Anticanonically balanced metrics and the Hilbert-Mumford criterion

§1. Basics of Kähler-Einstein metrics on Fano mfds

X: Fano mfd of cx dimension n.
i.e. cx mfd s.t. $-K_X$ is ample.

We're interested in Kähler-Einstein metrics (KE)

i.e. a Kähler metric $\omega_{KE} \in c_1(-K_X)$

satisfying $\mathrm{Ric}(\omega_{KE}) = \omega_{KE}$.

It doesn't always exist.

Example: $Bl_{pt}\mathbb{P}^2$

$Aut(X)$ nonreductive $\Rightarrow \nexists KE$ by Matsushima

Futaki invariant $\neq 0 \Rightarrow \nexists KE$ by Futaki.

Throught this minicourse, we assume $Aut(X)$ is discrete.

Important notation

① For a smooth hermitian metric h on $-K_X$, we write $d\mu_h$ for the volume form on X naturally defined via $\mathcal{H}om_{C^\infty_X}(-K_X \otimes \overline{(-K_X)}, \mathbb{C})$

$$\xrightarrow{\sim} K_X \otimes \overline{K_X}$$

Observe $d\mu_{e^{-\phi}h} = e^{-\phi} d\mu_h$

With this notation,

$$\mathrm{Ric}(\omega_h) = \omega_h \iff \omega_h^n = d\mu_h$$

(with appropriate normalisation)

② We fix a reference herm. metric h_0 on $-K_X$

$\omega_0 :=$ Kähler metric assoc. to h_0.

$$\mathcal{H} := \{ \phi \in C^\infty(X, \mathbb{R}) \mid \underbrace{\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi}_{\omega_\phi} > 0 \}.$$

$d\mu_0 = d\mu_{h_0}$, assume $\int_X d\mu_0 = 1$ by scaling.

③ $\mathrm{Vol}(X) := \int_X c_1(-K_X)^n$.

Formulate KE metric in terms of variational principle.

<u>Def</u>. The <u>Ding functional</u> $\mathcal{D} : \mathcal{H} \to \mathbb{R}$ is defined by $\mathcal{D}(\phi) := \mathcal{L}(\phi) - \mathcal{E}(\phi)$ where $\mathcal{L}, \mathcal{E} : \mathcal{H} \to \mathbb{R}$ are defined by

$$\mathcal{L}(\phi) := -\log \int_X e^{-\phi} d\mu_0$$

$$\mathcal{E}(\phi) := \frac{1}{(n+1)\,\mathrm{Vol}(X)} \sum_{j=0}^{n} \int_X \phi\, \omega_\phi^j \wedge \omega_0^{n-j}$$

<u>Def</u>. The <u>J-functional</u> $J : \mathcal{H} \to \mathbb{R}$ is

$$J(\phi) := -\mathcal{E}(\phi) + \frac{1}{\mathrm{Vol}(X)} \int_X \phi\, \omega_0^n$$

Thm ① $\phi \in \mathcal{H}$ is a crit. pt. of $\mathcal{D}$

$$\iff \omega_\phi^n = e^{-\phi} d\mu_0 \iff \omega_\phi \text{ is KE.}$$

② $\exists$ crit. pt of $\mathcal{D}$

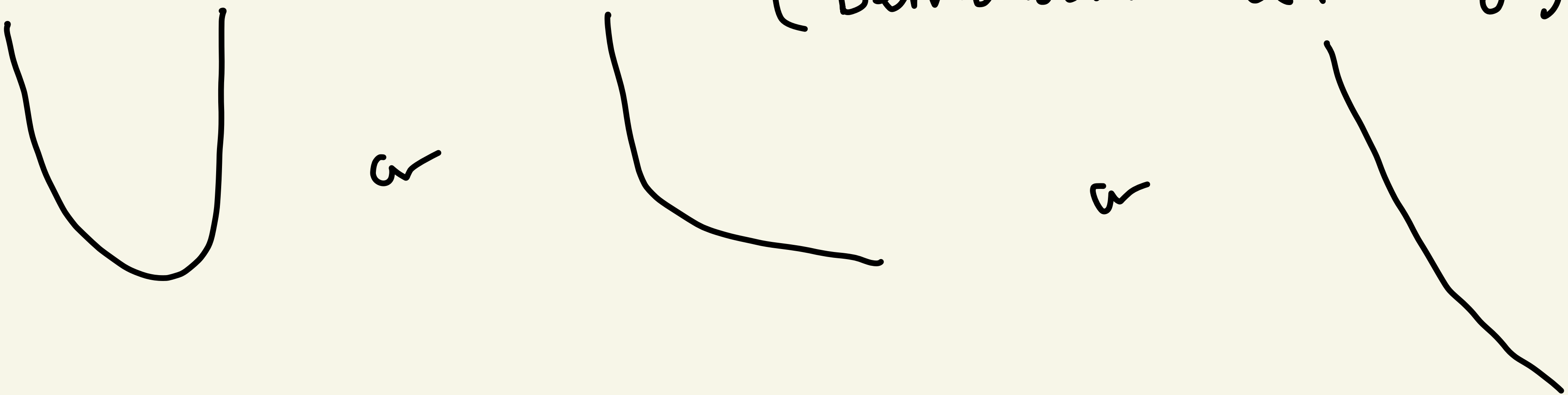
T. Darvas Geometric pluripotential theory.

