2020-11-17 Kähler geometry

Today (1) $\Longleftrightarrow$ (5) de Borbon-Legendre Theorem 1.2.

Lemma $H = \ker \eta$ = contact distribution

$$c_1(H) = \sum (1-\beta_a)[\Sigma_a] \qquad [\omega^T]$$

$$\Longleftrightarrow \quad c_1^B - \sum (1-\beta_a)[\Sigma_a]_B \in \mathbb{R}[d\eta]_B$$

<u>Proof</u>

$$0 \to \Omega_B^* \to \Omega^*(S)^T \xrightarrow{i(\xi)} \Omega_B^{*-1} \to 0$$

inclusion

$$\left\{ \begin{array}{l} i(\xi)\alpha = 0 \\ L_\xi \alpha = 0 \end{array} \right.$$

(T = torus generated by $\exp t\xi$)

$$H^p(\Omega^*) \cong H^p(\Omega^{*T})$$

Thus

$$\to H_B^p(S) \to H^p(S) \to H_B^{p-1}(S) \xrightarrow{\delta} H_B^{p+1}(S) \to \cdots$$

$$\delta[\alpha]_B = [d\eta]_B \cup [\alpha]$$

$$\begin{array}{ccc} & \Omega^*(S)^T & \longrightarrow & \Omega^*_B \\ & \psi & & \psi \\ & \eta \wedge \alpha & \longrightarrow & \alpha \\ & d \downarrow & & \\ [d\eta]_B \wedge \alpha \longrightarrow & d\eta \wedge \alpha & & \end{array}$$

$$(\eta(\xi) = 1.)$$

$p = 2$

$$(*) \quad \begin{array}{ccccccc} H^0_B(S) & \xrightarrow{\delta} & H^2_B(S) & \xrightarrow{i} & H^2(S) & \to & H^1_B(S) \\ \| & & & & & & \text{\rotatebox{90}{$\cong$}} \quad \text{BG Prop 7.2.3} \\ \mathbb{R} & & & & & & H^1(S) \\ & & & & & & \| \quad \text{Lerman} \\ & & & & & & 0 \end{array}$$

$$i\left(c_1^B - \sum (1-\beta_a)[\Sigma_a]_B\right) = c_1(H) - \sum (1-\beta_a)[\Sigma_a]$$

$$\text{RHS} = 0 \iff c_1^B - \sum (1-\beta_a)[\Sigma_a]_B \in \underbrace{\operatorname{Im} \delta}_{\mathbb{R}[d\eta]_B}$$

Prop If $\beta \in \mathcal{B}$ then

$$c_1(H) = \sum (1-\beta_a)[\Sigma_a].$$

proof Just as in compact toric mfd case

$$c_1^B = \sum [\Sigma_\alpha]_B \quad \text{in general.}$$

$$\therefore \quad c_1^B - \sum (1-\beta_a)[\Sigma_a]_B = \sum \beta_a [\Sigma_a].$$

Since $c_1(H) - \sum (1-\beta_a)[\Sigma_a] = i\left(c_1^B - \sum (1-\beta_a)[\Sigma_a]_B\right)$ then from $(*)$ we have only to show

$$\sum \beta_a [\Sigma_a] \in \mathbb{R}[d\eta]_B.$$

Consider

$$L : \bigoplus \mathbb{R}[\Sigma_a] \cong \mathbb{R}^d \longrightarrow H^2_B / \mathbb{R}[d\eta]_B$$

$$(\alpha_1[\Sigma_1], \cdots, \alpha_d[\Sigma_d]) \cong (\alpha_1, \cdots, \alpha_d) \qquad \cong \mathbb{R}^{d-n-1}$$

$\uparrow$ general fact in toric geom.

$$\pi : H^2_B \to H^2_B / \mathbb{R}[d\eta]_B.$$

$$(\alpha_1, \cdots, \alpha_d) \longmapsto \pi\left(\sum \alpha_a [\Sigma_a]_B\right)$$

We have only to show

$$\underset{\substack{\cap \\ \mathrm{Ker}\, L.}}{(\beta_1, \cdots, \beta_d)} \longmapsto 0$$

As was shown by Guillemin, l_a can be identified with a bundle metric of $[D_a] \to X$.

Put $L_a = [D_a]|_S \to S$.

$$\therefore \left[\frac{i}{2\pi} \partial\bar\partial \log l_a\right]\Big|_S = c_1(L_a) \cong [\Sigma_a] \in H^2(S).$$

<u>claim</u> $\sum \langle p, v_a \rangle \partial\bar\partial \log l_a = 0$ for $p \in \mathfrak{t}^*$.

