The dilogarithm and Chern-Simons invariants

Dan Freed
University of Texas at Austin

October 12, 2021
& 13

Joint work with Andy Neitzke
arXiv:2006.12565

Primary and secondary characteristic forms

Characteristic forms and geometric invariants

By Shiing-shen Chern and James Simons*

1. Introduction

This work, originally announced in [4], grew out of an attempt to derive a purely combinatorial formula for the first Pontrjagin number of a 4-manifold. The hope was that by integrating the characteristic curvature form (with respect to some Riemannian metric) simplex by simplex, and replacing the integral over each interior by another on the boundary, one could evaluate these boundary integrals, add up over the triangulation, and have the geometry wash out, leaving the sought after combinatorial formula. This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interest-

G	Lie group with $\pi_0 G$ finite
$\langle -, \cdots, - \rangle \colon \mathfrak{g}^{\otimes p} \longrightarrow \mathbb{R}$	G-invariant symmetric p-linear form on the Lie algebra $\mathfrak{g}$
$\pi \colon P \longrightarrow M$	principal G-bundle
$\mathcal{A}_P$	affine space of connections $\Theta \in \Omega^1_P(\mathfrak{g})$
$\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$	curvature of the connection Θ

Examples: G compact Lie group

$p = 2$

$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ G-invariant inner product

$G = \mathbb{C}^\times$

$p = 2$

$$\langle z_1, z_2 \rangle = -\frac{z_1 z_2}{4\pi^2}$$

Examples: G compact Lie group
$p = 2$
$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ G-invariant inner product

$G = \mathbb{C}^\times$
$p = 2$
$\langle z_1, z_2 \rangle = -\dfrac{z_1 z_2}{4\pi^2}$

Lemma: $\omega(\Theta) = \langle \Omega \wedge \cdots \wedge \Omega \rangle$ is a closed $2p$-form on M (Chern-Weil form)

Examples: G compact Lie group

$p = 2$

$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ G-invariant inner product

$G = \mathbb{C}^\times$

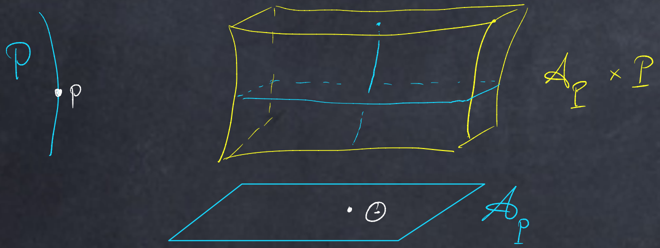
$p = 2$

$\langle z_1, z_2 \rangle = -\dfrac{z_1 z_2}{4\pi^2}$

Lemma: $\omega(\Theta) = \langle \Omega \wedge \cdots \wedge \Omega \rangle$ is a closed $2p$-form on M (Chern-Weil form)

Θ_P is the universal connection on $\mathcal{A}_P \times P \to \mathcal{A}_P \times M$ characterized by

$$\Theta_P\big|_{\{\Theta\}\times P} = \Theta, \qquad \Theta_P\big|_{\mathcal{A}_P \times \{p\}} = 0, \qquad \Theta \in \mathcal{A}_P,\ p \in P.$$



Examples: G compact Lie group
$p = 2$
$\langle -, - \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ G-invariant inner product

$G = \mathbb{C}^\times$
$p = 2$
$\langle z_1, z_2 \rangle = -\dfrac{z_1 z_2}{4\pi^2}$

Lemma: $\omega(\Theta) = \langle \Omega \wedge \cdots \wedge \Omega \rangle$ is a closed $2p$-form on M (Chern-Weil form)

Θ_P is the universal connection on $\mathcal{A}_P \times P \to \mathcal{A}_P \times M$ characterized by

$$\Theta_P\big|_{\{\Theta\}\times P} = \Theta, \qquad \Theta_P\big|_{\mathcal{A}_P \times \{p\}} = 0, \qquad \Theta \in \mathcal{A}_P,\ p \in P.$$

$\Omega_P = \Omega(\Theta_P)$ curvature (in $\Omega^2_{\mathcal{A}_P \times P}(\mathfrak{g})$)