$$\iff \exists\, c_1, c_2 > 0 \text{ s.t. } \mathcal{D}(\phi) \geq c_1 J(\phi) - c_2 \quad \forall \phi \in \mathcal{H}.$$

Guedj-Zeriahi Degenerate Monge-Ampère eq.

③ $\mathcal{D}$ is "convex along geodesics in $\mathcal{H}$".

(Berndtsson convexity).



§2. Anticanonically balanced metrics

<u>Def</u>. A hermitian metric h on $-K_X$ is said to be an (m-th) <u>Fubini-Study</u> metric if $\exists$ pos. def. herm. form H on $H^0(X, -mK_X)$, s.t. h is the pullback of the Fubini-Study metric on $\mathbb{P}(H^0(X, -mK_X)^\vee)$ def by H via the embedding

$$\iota : X \hookrightarrow \mathbb{P}(H^0(X, -mK_X)^\vee).$$

Throughout, we write $N_m := \dim H^0(X, -mK_X)$.

<u>Notation</u>. $\mathcal{B}_m := \{\text{pos. def. herm. forms on } H^0(X, -mK_X)\}$.

$\mathcal{H}_m := \{m\text{-th FS metrics}\} \subset \mathcal{H}$.

So we have a map $FS : \mathcal{B}_m \to \mathcal{H}_m$

via $X \hookrightarrow \mathbb{P}(H^0(X, -mK_X)^\vee)$.

This map is injective (proved by Lempert).

<u>Foundational fact</u> $\bigcup_{m \in \mathbb{N}} \mathcal{H}_m$ is dense in $\mathcal{H}$.

(Tian-Yau-Zelditch expansion).

Def. $\omega_h \in c_1(-K_X)$

The m-th anticanonical Bergman function is defined by

$$\rho_m(\omega_h) := \sum_{i=1}^{N_m} |\sigma_i|^2_{h^m} \in C^\infty(X, \mathbb{R}).$$

where $\{\sigma_i\}_{i=1}^{N_m}$ is an orthonormal basis for $H^0(X, -mK_X)$ wrt the inner product

$$\int_X h^m(\,,\,)\, d\mu_h.$$

Def. A Kähler metric $\omega_h \in c_1(-K_X)$ is said to be <u>anticanonically balanced (ACB)</u> at level m if $\rho_m(\omega_h) \equiv \text{const}$ on X.

Ma-Marinescu

Prop. ① ACB metric at level m is an m-th FS metric, i.e. $FS(H)$ for some $H \in \mathcal{B}_m$.

② $\exists$ ACB metric at level m

$\Longleftrightarrow \exists H \in \mathcal{B}_m$ s.t.

FS(H) is ACB.

$$\int_X \iota^*\left(\frac{z_i \bar{z}_j}{\sum_\ell |z_\ell|^2}\right) d\mu_{FS(H)} = \frac{1}{N_m}\delta_{ij}$$

Kronecker delta

$\{z_i\}_{i=1}^{N_m}$ is an H-onb (hmg coords on $\mathbb{P}(H^0(X,-mK_X)^\vee)$)

③ $h = FS(H)$ is ACB at level m

$\iff N_m \fint_X h^m(\cdot,\cdot)\, d\mu_h = H$,

where $\displaystyle \fint_X f\, d\mu_h = \frac{\int_X f\, d\mu_h}{\int_X d\mu_h}$

<u>Thm</u> (Berman-Witt Nyström)

Suppose $\mathrm{Aut}(X)$ discrete (can be removed by considering solitons).

If $(X, -K_X)$ admits a KE metric, then $\exists$ sequence $\{\omega_m\}_m$ of ACB metrics in $c_1(-K_X)$

each ω_m is ACB at level m.

s.t. $\omega_m \to \omega_{KE}$ $(m \to \infty)$ in the sense of currents.

Remark

1. This is an anticanonical version of Donaldson's thm, which approximates cscK metrics by balanced metrics

Difference: L^2 inner product in def of ρ_m is wrt ω_h^n (Donaldson) as opposed to $d\mu_h$ (BWN).

2 Convergence can be improved to C^∞. (Takahashi, Ioos).

Pick a reference $H_{0,m} \in \mathcal{B}_m \simeq GL(N_m,\mathbb{C})/U(N_m)$.

