# Square-tiled surfaces and interval exchanges: geometry, dynamics, combinatorics and applications

## Lecture 3. Explicit representatives of components of strata

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https://arxiv.org/abs/1011.0395)

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YMSC, Tsinghua University, October 11, 2022

# Stratification of the moduli space of holomorphic 1-forms

## Stratification of the moduli space of Abelian differentials

The moduli space of Abelian differentials $\mathcal{H}_g$ is the space of pairs (Riemann surface, holomorphic 1-form on it) considered up to a natural quotient. The moduli space of Abelian differentials $\mathcal{H}_g$ is a total space of a holomorphic vector bundle over the moduli space $\mathcal{M}_g$ of Riemann surfaces with a fiber $\mathbb{C}^g$.

The space $\mathcal{H}_g$ is stratified by degrees of zeroes of holomorphic one forms. A stratum $\mathcal{H}(m_1, \dots, m_n)$, where $m_1 + \cdots + m_n = 2g - 2$, in general does not fiber over $\mathcal{M}_g$. For example, the dimension of the smallest stratum (of the one, for which all zeroes have merged to a single zero of degree $2g-2$) is much less than the dimension of $\mathcal{M}_g$: such a differential cannot be found on a general Riemann surface:

$$2g - 1 = \dim \mathrm{P}\mathcal{H}(2g-2) < \dim \mathcal{M}_g = 3g - 3 \quad \text{for} \quad g > 2\,.$$

## Hyperelliptic components

Connected components of strata provide natural ergodic components of the Teichmüller flow, which explains importance of classification of components for problems of dynamics.

For every hyperelliptic Riemann surface it is easy to construct $2g+2$ holomorphic 1-forms having a single zero of maximal multiplicity $2g-2$ at one of the $2g+2$ Weierstrass points. For each Weierstrass point the corresponding 1-form is defined up to a multiplicative constant. An elementary dimension count shows that the resulting *hyperelliptic locus* in the stratum $\mathcal{H}(2g-2)$ has the dimension of the stratum, and, hence, forms a connected component of it.

Proof. We first construct a meromorphic quadratic differential $q$ on $\mathbb{C}\mathrm{P}^1$ having a single zero of degree $2g-3$ and $2g+1$ simple poles. Location of the zero and of the poles defines $q$ up to a multiplicative constant. We can choose freely a configuration of $2g+2$ points. A modular transformation sends three points to $0, 1, \infty$, which leaves $2g$ free parameters. The quadratic differential induced on the double cover ramified at all zeroes and poles of $q$ is a square of a globally defined holomorphic 1-form in $\mathcal{H}^{hyp}(2g-2)$. Thus, $\dim \mathcal{H}^{hyp}(2g-2) = 2g$. The ambient stratum has the same dimension $\dim \mathcal{H}(2g-2) = 2g$.

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A similar dimension count shows that the hyperelliptic locus has full dimension in exactly one other stratum, namely, in $\mathcal{H}(g-1,g-1)$. The two zeroes of any holomorphic differential in the hyperelliptic component $\mathcal{H}^{hyp}(g-1,g-1)$ are in involution. The complementary hyperelliptic locus, for which the two zeroes are fixed by the involution, has complex codimension $1$.

## Parity of the spin-structure

If multiplicities (orders) of all zeroes of a holomorphic 1-form $\omega$ are even, $\omega$ carries *odd* or *even spin-structure*. It is defined as the parity of the dimension of the linear system corresponding to the divisor $\frac{1}{2}K(\omega)$. Deformations of the pair (Riemann surface, holomorphic 1-form) inside the ambient stratum make change this dimension. However, by independent results of M. Atiyah and D. Mumford, the jumps of dimension are always even. Thus, the parity of the spin-structure depends only on the connected component of the ambient stratum $\mathcal{H}(2k_1, \dots, 2k_n)$.

Merging several zeroes of even degrees of the holomorphic 1-form into a single zero by a continuous deformation we again get a 1-form with zeroes of even degrees. Results of M. Atiyah and D. Mumford imply that it would define the same parity of the spin structure as the original 1-form.

Another way to define the parity of the spin-structure uses the flat metric inherited from the holomorhic 1-form and the induced Gauss map on smooth representatives of a basis of cycles on the Riemann surface.

## Index of a smooth closed path on a flat surface

Consider a simple smooth closed path $\rho$ on a flat surface avoiding conical singularities. At any point of the surfaces we know where is the “direction to the North”. Hence, at any point $z = \rho(t)$ we can apply a compass and measure the direction of the tangent vector $\dot{z}$. Moving along $\rho$ we make the tangent vector turn in the compass.

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We get a map $G(\rho) : S^1 \to S^1$ from the parameterized closed path to the circumference of the compass. This map is called the *Gauss map*.

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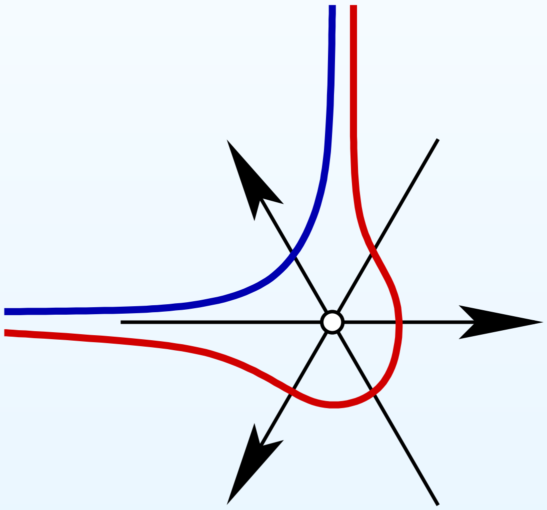
We define the *index* $ind(\rho)$ of the path $\rho$ as a degree of the corresponding Gauss map (or, in other words as the algebraic number of turns of the tangent vector around the compass) taken modulo $2$.

$$ind(\rho) = \deg G(\rho) \mod 2$$

## Parity of the spin structure

It is easy to see that $ind(\rho)$ does not depend on parametrization. Moreover, it does not change under small deformations of the path.

**Exercise.** If a conical point $P$ has a cone angle which is an odd multiple of $2\pi$, then bypassing $P$ on one side or on the other we get the same $ind(\rho)$.



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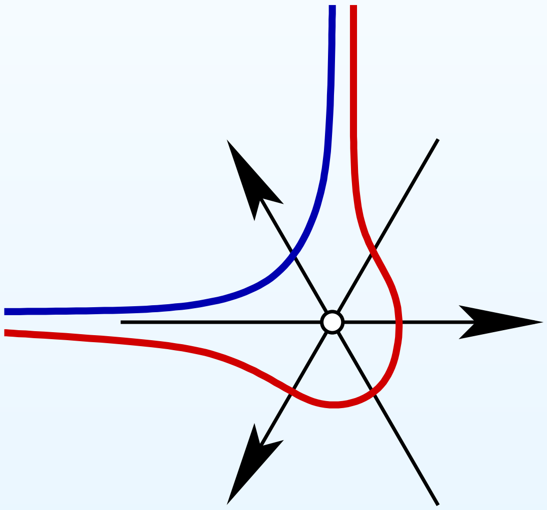


Consider a collection of simple closed smooth paths $a_1, b_1, \ldots, a_g, b_g$ representing a symplectic basis of homology $H_1(S, \mathbb{Z}/2\mathbb{Z})$. We define a *parity of the spin-structure* of a flat surface $S \in \mathcal{H}(2d_1, \ldots, 2d_n)$ as

$$\phi(S) = \sum_{i=1}^{g} \left(ind(a_i) + 1\right)\left(ind(b_i) + 1\right) \mod 2$$

**Exercise.** Compute a parity of the spin structure for a flat torus.

# Classification Theorem: genus four and higher

**Theorem (M. Kontsevich, A.Z.)** *General case: $g \geq 4$.*

• *The stratum $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2g-2)$, and two other components: $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures.*

• *The stratum $\mathcal{H}(2l, 2l)$, $l \geq 2$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2l, 2l)$, and two other components: $\mathcal{H}^{even}(2l, 2l)$ and $\mathcal{H}^{odd}(2l, 2l)$.*

• *All the other strata of the form $\mathcal{H}(2l_1, \ldots, 2l_n)$, where all $l_i \geq 1$, have two connected components: $\mathcal{H}^{even}(2l_1, \ldots, 2l_n)$ and $\mathcal{H}^{odd}(2l_1, \ldots, 2l_n)$, corresponding to even and odd spin structures.*

• *The strata $\mathcal{H}(2l-1, 2l-1)$, $l \geq 2$, have two connected components; one of them, $\mathcal{H}^{hyp}(2l-1, 2l-1)$, is hyperelliptic; the other one, $\mathcal{H}^{nonhyp}(2l-1, 2l-1)$, is not.*

• *All other strata of Abelian differentials on complex curves of genera $g \geq 4$ are nonempty and connected.*

## Classification Theorem: low genera

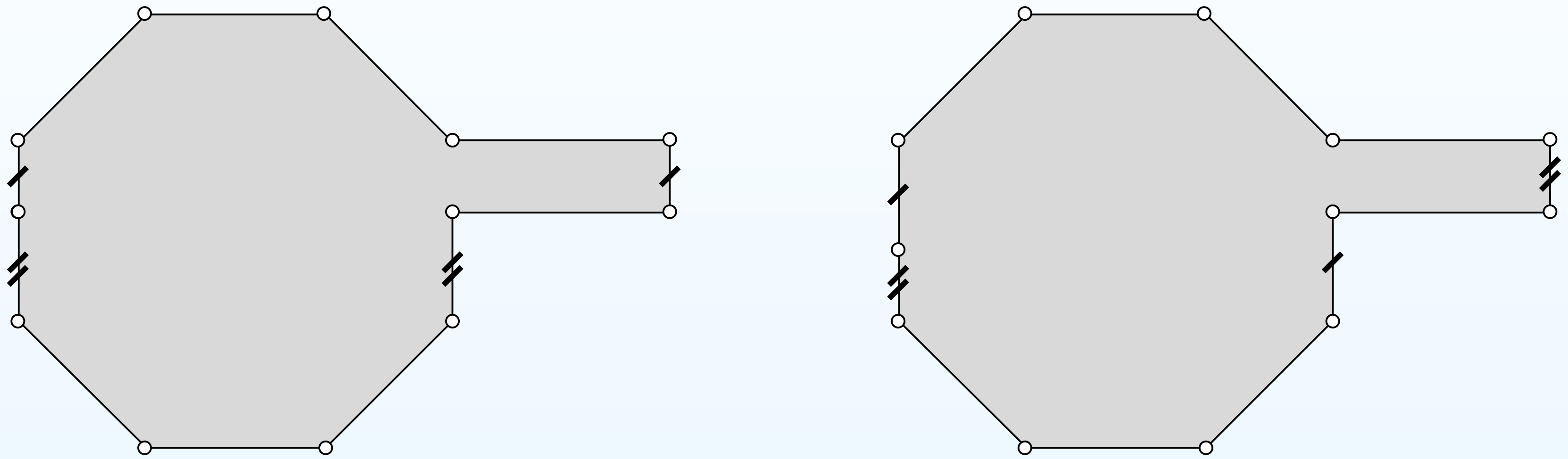
The theorem below shows that in genera $g = 2, 3$ some components are missing with respect to the general case.