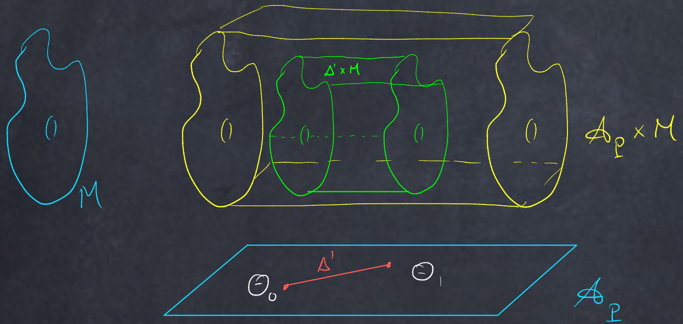
$\omega(\Theta_P)$ Chern-Weil form (in $\Omega^{2p}_{\mathcal{A}_P \times M}$)

$\Theta_0, \Theta_1 \in \mathcal{A}_P$ connections on $P \to M$

$\Delta^1 \to \mathcal{A}_P$ affine map with endpoints Θ_0, Θ_1

Definition: The Chern-Simons $(2p-1)$-form is

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_P) \qquad \in \Omega_M^{2p-1}$$



$\Theta_0, \Theta_1 \in \mathcal{A}_P$ connections on $P \to M$

$\Delta^1 \to \mathcal{A}_P$ affine map with endpoints Θ_0, Θ_1

Definition: The Chern-Simons $(2p-1)$-form is

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_P) \qquad \in \Omega_M^{2p-1}$$

Stokes' formula: $$d\alpha(\Theta_0, \Theta_1) = \omega(\Theta_1) - \omega(\Theta_0)$$

$\Theta_0, \Theta_1 \in \mathcal{A}_P$ connections on $P \to M$

$\Delta^1 \to \mathcal{A}_P$ affine map with endpoints Θ_0, Θ_1

Definition: The Chern-Simons $(2p-1)$-form is

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_P) \qquad \in \Omega_M^{2p-1}$$

Stokes' formula: $d\alpha(\Theta_0, \Theta_1) = \omega(\Theta_1) - \omega(\Theta_0)$

Remark: This Chern-Simons form is a secondary invariant of *two* connections

$\Theta_0, \Theta_1 \in \mathcal{A}_P$ connections on $P \to M$

$\Delta^1 \to \mathcal{A}_P$ affine map with endpoints Θ_0, Θ_1

Definition: The Chern-Simons $(2p-1)$-form is

$$\alpha(\Theta_0, \Theta_1) = \int_{\Delta^1} \omega(\Theta_P) \qquad \in \Omega_M^{2p-1}$$

Stokes' formula: $d\alpha(\Theta_0, \Theta_1) = \omega(\Theta_1) - \omega(\Theta_0)$

Remark: This Chern-Simons form is a secondary invariant of *two* connections

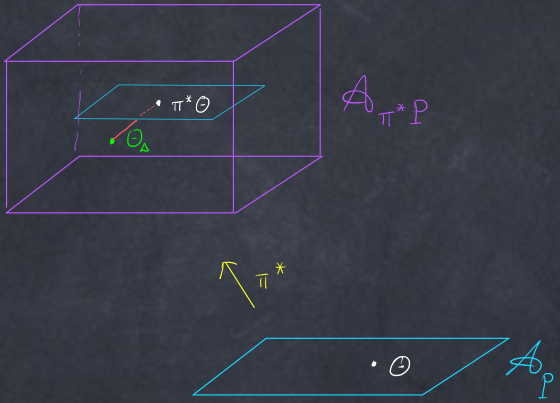
Remark: The de Rham cohomology class of $\omega(\Theta)$ in $H^{2p}_{\mathrm{dR}}(M) \cong H^{2p}(M;\mathbb{R})$ is independent of Θ

There is also a Chern-Simons $(2p-1)$-form which is a secondary invariant of *one* connection

There is also a Chern-Simons $(2p-1)$-form which is a secondary invariant of *one* connection

