Mirror symmetry: categories and constructions

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Plan: start with quick introduction to :

- mirror symmetry (MS) &
- homological mirror symmetry (HMS)

mathematically, MS gives conjectural relations between :

- algebraic geometry (AG) &
- symplectic geometry (SG)

next, we will review relevant geometry.

finally, we introduce categories which allow us to state HMS, namely :

- derived category of coherent sheaves ⊛
- derived Fukaya category

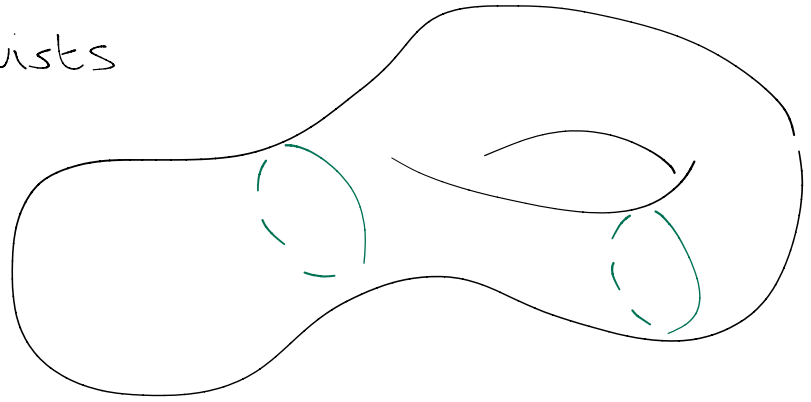
a typical HMS statement is an equivalence between such categories.

note: category ⊛ is natural for study of homology and cohomology, explaining the name "HMS".

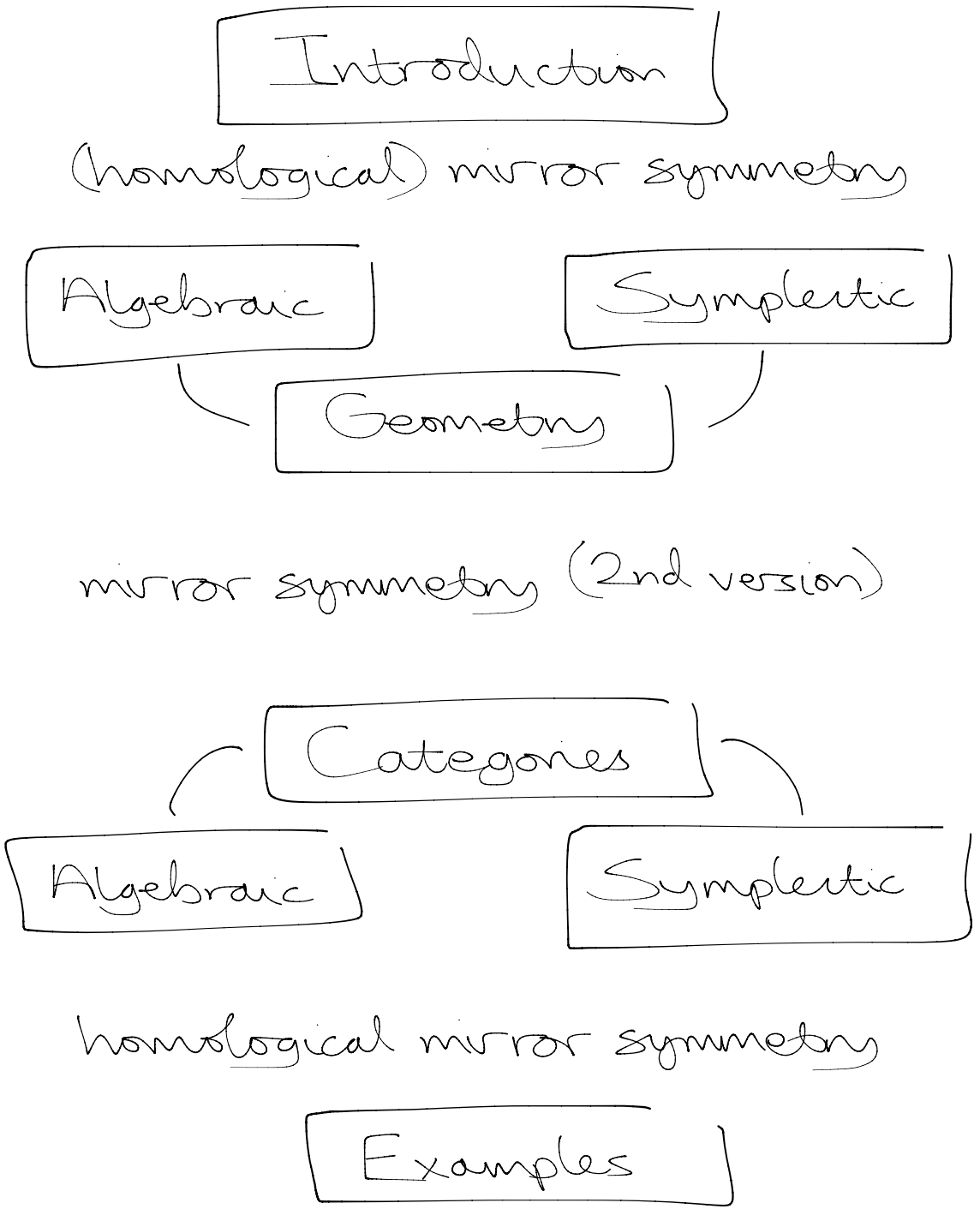
note: we will not prove equivalences, but instead compare structures between the categories.