$\because$ Guillemin showed the symplectic potential obtained by Delzant construction is

$$G = \frac{1}{2}\sum_a l_a \log l_a$$ "canonical metric"

$$x = DG(y)$$

inverse of Legendre Transform

$l_a(p) = \langle v_a, p \rangle$

$$= \frac{1}{2}\sum v_a (1 + \log l_a(y))$$

Fixed $p \in t^*$

$$\langle p, x \rangle = \frac{1}{2}\sum \langle v_a, p \rangle \log l_a(y) + \overbrace{\sum \langle v_a, p \rangle}^{\text{const}}$$

linear function of x, pluriharmonic

$\Downarrow$

$\partial\bar\partial \, \mathrm{LHS} = 0$

$$\therefore \frac{1}{2}\sum_a \langle v_a, p \rangle \partial\bar\partial \log l_a(y) = 0.$$

☺ claim

By this claim and $\beta_a = \langle v_a, p \rangle$

$$\sum \beta_a [\Sigma_a] = 0 \quad \text{in } H^2(S).$$

Hence by (*)

$$\sum \beta_a [\Sigma_a]_B \in \mathbb{R}[d\eta]_B.$$

we are done. ☺

Prop 4.8 $\leftarrow$ de Borbon-Legendre. (5)

If $\beta_a = l_a(p)$, $a = 1, \dots, d$ for $\exists\, p \in C$.

then $c_1^B - \sum (1-\beta_a)[\Sigma_a]_B > 0$.

proof

$$c_1^B = \sum [\Sigma_a]_B \quad \leftarrow \text{Toric property.}$$

so

$$c_1^B - \sum (1-\beta_a)[\Sigma_a]_B = \sum \beta_a [\Sigma_a]_B$$

$$\in \mathbb{R}[d\eta]_B$$

(last Prop).

$$\therefore \quad \sum \beta_a [\Sigma_a] = \tau \underbrace{[d\eta]_B}_{\text{wT } > 0.} \qquad \tau \in \mathbb{R}.$$

$\beta_a = l_a(p) > 0$ since $p \in C$.

$$\int_S \left(\sum \beta_a [\Sigma_a]\right) \wedge \eta \wedge (d\eta)^{m-1} = \tau \int_S \eta (d\eta)^m$$

LHS $= \sum \beta_a \operatorname{vol}([\Sigma_a]) > 0$

RHS $= \tau \operatorname{vol}(S)$

$\therefore \tau > 0$.

Thus we have proved

(1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) in Theorem 1.2.

(1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) follows from Theorem 1.1.

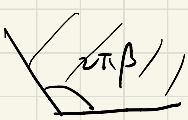
Now we prove to Theorem 1.1.

- Abreu - Guillemin theory for cone metrics.
  - scalar curvature formula. (Abreu formula)
  - Futaki inv (Donaldson)
  - Martelli - Sparks - Yau vanishing of first variation p = barycenter $\Longleftrightarrow$ vanishing Futaki
  - Wang - Zhu estimates, Donaldson's analysis.

  $\rightsquigarrow$ existence of transverse K-E metric

Abreu - Guillemin theory for cone angle metrics.

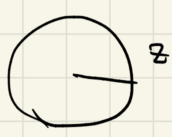
$\mathbb{C} \ni w$



$w = z^\beta = re^{i\beta\theta}$

$w = z^\beta$

$\mathbb{C}_\beta \ni z$



$z = r^{\frac{1}{\beta}} e^{i\theta}$

$$g_\beta = |dw|^2 = \beta^2 |z|^{2\beta-2} |dz|^2$$
$$= \beta^2 r^{\frac{2\beta-2}{\beta}} \left| \frac{1}{\beta} r^{\frac{1}{\beta}-1} dr\, e^{i\theta} + r^{\frac{1}{\beta}} e^{i\theta} d\theta \right|^2$$
$$= \beta^2 r^{\frac{2\beta-2}{\beta}} \left(\frac{1}{\beta^2} r^{\frac{2-2\beta}{\beta}} dr^2 + r^{\frac{2}{\beta}} d\theta^2 \right)$$
$$= dr^2 + \beta^2 r^2 d\theta^2$$

Prop $J\left(r\frac{\partial}{\partial r}\right) = \beta^{-1}\frac{\partial}{\partial\theta}$: This is the Reeb vector field.

$\because$ If $J\frac{\partial}{\partial r} = A\frac{\partial}{\partial\theta}$ since g is J-inv.

$$1 = dr^2\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \beta^2 r^2 d\theta^2\left(J\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\right)$$
$$= \beta^2 r^2 A^2 \qquad \therefore A = \frac{1}{\beta r}$$

$$J\left(r\frac{\partial}{\partial r}\right) = \beta^{-1}\frac{\partial}{\partial\theta} \qquad \because$$

Prop $\omega_\beta = \beta r\, dr \wedge d\theta$ (excercise)

Prop (y, θ) action-angle coord. $\omega_\beta = dy \wedge d\theta$.

$$g_\beta = \frac{1}{2\beta y} dy^2 + 2\beta y\, d\theta^2$$
$$= G'' dy^2 + (G'')^{-1} d\theta^2$$

$$G = \frac{1}{2\beta} \, y \log y$$

(Excercis)