We construct it from the previous Chern-Simons form $\alpha(-,-)$ using a pullback bundle:

$$\begin{array}{ccc} \pi^*P = P \times_M P & \longrightarrow & P \\ \pi^{(2)} \Big\downarrow \Big\uparrow \Delta & & \Big\downarrow \pi \\ P & \xrightarrow{\quad \pi \quad} & M \end{array}$$



$$\alpha(\Theta) = \alpha(\Theta_\Delta, \pi^*\Theta) \qquad \in \Omega_P^{2p-1}$$

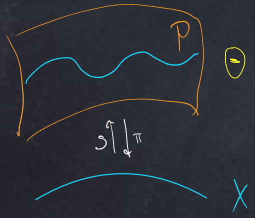
Stokes': $d\alpha(\Theta) = \pi^*\omega(\Theta) \qquad \in \Omega_P^{2p}$

X^{2p-1} compact, oriented

$\pi\colon P \longrightarrow X$ principal G-bundle with connection Θ

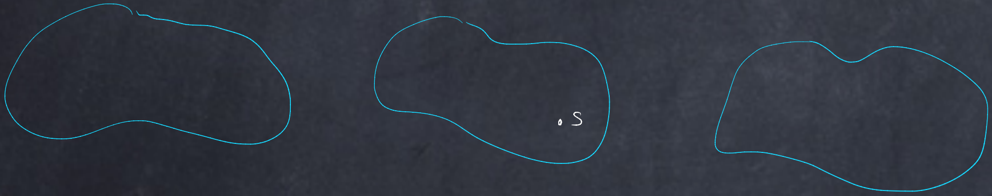
$s\colon X \longrightarrow P$ section of π (may not exist)

$\Gamma(\Theta, s) = \int_X s^*\alpha(\Theta)$ integral of Chern-Simons form



X^{2p-1}	compact, oriented
$\pi\colon P \longrightarrow X$	principal G-bundle with connection Θ
$s\colon X \longrightarrow P$	section of π (may not exist)
$\Gamma(\Theta, s) = \int_X s^*\alpha(\Theta)$	integral of Chern-Simons form

$\Gamma(\Theta, s)$ is independent of the homotopy class of s, since $\omega(\Theta) = 0$ for degree reasons



$\Gamma(P \xrightarrow{\pi} X) \circlearrowleft \mathrm{Map}(X, G)$

X^{2p-1} compact, oriented

$\pi\colon P \longrightarrow X$ principal G-bundle with connection Θ

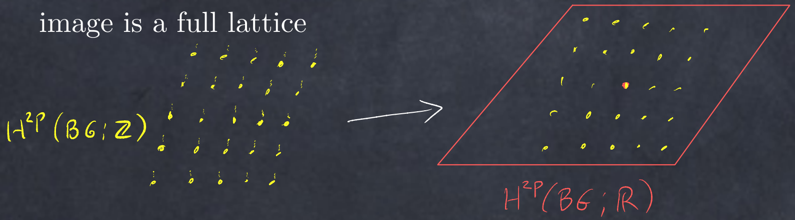
$s\colon X \longrightarrow P$ section of π (may not exist)

$\Gamma(\Theta, s) = \int_X s^*\alpha(\Theta)$ integral of Chern-Simons form

$\Gamma(\Theta, s)$ is independent of the homotopy class of s, since $\omega(\Theta) = 0$ for degree reasons

$\langle -, \cdots, - \rangle\colon \mathfrak{g}^{\otimes p} \to \mathbb{R}$ represents a class in $H^{2p}(BG;\mathbb{R})$

$H^{2p}(BG;\mathbb{Z}) \to H^{2p}(BG;\mathbb{R})$ image is a full lattice



X^{2p-1} compact, oriented

$\pi\colon P \longrightarrow X$ principal G-bundle with connection Θ

$s\colon X \longrightarrow P$ section of π (may not exist)

$\Gamma(\Theta, s) = \int_X s^*\alpha(\Theta)$ integral of Chern-Simons form

$\Gamma(\Theta, s)$ is independent of the homotopy class of s, since $\omega(\Theta) = 0$ for degree reasons

$\langle -, \cdots, - \rangle\colon \mathfrak{g}^{\otimes p} \to \mathbb{R}$ represents a class in $H^{2p}(BG;\mathbb{R})$