**Theorem**

- *The moduli space of Abelian differentials on a complex curve of genus* $g = 2$ *contains two strata:* $\mathcal{H}(1,1)$ *and* $\mathcal{H}(2)$. *Each of them is connected and coincides with its hyperelliptic component.*
- *Each of the strata* $\mathcal{H}(2,2)$, $\mathcal{H}(4)$ *of the moduli space of Abelian differentials on a complex curve of genus* $g = 3$ *has two connected components: the hyperelliptic one, and one having odd spin structure. The other strata are connected for genus* $g = 3$.

## Exercise

- Check that the following two flat surfaces belong to the stratum $\mathcal{H}(4)$.



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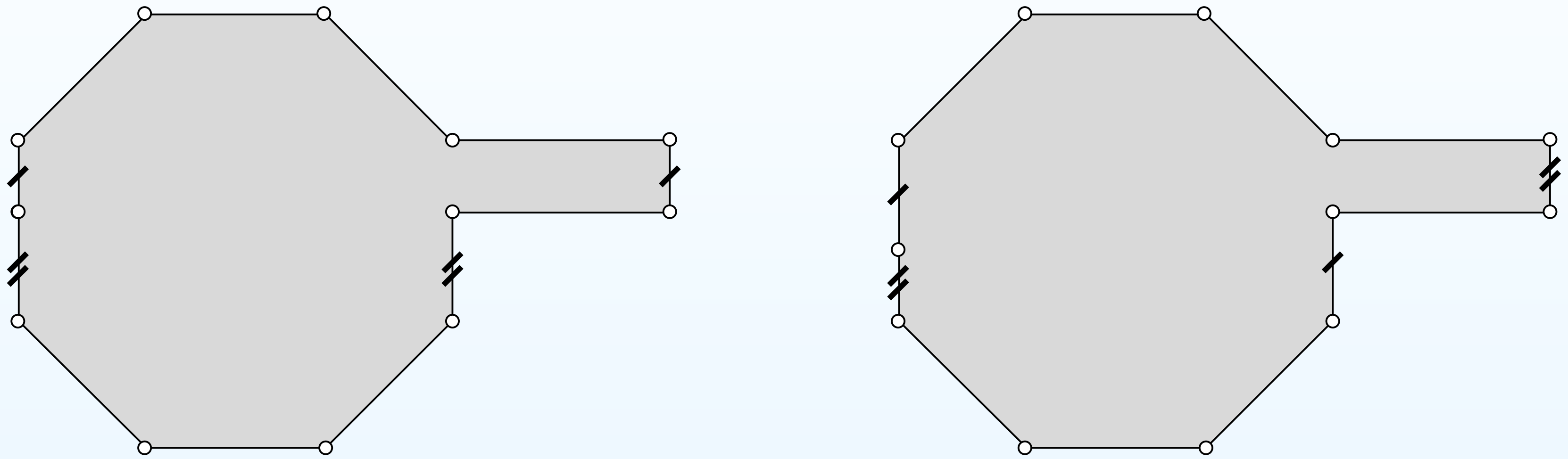

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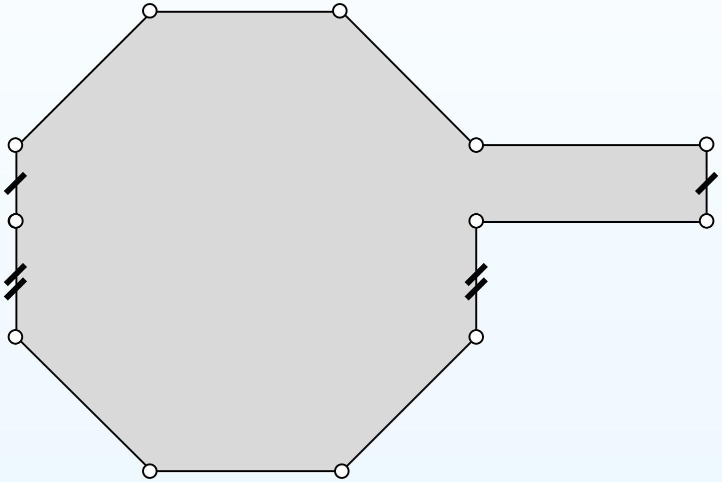

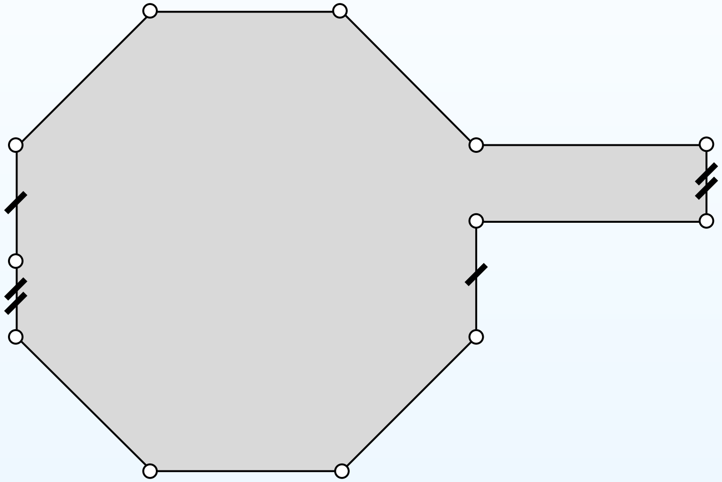

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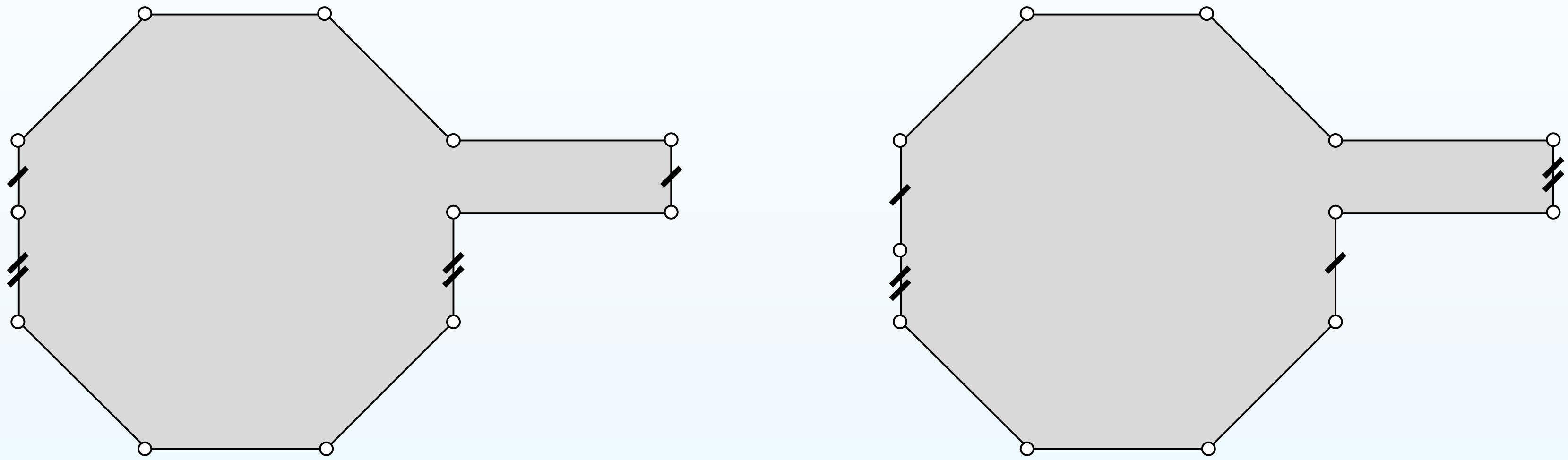


