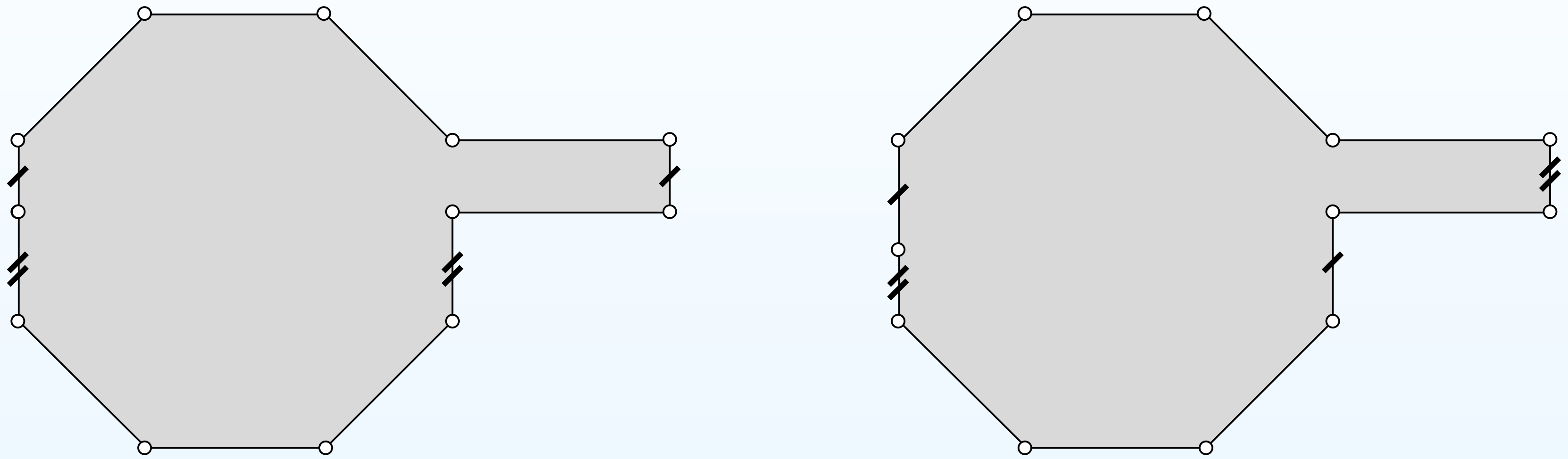
The theorem below shows that in genera $g = 2, 3$ some components are missing with respect to the general case.

Theorem

- *The moduli space of Abelian differentials on a complex curve of genus* $g = 2$ *contains two strata:* $\mathcal{H}(1,1)$ *and* $\mathcal{H}(2)$*. Each of them is connected and coincides with its hyperelliptic component.*
- *Each of the strata* $\mathcal{H}(2,2)$*,* $\mathcal{H}(4)$ *of the moduli space of Abelian differentials on a complex curve of genus* $g = 3$ *has two connected components: the hyperelliptic one, and one having odd spin structure. The other strata are connected for genus* $g = 3$*.*

Exercise

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.