<u>Def</u>. The m-th <u>quantised Ding functional</u>

$\mathcal{D}_m : \mathcal{B}_m \to \mathbb{R}$ is def by

$$\mathcal{D}_m(H) := \mathcal{L}(FS(H)) - \mathcal{E}_m(H)$$

Kähler potential

where $\mathcal{E}_m(H) := -\frac{1}{m N_m} \log\det(H H_{0,m}^{-1})$.

<u>Lemma</u> For each fixed $m \in \mathbb{N}$:

1. The crit. pt. of $\mathcal{D}_m$ is precisely the ACB metric at level m.

2. Define $J_m : \mathcal{B}_m \to \mathbb{R}$ by

$$J_m(H) := -\mathcal{E}_m(H) + \frac{1}{\mathrm{Vol}(X)} \int_X FS(H)\, \omega^n_{FS(H_{0,m})},$$

$\mathcal{D}_m$ admits a crit. pt. iff $\exists\, c_1, c_2 > 0$ s.t.

$$\mathcal{D}_m(H) \geq c_1 J_m(H) - c_2.$$

Bergman geodesic ray

3. Let $H_t := e^{-tA^*} H_{0,m} e^{-tA}$ be a ray in $\mathcal{B}_m$ generated by $A \in \mathfrak{gl}(H^0(X, -mK_X))$, which is in fact a geodesic ray wrt the biinvariant metric on $\mathcal{B}_m$.

$\mathcal{D}_m$ is convex along Bergman geodesic rays.
(Berndtsson convexity).

<u>Cor</u>. ACB metrics are unique (mod Aut(X)).

<u>Very rough sketch pf of BWN</u>

$\exists$ KE $\Longleftrightarrow$ $\mathcal{D}(\phi) \geq c_1 J(\phi) - c_2 \quad \forall \phi \in \mathcal{H}$.

Set $H_{0,m} := N_m \int_X h_{KE}^m(\, , \,) d\mu_{KE}$

Show that there exists a sequence
$\{C_m'\}_{m\in\mathbb{N}} \subset \mathbb{R}_{>0}$, $C_m' \to 0 \quad (m\to\infty)$
s.t.

$$J(FS(H)) \leq (1+C_m')J_m(H) + C_m' \quad \forall H \in \mathcal{B}_m.$$

This is based on a comparison between $\mathcal{E}_m(H)$ and $\mathcal{E}(FS(H))$.

Using this uniform estimate and TYZ expansion, they conclude $\mathcal{D}_m(H) \geq \tilde{C}_1 J_m(H) - \tilde{C}_2$

for some $\tilde{C}_1, \tilde{C}_2 > 0$

uniformly for $H \in \mathcal{B}_m$

for all suff. large m.

$\Rightarrow \exists$ crit. pt. of $\mathcal{D}_m$

§3. Main results for this minicourse

<u>Thm.</u> $(X, -K_X)$ admits an m-th ACB metric

iff $\mathrm{Ding}(\mathcal{X}, \mathcal{L}) + \mathrm{Chow}_m(\mathcal{X}, \mathcal{L}) \geq 0$

for all very ample test configurations

$(\mathcal{X}, \mathcal{L})$ of exponent m,

with equality iff $(\mathcal{X}, \mathcal{L})$ is trivial.

<u>Important remarks</u>

1. The alg. geom. condition in the above statement is the <u>F-stability</u> def by Saito-Takahashi, who proved $\Rightarrow$ of the above thm.

2. Rubinstein-Tian-K. Zhang proved that

$\exists$ ACB metric at level m $\Longleftrightarrow$ Fujita-Odaka's δ_m-invariant > 1

So we have

$$\begin{array}{ccc} \exists \text{ ACB metric at level } m & \overset{\text{Thm}}{\Longleftrightarrow} & \text{F-stability at level } m. \\ \text{RTZ} \Big\Updownarrow & \swarrow\!\!\nearrow & \\ \delta_m > 1 & & \text{no alg. geom. proof known.} \end{array}$$

$X = \mathbb{P}^n$

$\delta_m^T = 1$

Recall also the following famous result.

$$
\begin{tikzcd}
\exists \text{ KE metric} \arrow[rr, Leftrightarrow, "\text{YTD}", "\text{Berman-Boucksom-Jonsson, CDS, Tian}"'] \arrow[dr, Leftrightarrow, "\text{direct proof by K. Zhang}"'] & & \text{uniform Ding stability} \arrow[dl, Leftrightarrow, "\text{Blum-Jonsson, Fujita-Odaka (purely AG result)}"] \\
& \text{Fujita-Odaka's } \delta\text{-invariant} > 1 &
\end{tikzcd}
$$