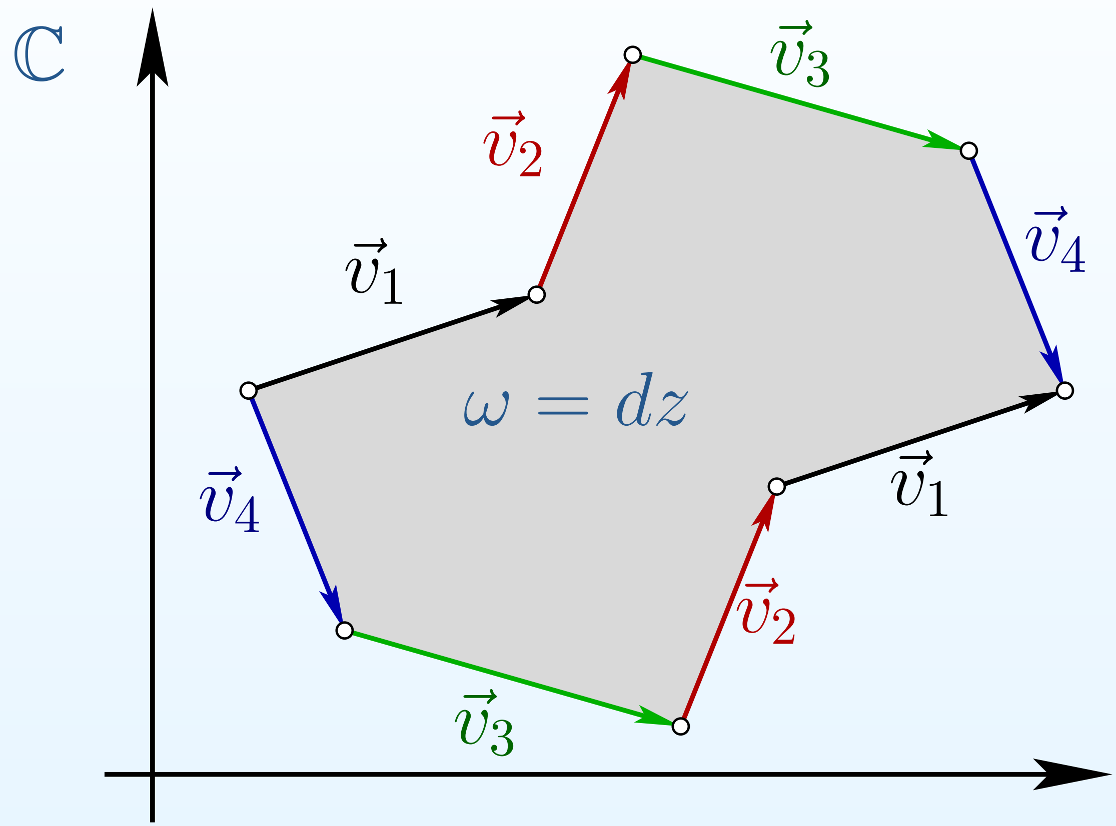
- Compute the parity of the spin structure for these surfaces (and notice that it is not the same).

- Determine which of the two surfaces is hyperelliptic.

- (*non obligatory*) Find the hyperelliptic involution of this surface in geometric terms. Find the Weierstrass points (the fixed points of the hyperelliptic involution). Check that there are $2g + 2$ such points.

# Explicit representatives of connected components

## Holomorphic 1-form associated to a flat structure




The form $\omega = dz$ has zeroes exactly at those points of $S$ where the flat structure has conical singularities.

## Flat metric defined by a holomorphic 1-form

Reciprocally a pair (Riemann surface $X$, holomorphic 1-form $\omega$) uniquely defines a flat structure. Namely, in a simple-connected neighborhood $U \subset X$ one can choose a coordinate $z$ such that $\omega = dz$. This is the flat coordinate. For a pair of such overlapping neighborhoods with coordinates $z$ and $z'$ one has $z' = z + const$ on the overlaps, so the change of coordinates is a parallel translation.

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In a neighborhood of zero a holomorphic 1-form can be represented as $w^d\, dw$, where $d$ is the **degree** of zero. The form $\omega$ has a zero of degree $d$ at a conical point with cone angle $2\pi(d+1)$.

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The moduli space $\mathcal{H}_g$ of pairs (complex structure, holomorphic 1-form) is naturally stratified by the strata $\mathcal{H}(d_1, \dots, d_n)$ enumerated by unordered partitions $d_1 + \dots + d_n = 2g - 2$.

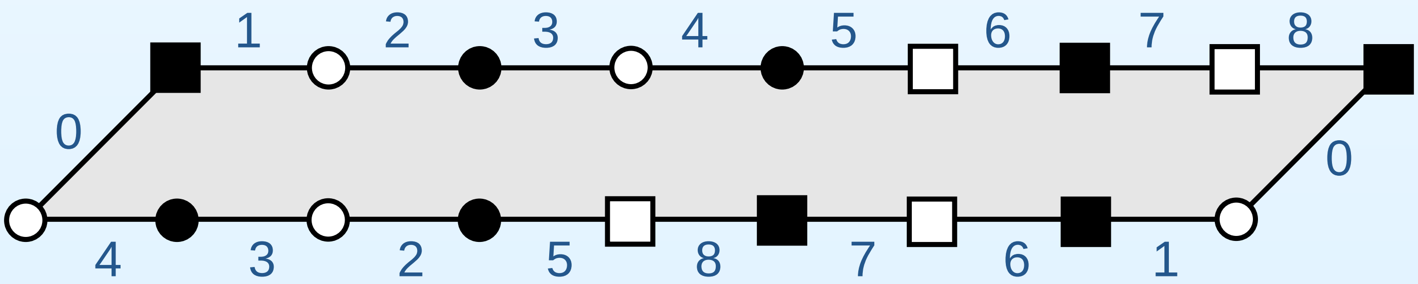
Any holomorphic 1-forms corresponding to a fixed stratum $\mathcal{H}(d_1, \dots, d_n)$ has exactly $n$ zeroes of degrees $d_1, \dots, d_n$.

## Polygon which mimics a parallelogram

In case when the permutation $\pi \in S_n$ defining the polygon has the property $\pi^{-1}(n) = 1$, we can choose all vectors $\vec{V}_2, \ldots, \vec{V}_n$ horizontal. The resulting polygon will have the shape of a parallelogram with extra vertices added to horizontal sides.

It would be convenient to shift enumeration of elements in the permutation $\pi$ starting it from $0$. Under this convention the polygon below corresponds to the permutation $\pi$, where

$$\pi^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 5 & 8 & 7 & 6 & 1 & 0 \end{pmatrix}$$



## Strebel differentials

A holomorphic 1-form $\omega = dz = dx + idy$ on such a special polygon has the following property: all nonsingular leaves of the foliation defined by its imaginary part $dy$ are closed circles. A holomorphic form having this property is called *Strebel differential*. In our case the union of all regular leaves forms a single flat cylinder. Record the cyclic order of saddle connections on the top boundary component of this single cylinder. The ambient stratum and the connected component of the stratum are uniquely determined by the cyclic order in which the saddle connections show up on the bottom boundary component.

*Our goal can be now formulated as follows. For each connected component of each stratum present a cyclic permutation realizing a one-cylinder Strebel holomorphic 1-form in the given component.*

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## What for?

● For example, for numerical experiments like computation of Siegel–Veech constants, Lyapunov exponents, cylinder decomposition statistics of square-tiled surfaces etc. Example with our paper with A. Eskin and H. Masur.

● To study geometry and topology of the embodying connected component of the stratum. For example, letting all saddle connections be integer, we get a square-tiled surface. One can efficiently construct the associated arithmetic Teichmüller disc (as the quotient of the $\mathrm{SL}(2,\mathbb{R})$-orbit passing through the corresponding integer point of the stratum). One can compute certain geometric characteristics of the resulting Teichmüller curve, as, for example, the degree of the Hodge bundle.

## Idea of construction

We construct such a representative in two steps. We use an alternative representation of one-cylinder differentials now cutting the single cylinder filled with horizontal circles not by a singular horizontal leaf on the boundary of the cylinder (i.e. not by a chain of horizontal saddle connections), but by a regular horizontal leaf — by a "waist curve" of the cylinder. We construct representatives of the strata $\mathcal{H}(1,\dots,1)$, $\mathcal{H}^{even}(2,\dots,2)$, $\mathcal{H}^{odd}(2,\dots,2)$ explicitly. Given any connected component of any stratum we contract appropriate saddle connections of appropriate differential from the above list thus merging groups of zeroes. We shall see that in this way we can get to any component of any stratum.

Consider an explicit example in which we first construct a representative of the stratum $\mathcal{H}(1,1,1,1)$ and then deform it into a representative of the stratum $\mathcal{H}(3,1)$, by merging three simple zeroes into one.

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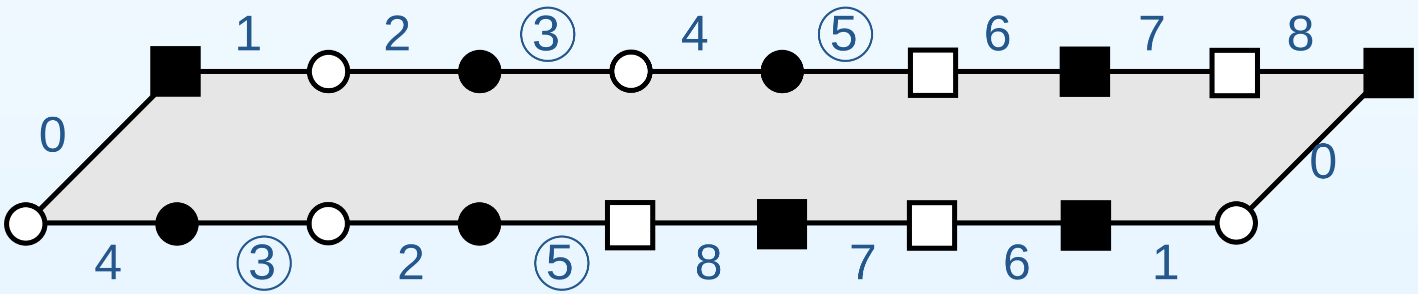
## Getting from $\mathcal{H}(1,1,1,1)$ to $\mathcal{H}(3,1)$