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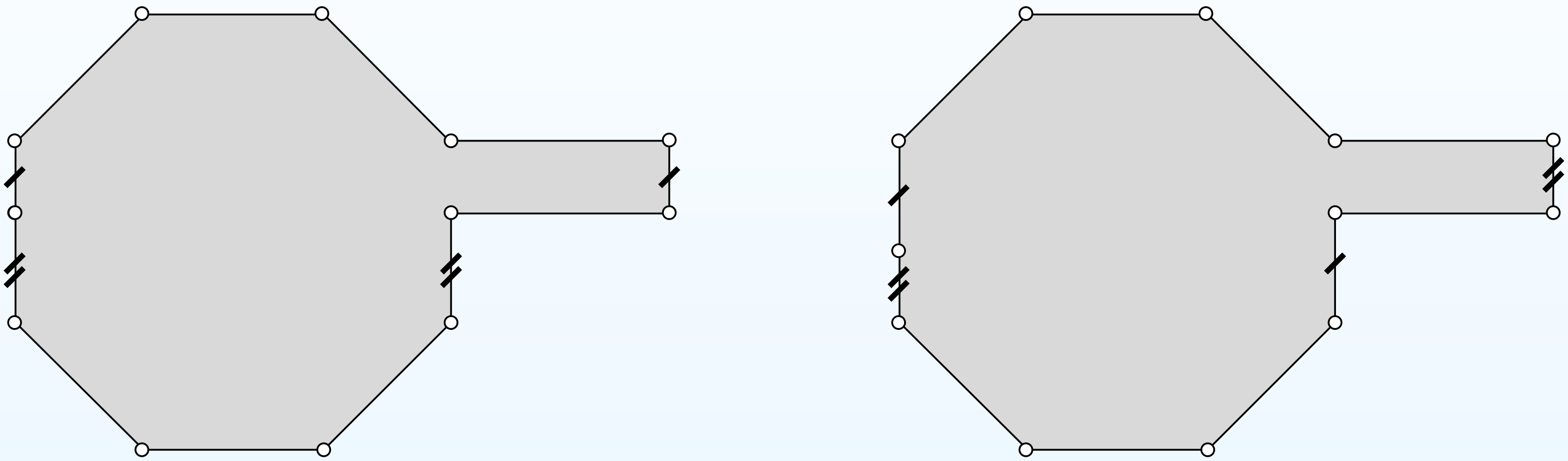

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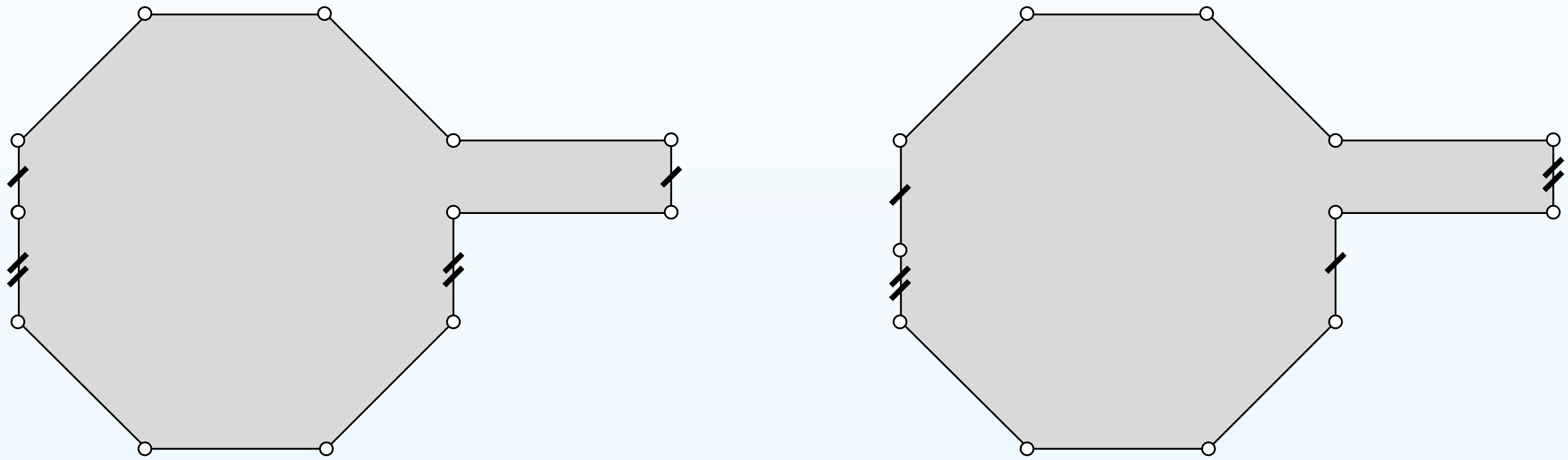

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- Determine which of the two surfaces is hyperelliptic.