example: braid group actions from Dehn twists in symplectic geometry

⇝ braid group actions on ⊛



Outline


Introduction
(homological) mirror symmetry
Algebraic
Symplectic
Geometry
mirror symmetry (2nd version)
Categories
Algebraic
Symplectic
homological mirror symmetry
Examples

1 Introduction

Credit: Recalls lectures of Joyce, '07.

Note: Though this is math course, discuss some physics.

Reference: Cox-Katz book [CK], introduction.

String theory

A theoretical model of particles (0-dimensional) as small vibrating strings (1-dimensional) moving in spacetime manifold M.

Studying the quantum version of this model is hard.

It works best when $\dim(M)=10$, and we include "supersymmetry".

Remark: It remains unclear how this might relate to reality! Anyway, it's led to lots of beautiful progress in mathematics.

As a simple model, we may take $M = \mathbb{R}^4 \times X$

$\mathbb{R}^4$: 4-dimensional spacetime (Minkowski space)

X: 6-dimensional, compact manifold

Note: Idea is that lengths in X are very small, so we cannot observe directly

Supersymmetry $\Rightarrow$ X Calabi-Yau

(for more detail, see [CK, §1.3])

Def: X Calabi-Yau. X is complex manifold with

- Kähler structure
- compact
- trivial canonical bundle ω_X, where

$$\omega_X = \det T_X^\vee \quad \text{(for } T_X \text{ tangent bundle)}$$

Note: there are many versions of this definition.

Rem: If $\dim_{\mathbb{C}} X = n$, say X is "n-fold"

Then $\dim_{\mathbb{R}} X = 2n$, say "$2n$-manifold"

So in the above, take X **Calabi-Yau 3-fold (CY3)**

From a CY3 X, physics gives a quantized theory of string motion called a

Super conformal field theory (SCFT)

It seems difficult to make this theory mathematically rigorous. However, important geometric objects naturally appear in the theory, such as

- Dolbeault cohomology groups $H^{p,q}(X)$
- Curve counting invariants (Gromov-Witten)

Now it happens that two CY3s X and X' may have isomorphic SCFTs. Indeed there is an automorphism Φ of $\{\text{SCFTs}\}$.

We say X and X' are mirror partners, or X is mirror to X', if their SCFTs are interchanged by Φ.