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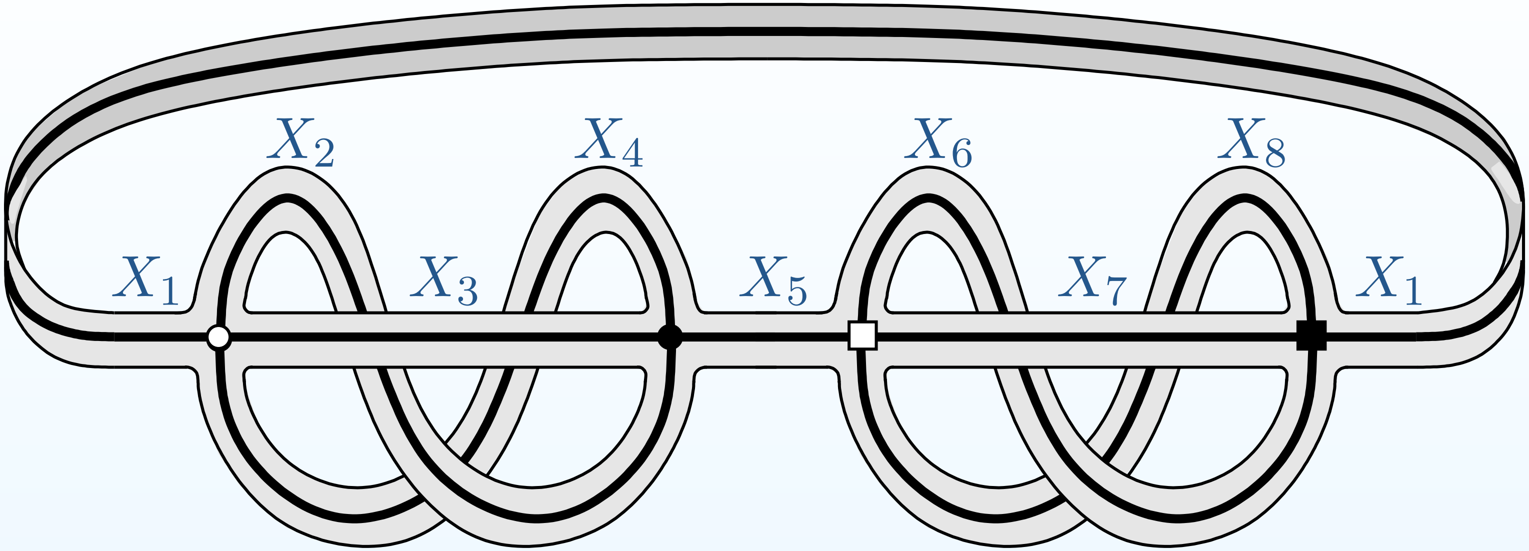


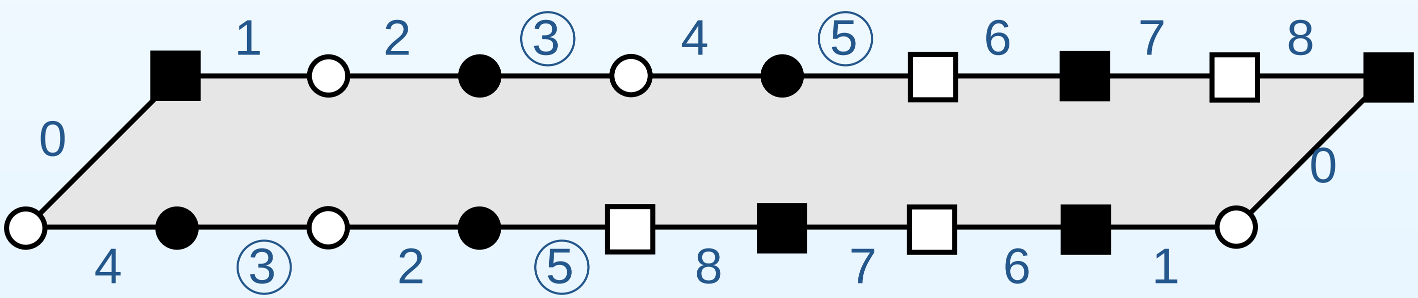

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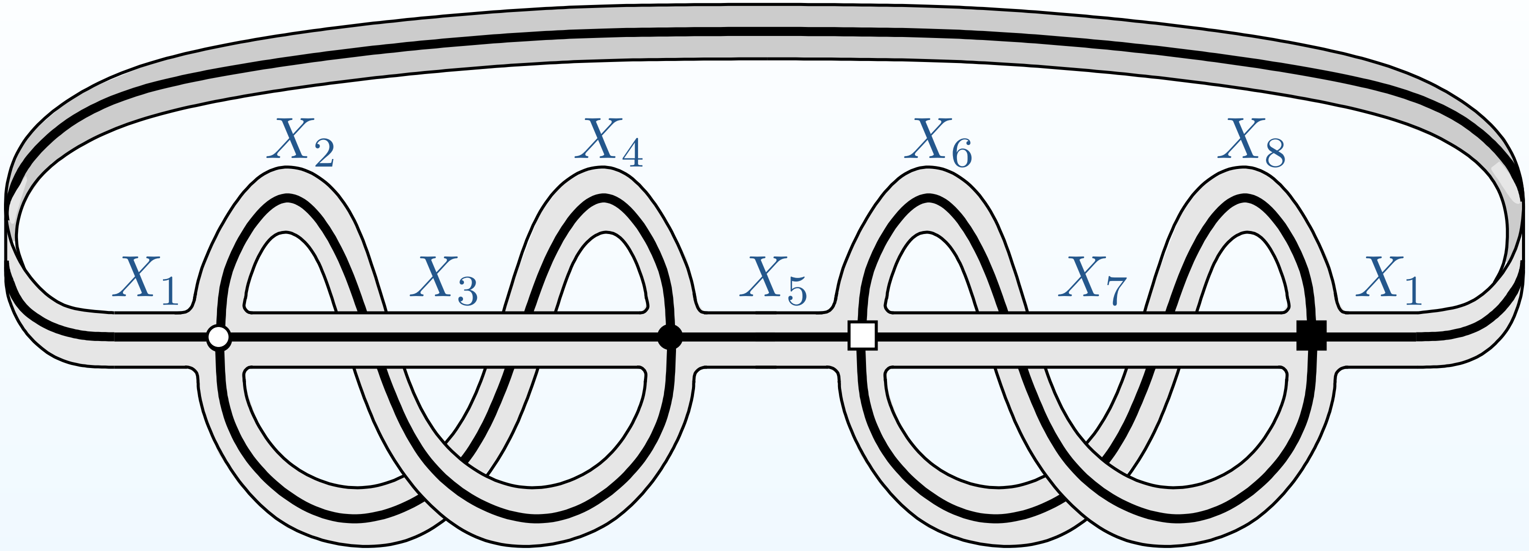
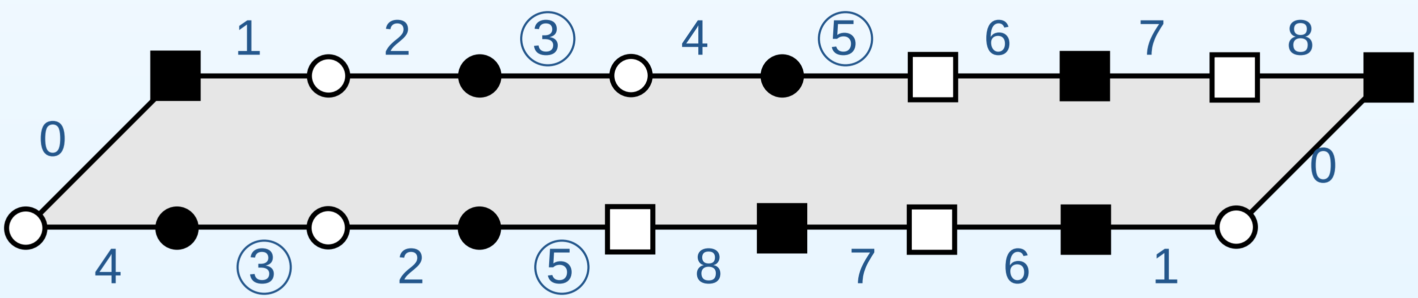
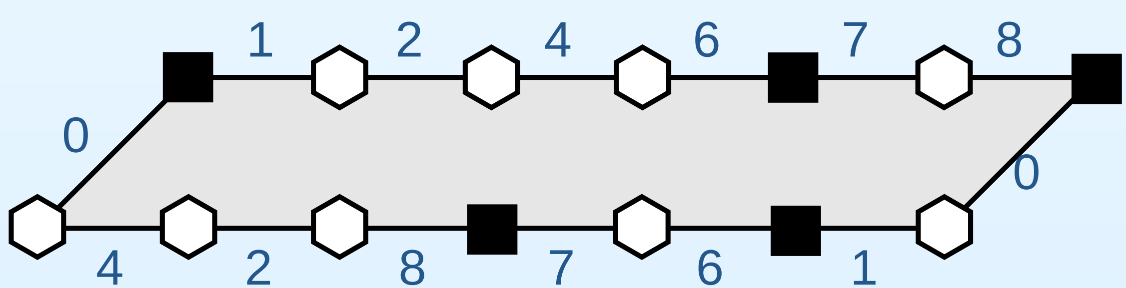
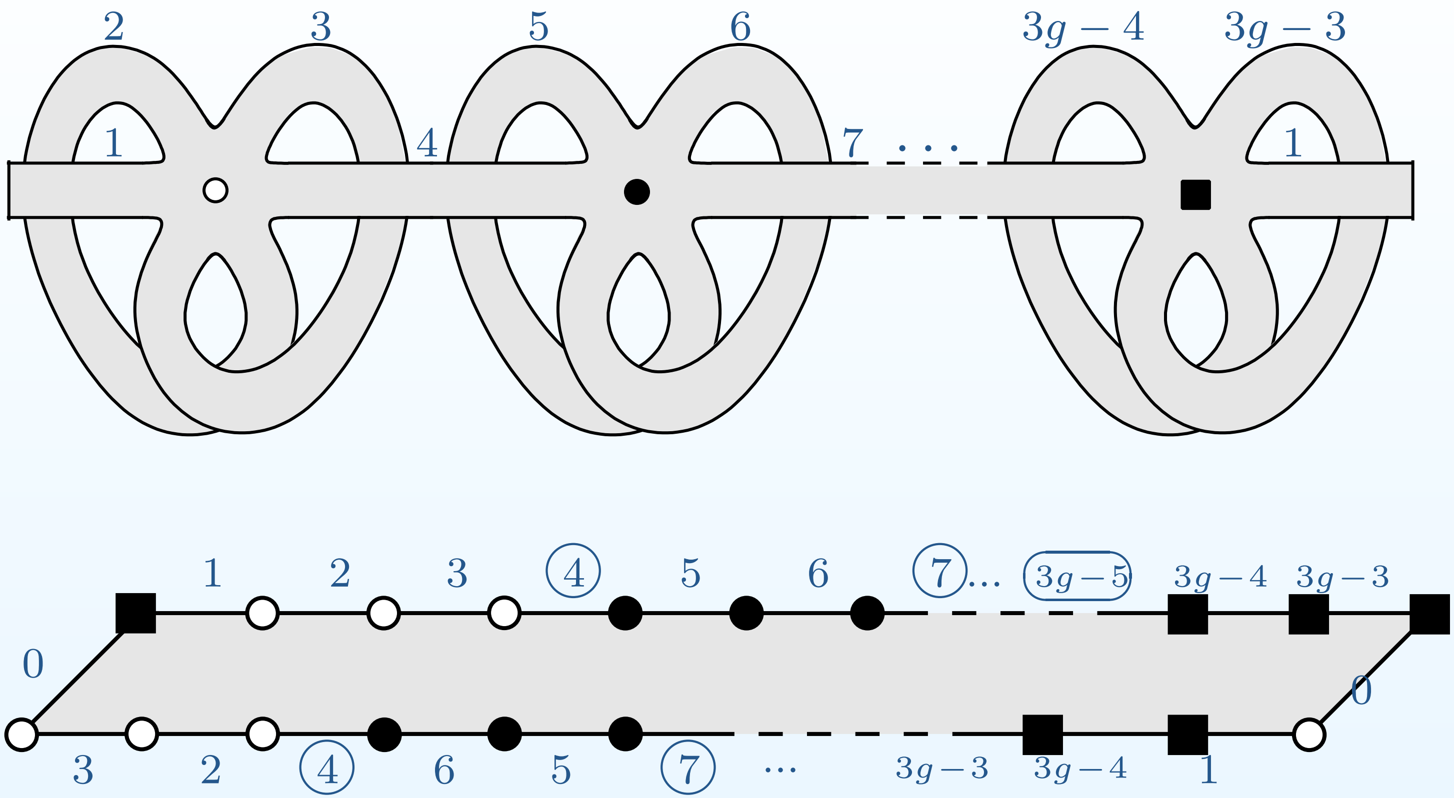