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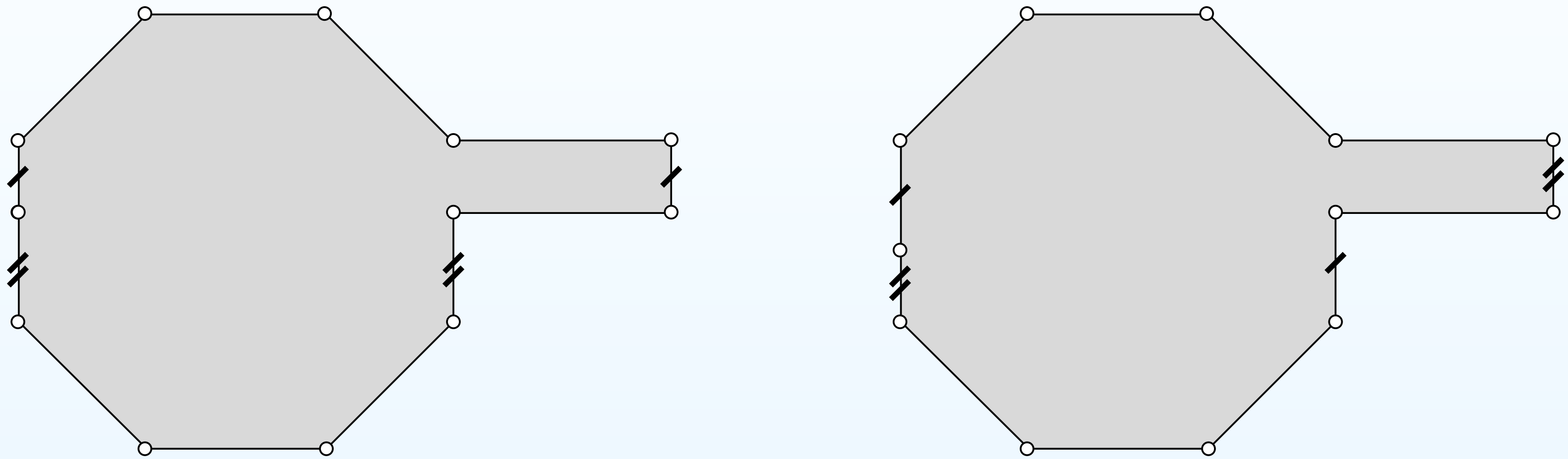

- Compute the parity of the spin structure for these surfaces (and notice that it is not the same).

- Determine which of the two surfaces is hyperelliptic.

- (*non obligatory*) Find a hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

Scheme of the proof

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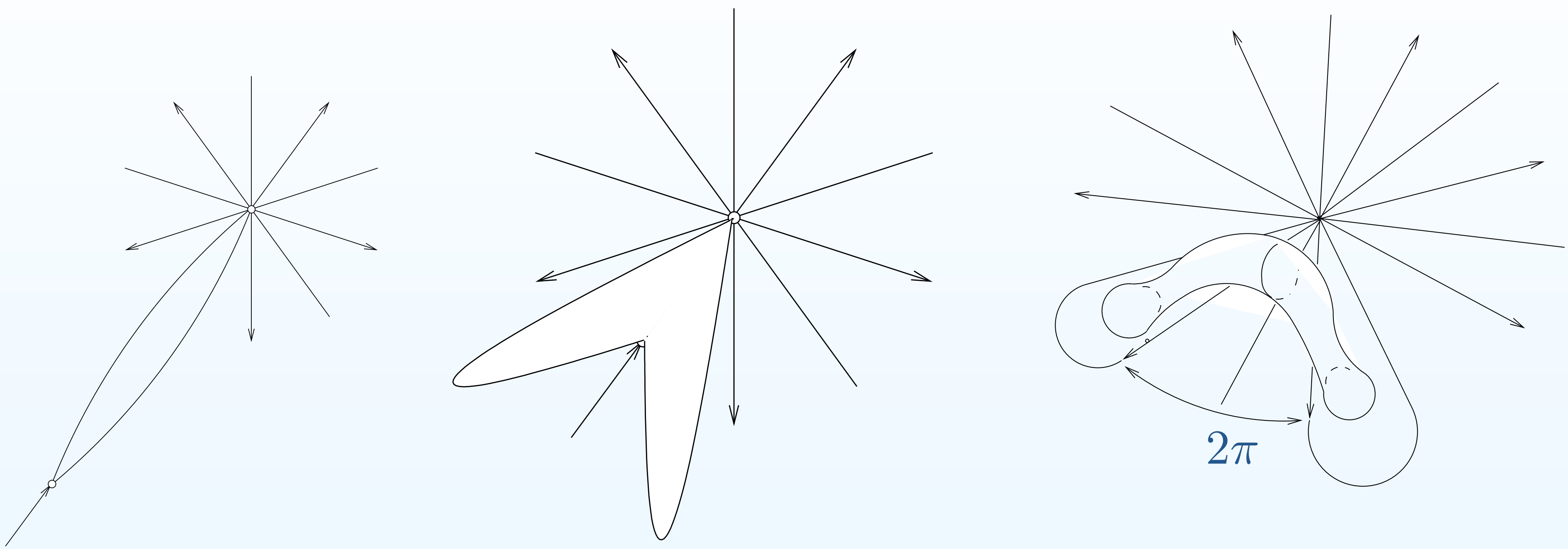
Induction in g only for the minimal stratum $\mathcal{H}(2g-2)$. Base of induction: $\mathcal{H}(0)$ and $\mathcal{H}(2)$ are connected. Then
(i) To every flat surface in $\mathcal{H}(2g-2)$ one can apply a local surgery (called "bubbling a handle") producing a surface in $\mathcal{H}(2g)$. One can bubble a handle at any surface from a continuous family of flat surfaces in $\mathcal{H}(2g-2)$ getting a continuous family of flat surfaces in $\mathcal{H}(2g)$.
(ii) Every connected component of $\mathcal{H}(2g)$ is accessible from some component of $\mathcal{H}(2g-2)$ by "bubbling a handle".