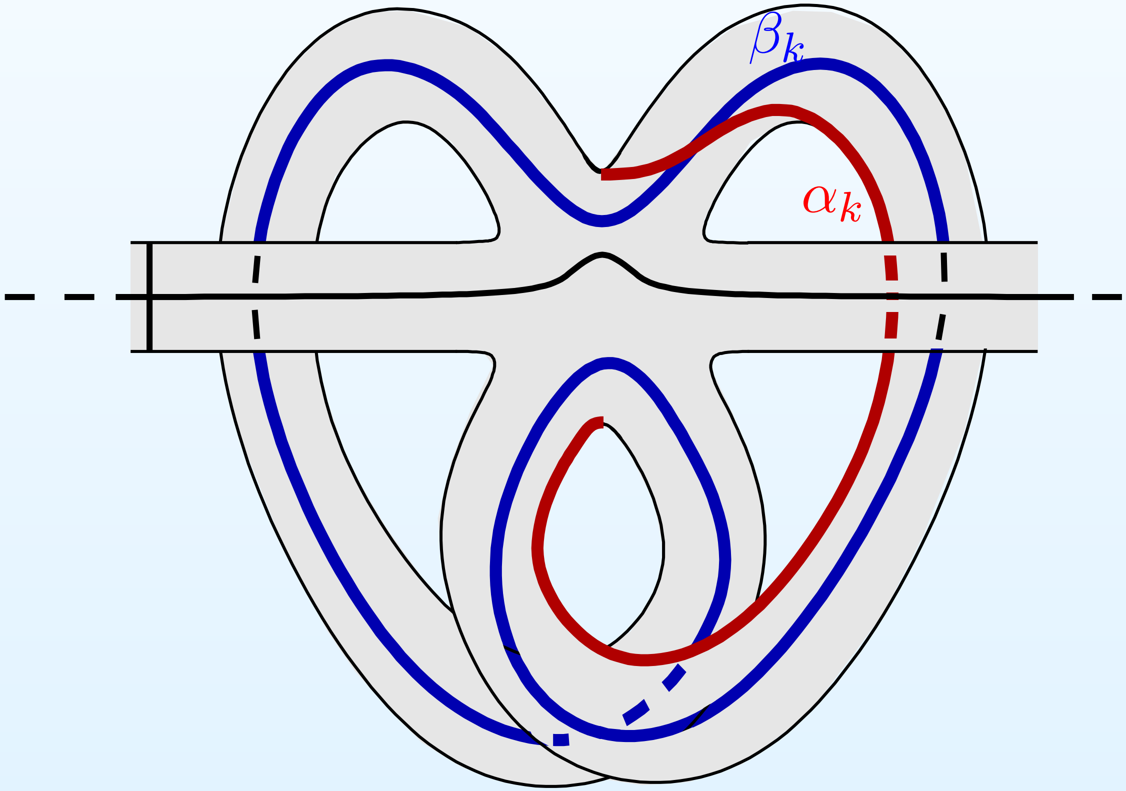
## Representative of $\mathcal{H}^{odd}(2,\ldots,2)$




A one-cylinder Strebel differential from the component $\mathcal{H}^{odd}(2,\ldots,2)$ is represented by a ribbon graph on top and by a cylinder (on the bottom). Any subcollection of saddle connections with marked indices $4, 7\ldots, 3g-5$ is suitable for contraction.

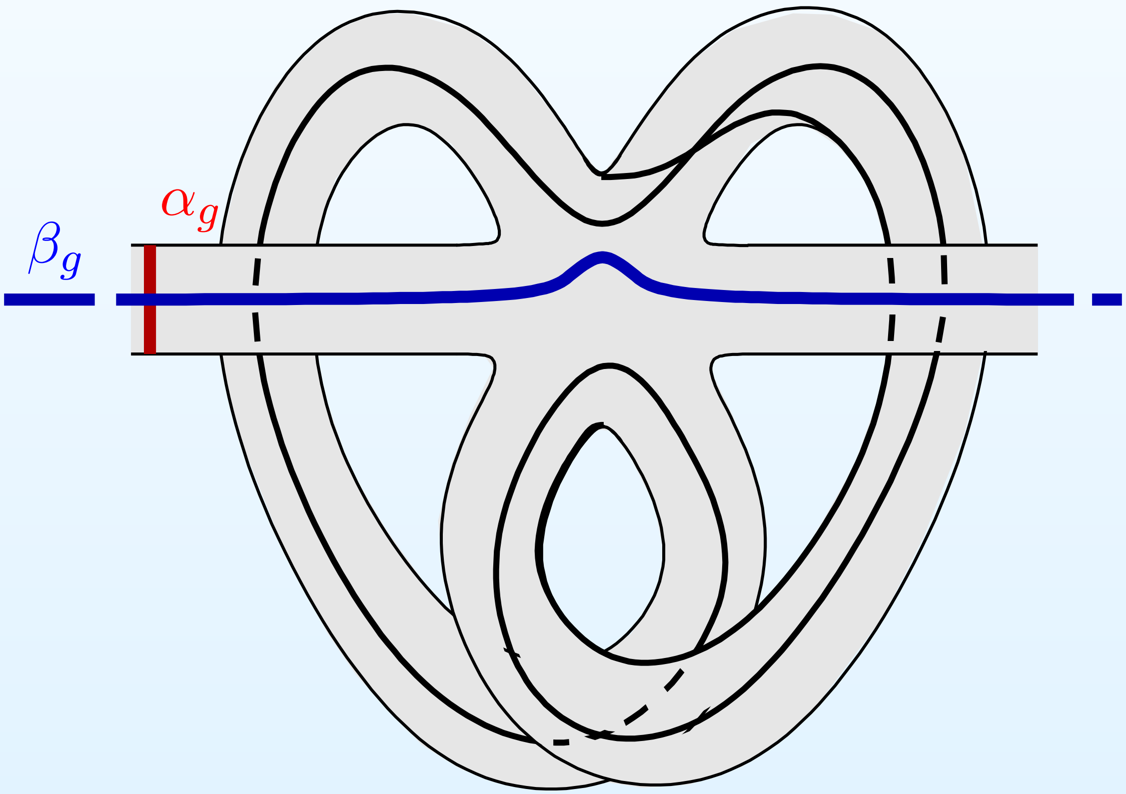
## Basis of cycles

For each of the $g-1$ repetitive fragments of our surface we construct a pair of smooth closed curves $\alpha_k, \beta_k$ as in the picture. By construction each of the curves is everywhere transverse to the vertical foliation and curves $\alpha_k$ and $\beta_k$ have a single transverse intersection.

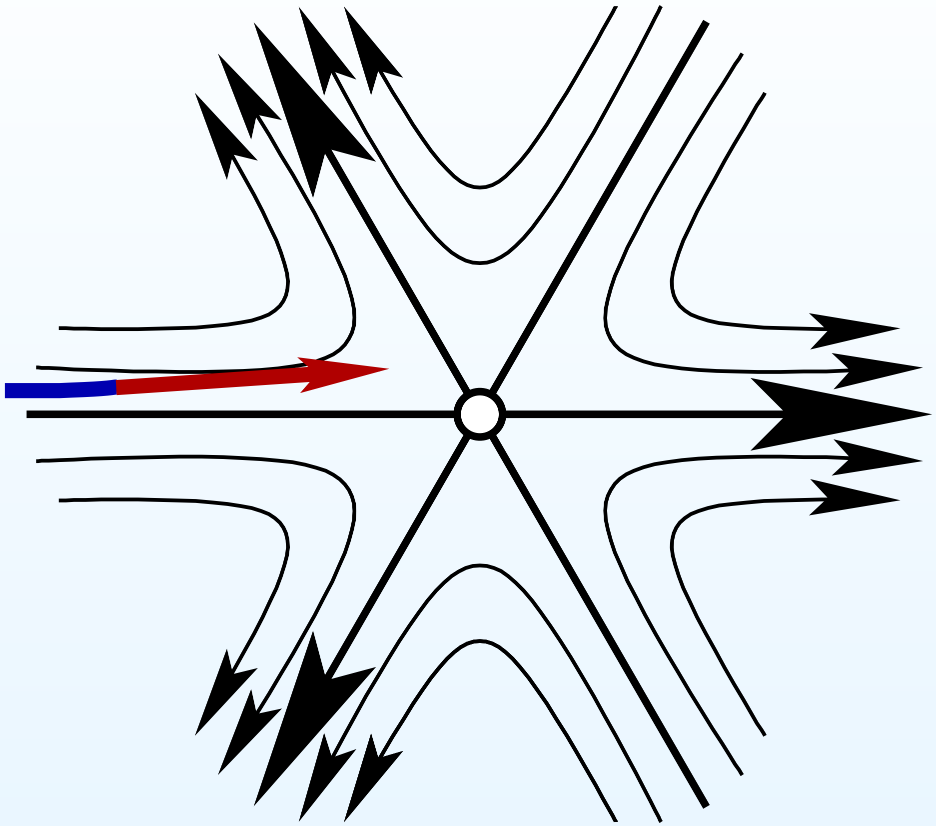


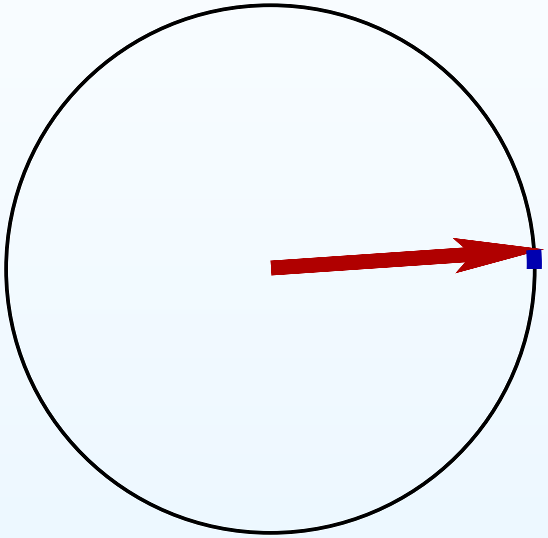
## Basis of cycles

We construct one more pair of smooth closed curves $\alpha_g, \beta_g$ as in the picture. We can choose $\alpha_g$ to be a closed leaf of the vertical foliation and make $\beta_g$ follow the core horizontal leaf in the complement of neighborhoods of zeroes and bypass the zeroes as in the picture.



## Index of the curve $\beta_g$





Following the blue path we make its image under the Gauss map perform a complete turn around the unit circle in the counterclockwise direction.

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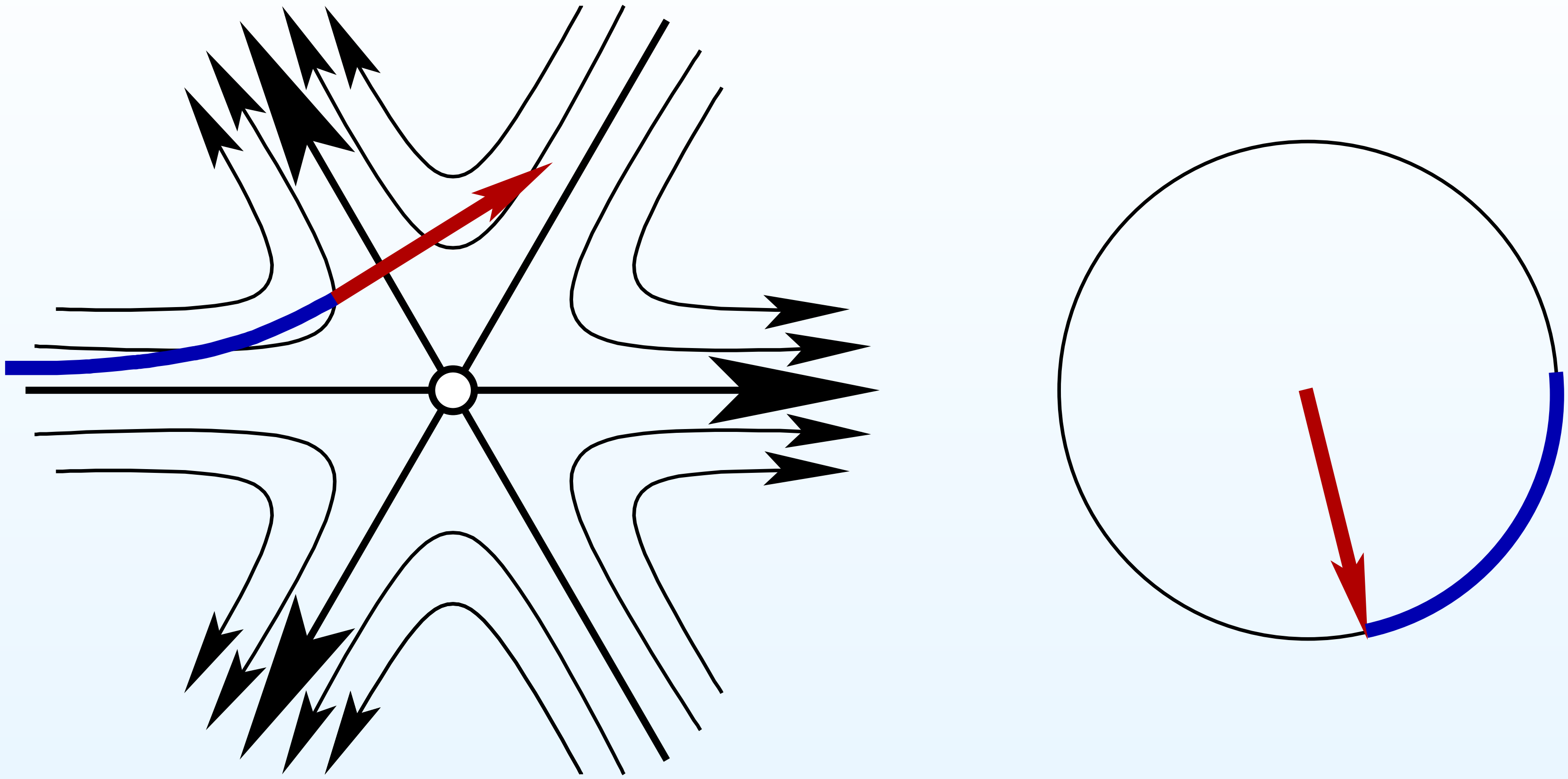

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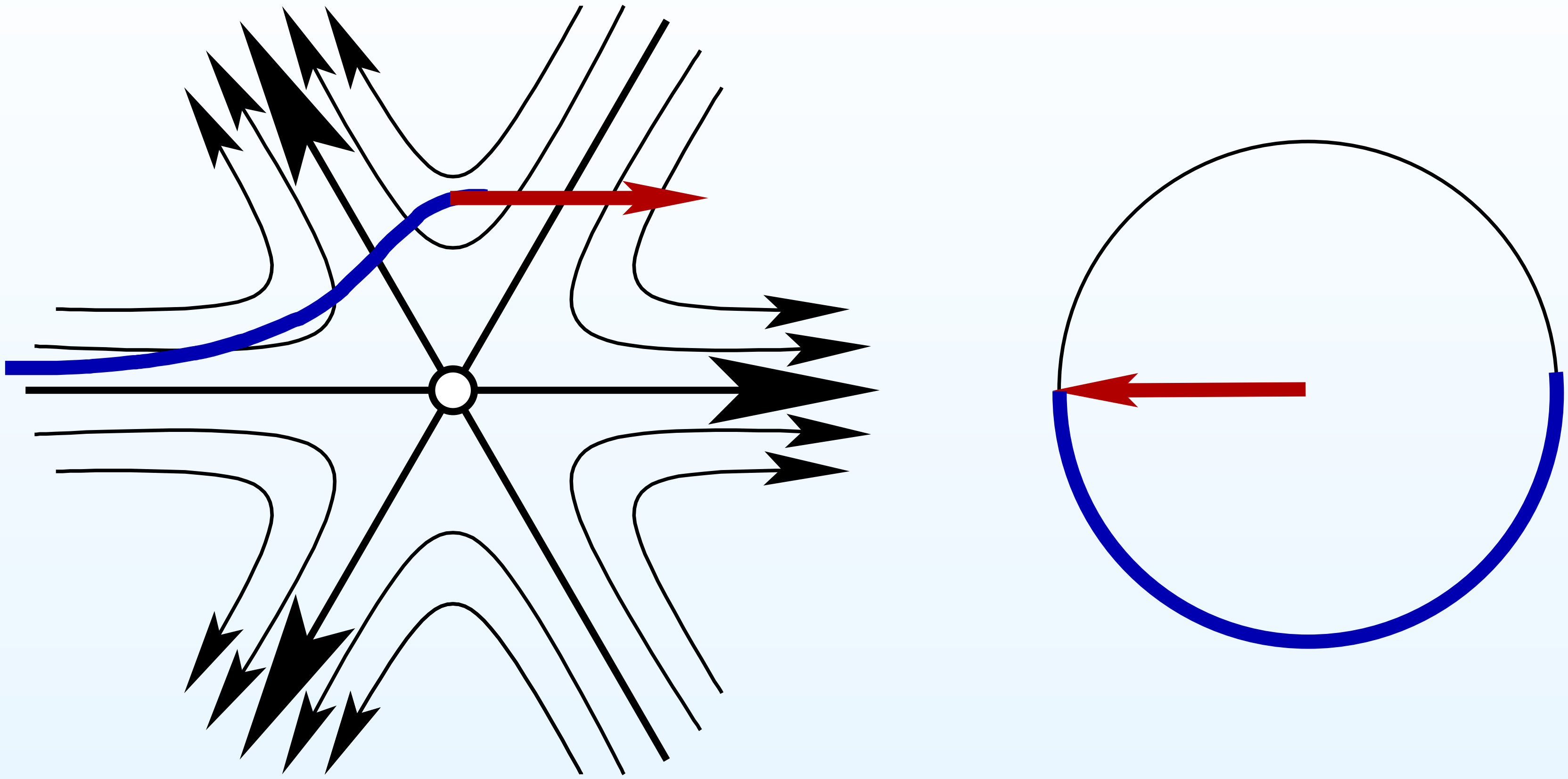