Induction in the length of partition $d_1+\cdots+d_n=2g-2$. Base of induction: $d_1=2g-2$. A Lemma from deformation theory tells that when pass from $\mathcal{H}(d_1+d_2,d_3,\dots,d_n)$ to $\mathcal{H}(d_1,d_2,d_3,\dots,d_n)$ the number of connected components might stay unchanged or decrease, but never increase. We prove that $\mathcal{H}^{even}(2l_1,\dots,2l_n)$ and $\mathcal{H}^{odd}(2l_1,\dots,2l_n)$ admit non hyperelliptic representatives for $g\geq 4$. We connect $\mathcal{H}^{even}(2g-2)$ with $\mathcal{H}^{odd}(2g-2)$ and with $\mathcal{H}^{hyp}(2g-2)$ inside any stratum of the form $\mathcal{H}(2l_1-1,2(g-l_1)-1)$ by explicit paths. Similar constructions complete the proof.

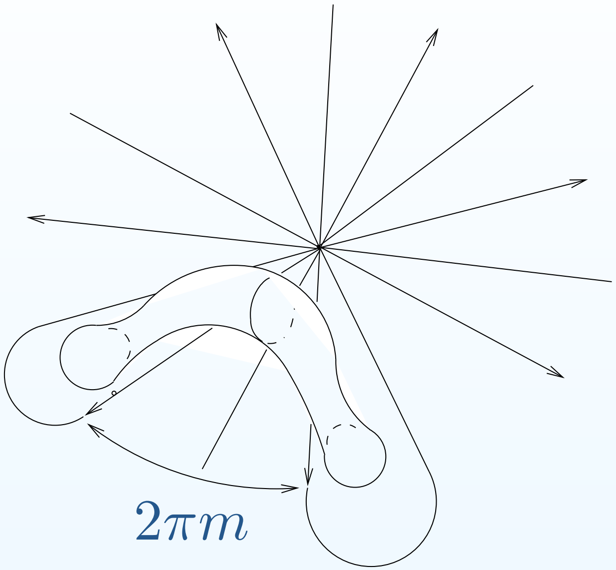
Bubbling a handle



Similarly, we can break the zero into two zeroes joined by a short saddle connection, slit the saddle connection, create a pair of loops and paste in a flat cylinder. Such local constructions allows to get all possible angles $6\pi, 10\pi, \dots$ between the “petals”.

We can chose any direction for the newborn separatrix loops. We can also continuously turn this direction creating a 1-parameter family of flat surfaces of genus $g+1$ with a bubbled handle.

Parity of spin under bubbling a handle



Lemma. *Let an Abelian differential* $\hat\omega \in \mathcal{H}^{num}(2(l_1+1), 2l_2, ..., 2l_n)$ *on a surface of genus* $g+1$ *be obtained from an Abelian differential* $\omega \in \mathcal{H}^{num}(2l_1, 2l_2, ..., 2l_n)$ *on a surface of genus* g *by "bubbling a handle". Let* $2\pi m$ *be the angle of one of the two sectors complementary to the handle. The parities of the spin structures determined by* ω *and by* $\hat\omega$ *are related in the following way:*

$$\varphi(\hat\omega) - \varphi(\omega) = m + 1 \pmod 2$$

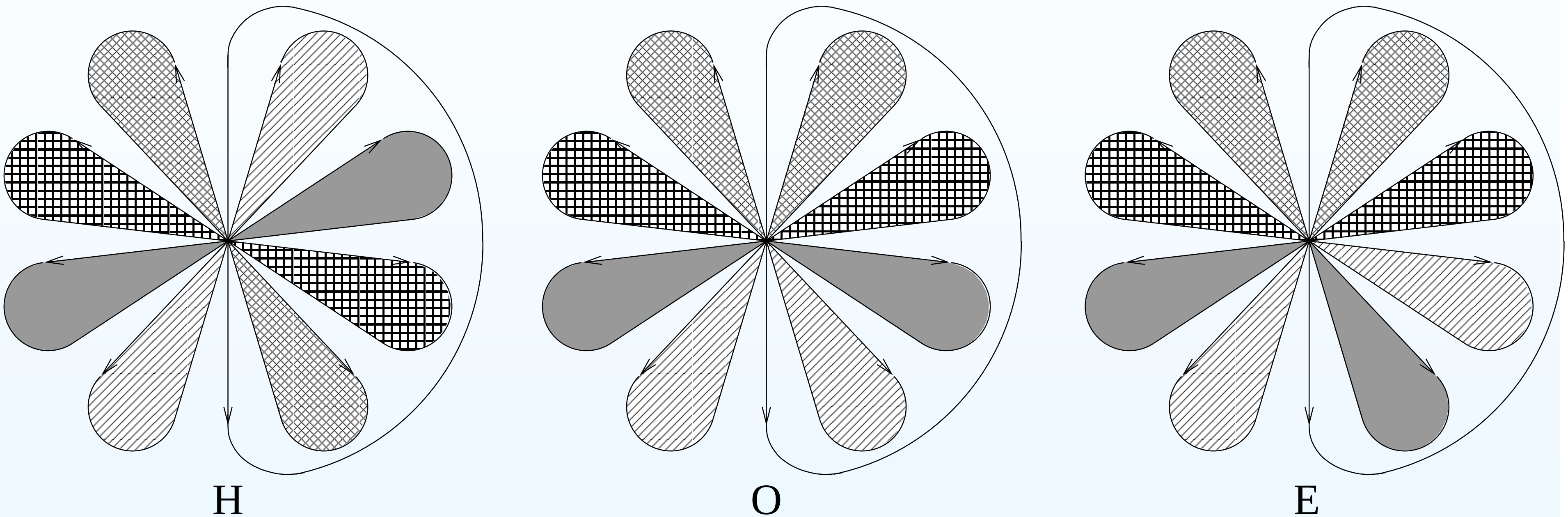
Unbubbling a handle

Lemma. *Any connected component of any stratum of Abelian differentials in genus $g+1$ contains a flat surface obtained from a flat surface in a stratum in genus g by bubbling a handle.*

The proof is elementary, but uses interval exchange transformations and Rauzy moves.

Actually, the fact is less trivial, than it seems to be at the first glance. Also, the fact that one can always merge two zeroes into one is not completely trivial (contrary to breaking up a zero). For example, for one of the two connected components of the stratum $\mathcal{Q}(9,-1)$ one cannot merge the zero and the simple pole into a single zero of order 8: merging the zero and the simple pole we necessarily get a degenerate surface. Thus, this component of $\mathcal{Q}(9,-1)$ is not adjacent to any component of $\mathcal{Q}(8)$. Fortunately, the situation is much easier with Abelian differentials and we can freely break zeroes and merge pairs of zeroes.

Representatives of components of $\mathcal{H}(2g-2)$



The separatrix diagrams represent the following components of $\mathcal{H}(8)$:
H) hyperelliptic component; O) odd spin structure; E) even spin structure.