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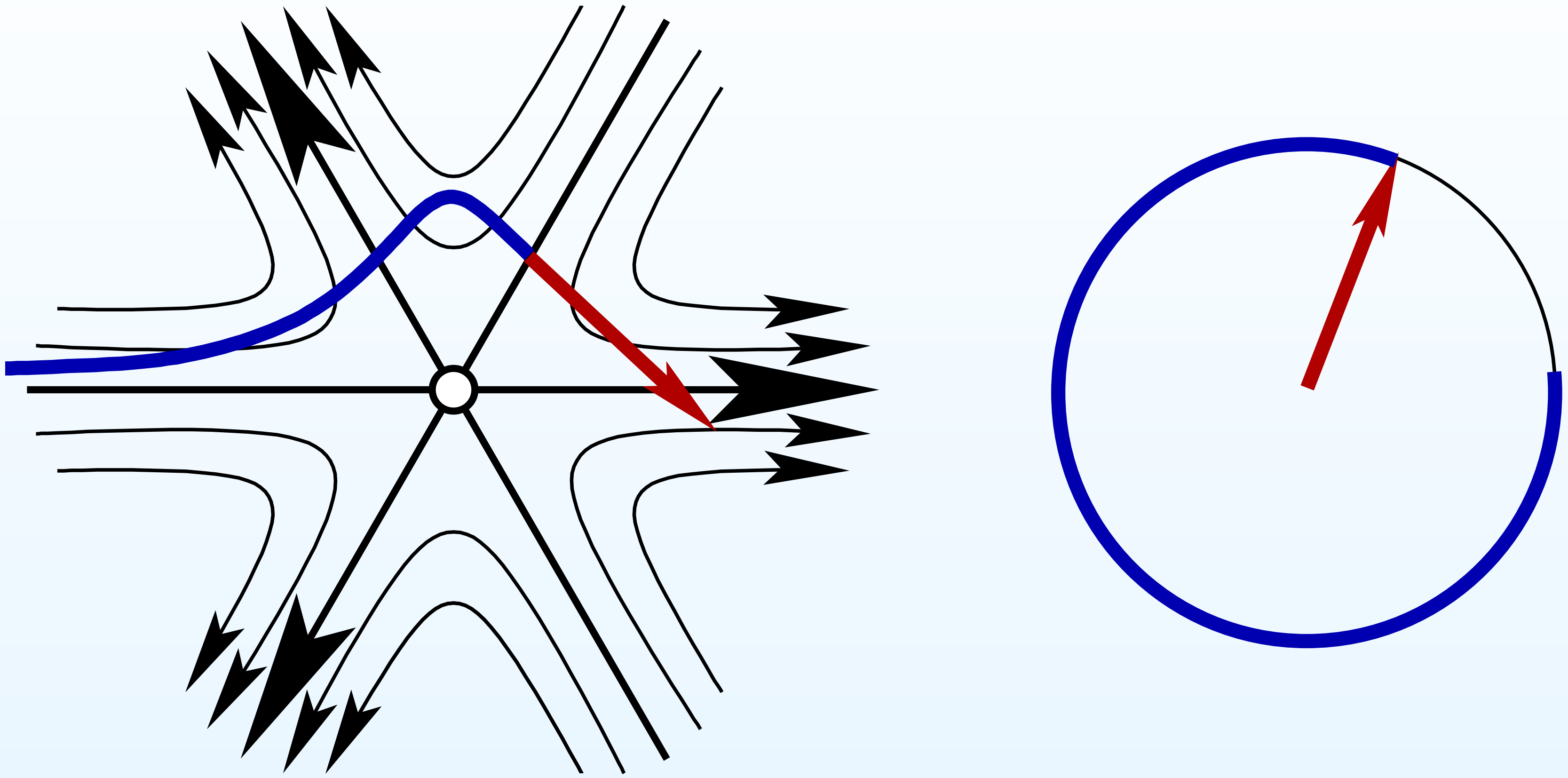

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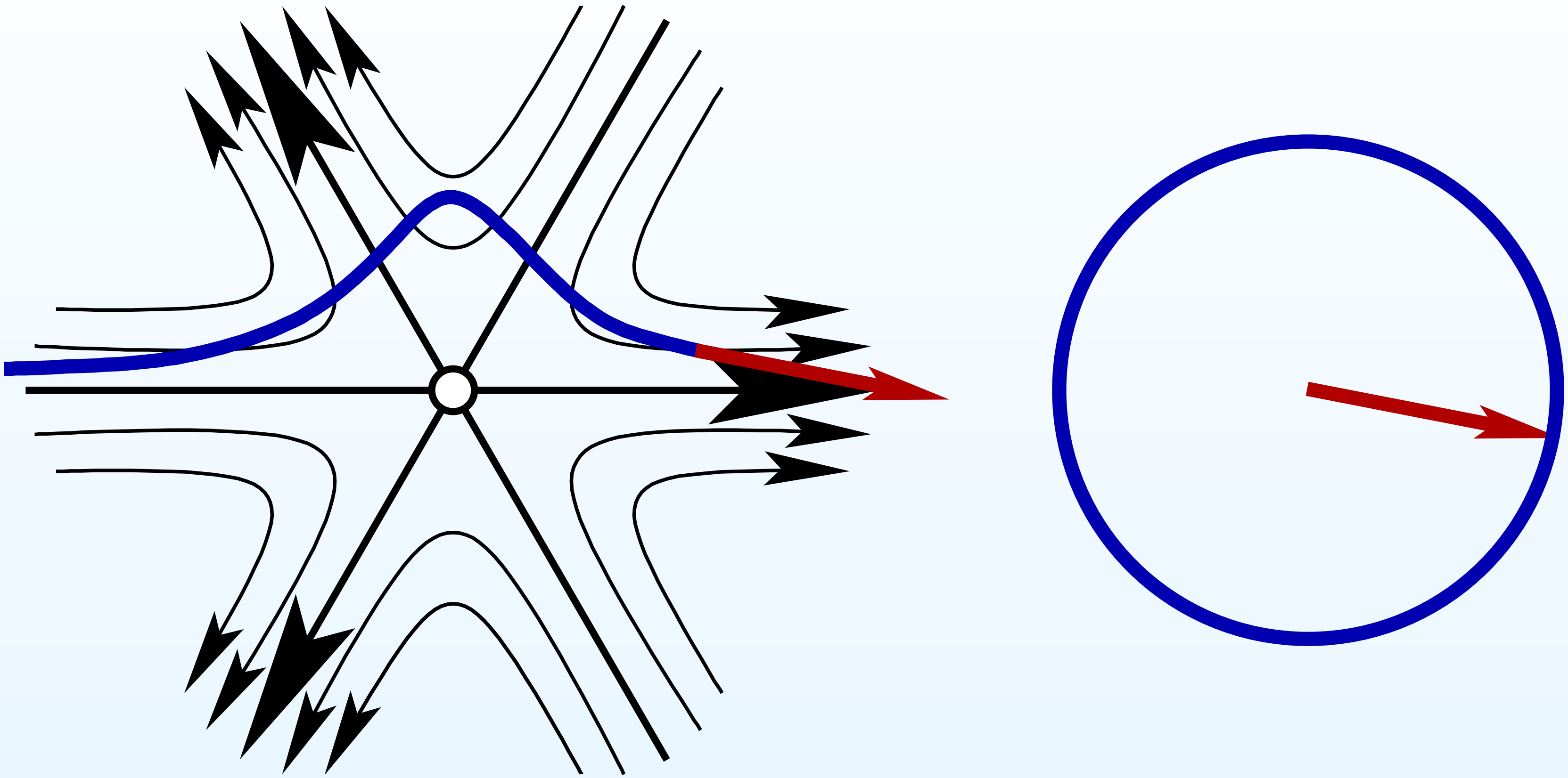