Theorem. *Any connected component of the stratum $\mathcal{H}(2g-2)$ for $g \geq 2$ contains an Abelian differential with a horizontal foliation represented by one of the diagrams H, O, E.*

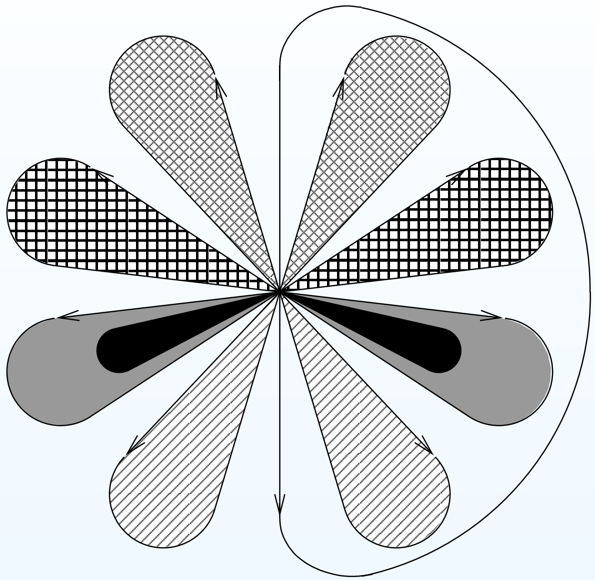
Proof

Use Lemma saying that in any connected component of $\mathcal{H}(2g)$, $g \geq 2$, one can find an Abelian differential $\hat{\omega}$ obtained from some Abelian differential $\omega \in \mathcal{H}(2g-2)$ by “bubbling a handle”.

“Remove” the corresponding handle. By the induction assumption we can deform continuously the corresponding Abelian differential ω inside $\mathcal{H}(2g-2)$ to fit one of the diagrams H, O, or E. Now we can “bubble the removed handle” along the path in the stratum $\mathcal{H}(2g-2)$. The Theorem now follows from the Lemma below.

Lemma. *Consider an Abelian differential* $\hat{\omega} \in \mathcal{H}(2g)$ *obtained by “bubbling a handle” at the zero of an Abelian differential* $\omega \in \mathcal{H}(2g-2)$ *having the horizontal foliation of one of the types* H, O, E *in genus* g*. There exist a continuous path in* $\mathcal{H}(2g)$ *joining* $\hat{\omega}$ *with an Abelian differential having the horizontal foliation of one of the types* H, O, E *in genus* $g+1$*.*

One of the cases to consider (case O)

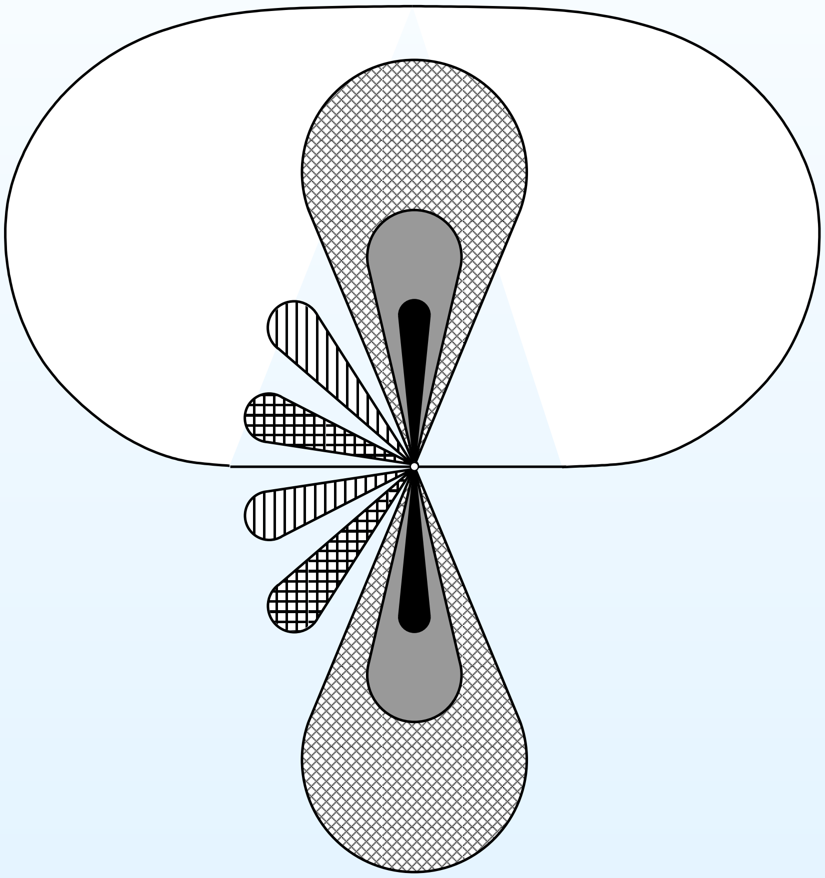


The pair of petals corresponding to a newly bubbled handle is colored in black. By a continuous deformation we can turn it to make black petals symmetric with respect to the vertical axes. If we get a diagram of type O again we are done.