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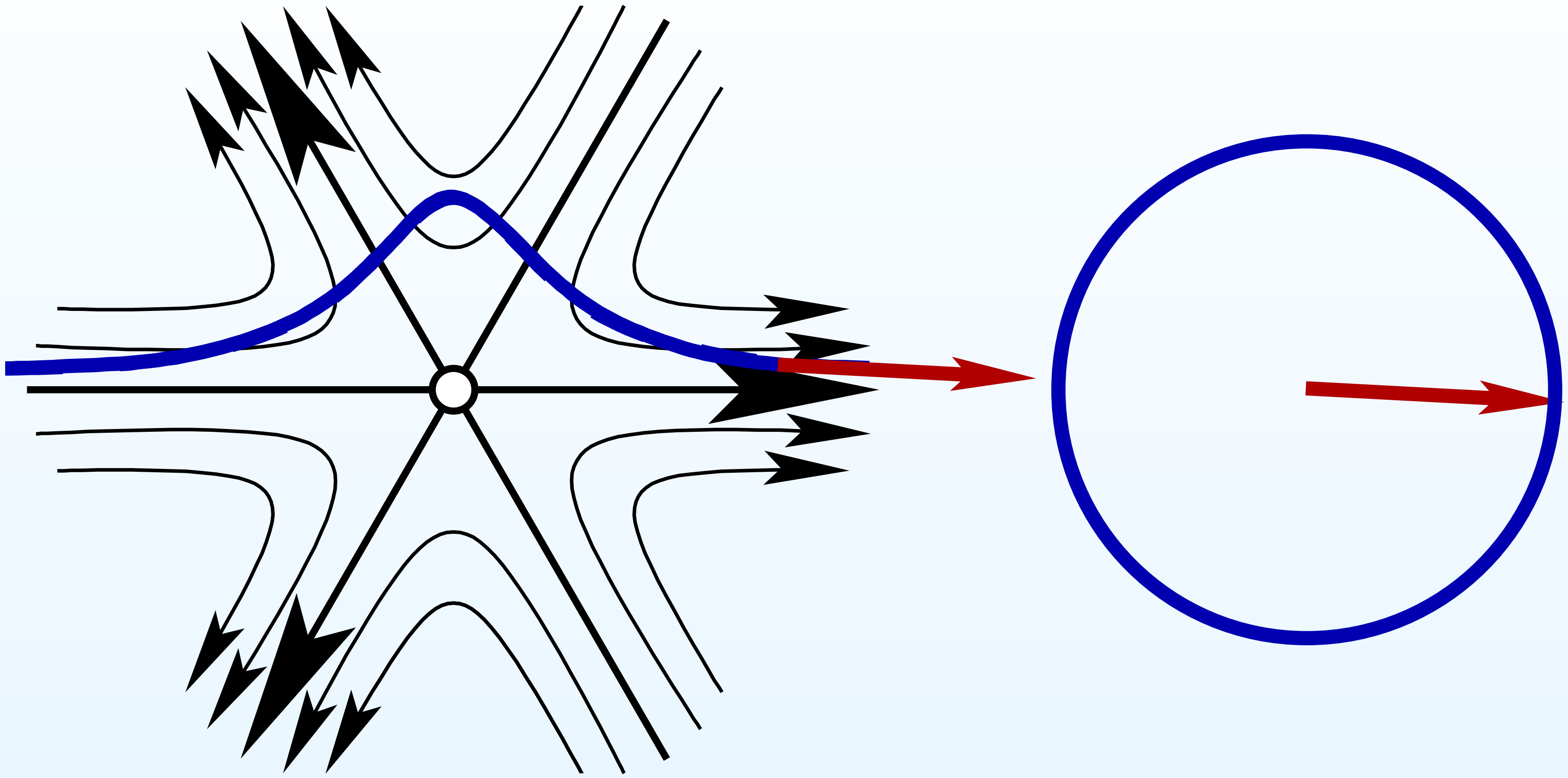

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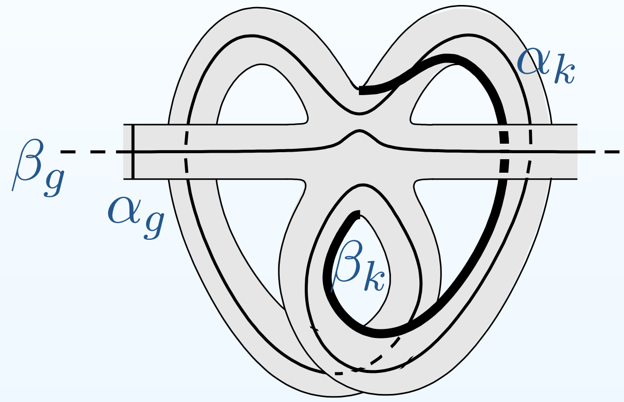
## Computation of the parity of the spin-structure

Parity of the spin-structure: $\varphi(S) := \sum_{i=1}^{g} \big(ind(\alpha_i)+1\big)\big(ind(\beta_i)+1\big) \mod 2\,,$

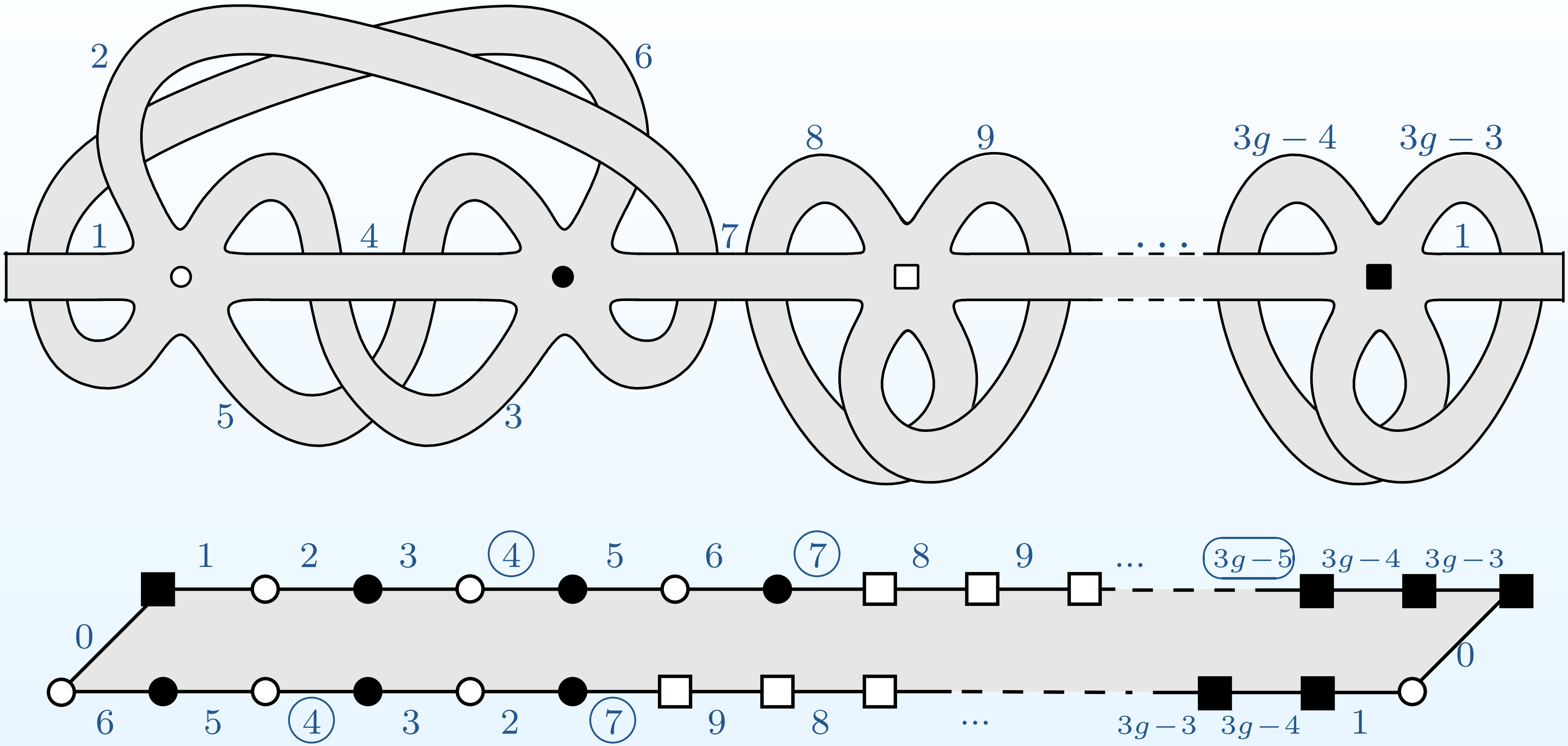
Cycles $\alpha_j, \beta_j$
for $j = 1, \ldots, g$
form a “canonical”
basis of cycles.




All curves except $\beta_g$ are transverse to the vertical foliation, and, hence, $ind(\alpha_1) = ind(\beta_1) = \cdots = ind(\alpha_g) = 0$. We have, $ind(\beta_g) = g - 1$, since each time when $\beta_g$ turns around one of the $g-1$ zeroes, the image of the Gauss map makes a complete turn around a circle. Hence,

$$\varphi(S) = \sum_{i=1}^{g-1}(0+1)(0+1) + (0+1)\big((g-1)+1\big) = g-1+g \equiv 1 \mod 2$$

## Representative of $\mathcal{H}^{even}(2,\ldots,2)$




A one-cylinder Strebel differential from the component $\mathcal{H}^{even}(2,\ldots,2)$ is represented by a ribbon graph on top and by a cylinder (on the bottom). Any subcollection of saddle connections with marked indices $4, 7\ldots, 3g-5$ is suitable for contraction.

## Hyperelliptic components

**Theorem.** *Let $\omega$ be an Abelian Jenkins–Strebel differential with a single cylinder. If it belongs to a hyperelliptic connected component, then a natural cyclic structure on the set of horizontal saddle connections of $\omega$ has the following form:*

$$\begin{array}{ccccc} 1 & 2 & \dots & k-1 & k \\ k & k-1 & \dots & 2 & 1 \end{array}.$$

*The Abelian differential $\omega$ belongs to $\mathcal{H}^{hyp}(2g-2)$ when $k = 2g-1$ is odd and to $\mathcal{H}^{hyp}(g-1, g-1)$ when $k = 2g$ is even.*

**Idea of the proof.** A hyperelliptic involution $\tau$ acts on any Abelian differential $\omega$ as $\tau^*\omega = -\omega$. Hence, the flat isometry $\tau$ acts on the horizontal cylinder by an orientation-preserving involution interchanging the boundaries of the cylinder. Since zeroes are mapped to zeroes, horizontal saddle connections are isometrically mapped to horizontal saddle connections. Since their lengths might be different, every saddle connection $X_i$ is mapped to $X_i$ for $i = 1, \dots, k$. This proves that the cyclic orders of the horizontal saddle connections on two components of the cylinder are inverse to each other.