Otherwise, we get a diagram as in the picture. Reversing arrows, if necessary, we may assume that the pair of simple separatrix loops next to the top vertical ray is free of black petals. Remove it. The resulting diagram has even parity of the spin structure. By the induction assumption we can deform it to the diagram E in genus g. “Recall” the removed handle. We can assume that it is located near the marked ray. The diagram thus obtained is diagram E in genus $g + 1$.

Paths joining components of smaller strata in larger strata

A path in $\mathcal{H}(k, 2g-2-k)$ joining hyperelliptic and nonhyperelliptic components of $\mathcal{H}(2g-2)$.



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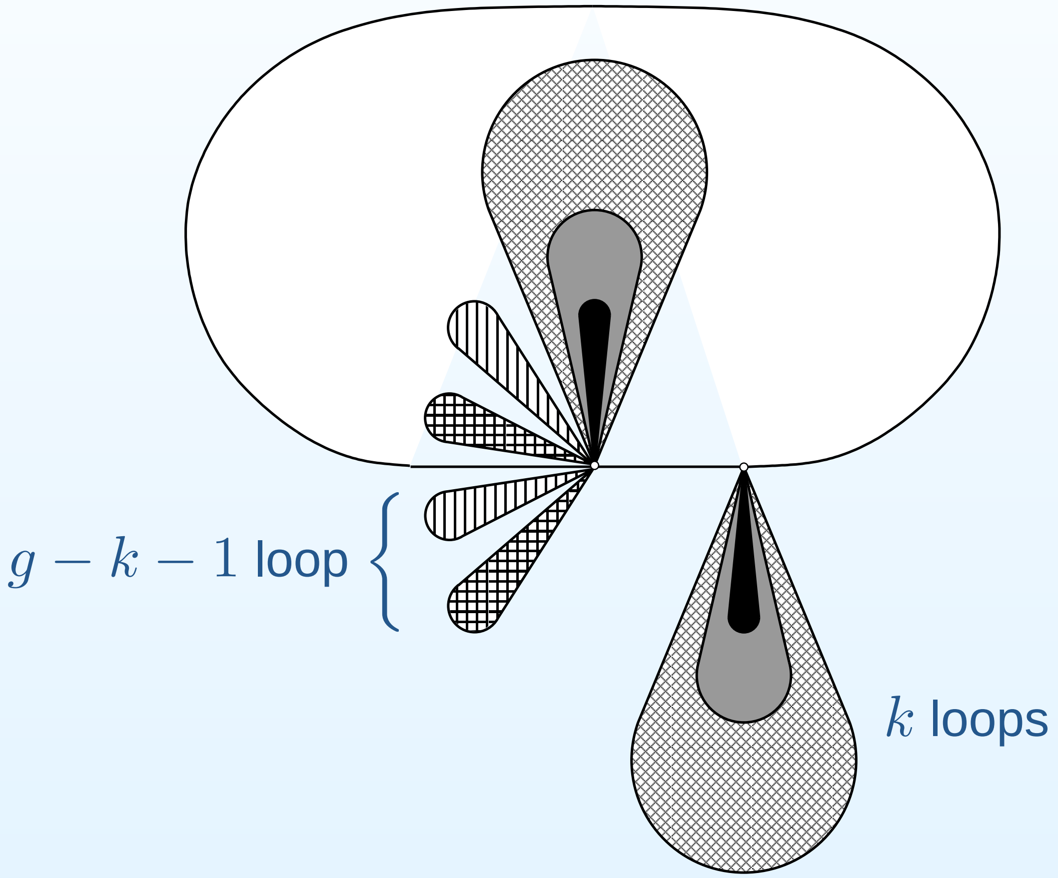

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