$H^{2p}(BG;\mathbb{Z}) \to H^{2p}(BG;\mathbb{R})$ image is a full lattice

Integrality assumption: : $\langle -, \cdots, - \rangle$ is the image of a *level* $\lambda \in H^{2p}(BG;\mathbb{Z})$

Then $\mathscr{F}(X;\Theta) = \exp\bigl(2\pi\sqrt{-1}\,\Gamma(\Theta, s)\bigr) \in \mathbb{C}^\times$ is well-defined (Chern-Simons invariant)

Spin refinement

Example: $G = \mathbb{C}^\times$, $p = 2$, $P \to M$ principal $\mathbb{C}^\times$-bundle with connection Θ

Then $\omega(\Theta) \in \Omega^4_M$ represents $c_1(P)^2 \in H^4(M;\mathbb{Z})$

Theorem: If M is a closed, *spin* 4-manifold, then $\langle c_1(P)^2, [M]\rangle \in 2\mathbb{Z}$

If X is a closed, *spin* 3-manifold, then $\mathscr{F}(X;\Theta) \in \mathbb{C}^\times$ has a canonical $\sqrt{}$

This square root is the *spin Chern-Simons invariant* $\mathscr{S}(X;\Theta) \in \mathbb{C}^\times$

Spin refinement

Example: $G = \mathbb{C}^\times$, $p = 2$, $P \to M$ principal $\mathbb{C}^\times$-bundle with connection Θ

Then $\omega(\Theta) \in \Omega^4_M$ represents $c_1(P)^2 \in H^4(M;\mathbb{Z})$

Theorem: If M is a closed, *spin* 4-manifold, then $\langle c_1(P)^2, [M]\rangle \in 2\mathbb{Z}$

If X is a closed, *spin* 3-manifold, then $\mathscr{F}(X;\Theta) \in \mathbb{C}^\times$ has a canonical $\sqrt{}$

This square root is the *spin Chern-Simons invariant* $\mathscr{S}(X;\Theta) \in \mathbb{C}^\times$

If $X = \partial M$ for M a compact spin 4-manifold, and Θ extends to a $\mathbb{C}^\times$-connection Ξ on M:

$$\mathscr{F}(X;\Theta) = \exp\left(2\pi\sqrt{-1}\int_M \omega(\Xi)\right) \qquad \mathscr{S}(X;\Theta) = \exp\left(2\pi\sqrt{-1}\int_M \frac{1}{2}\omega(\Xi)\right)$$

Spin refinement

Example: $G = \mathbb{C}^\times$, $p = 2$, $P \to M$ principal $\mathbb{C}^\times$-bundle with connection Θ

Then $\omega(\Theta) \in \Omega^4_M$ represents $c_1(P)^2 \in H^4(M;\mathbb{Z})$

Theorem: If M is a closed, *spin* 4-manifold, then $\langle c_1(P)^2, [M]\rangle \in 2\mathbb{Z}$

If X is a closed, *spin* 3-manifold, then $\mathscr{F}(X;\Theta) \in \mathbb{C}^\times$ has a canonical $\sqrt{}$

This square root is the *spin Chern-Simons invariant* $\mathscr{S}(X;\Theta) \in \mathbb{C}^\times$

If $X = \partial M$ for M a compact spin 4-manifold, and Θ extends to a $\mathbb{C}^\times$-connection Ξ on M:

$$\mathscr{F}(X;\Theta) = \exp\left(2\pi\sqrt{-1}\int_M \omega(\Xi)\right) \qquad \mathscr{S}(X;\Theta) = \exp\left(2\pi\sqrt{-1}\int_M \frac{1}{2}\omega(\Xi)\right)$$

The spin refinement is defined based on a cohomology theory E which fits into

$$\cdots \longrightarrow H^q(-;\mathbb{Z}) \longrightarrow E^q(-) \longrightarrow H^{q-2}(-;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta\circ Sq^2} H^{q+1}(-;\mathbb{Z}) \longrightarrow \cdots$$

The map $H^{2p}(BG;\mathbb{Z}) \to E^{2p}(BG)$ maps levels to "spin levels"

Application to conformal immersions (Chern-Simons)

This appears in the 1973 Chern-Simons paper, here reinterpreted a bit

Application to conformal immersions (Chern-Simons)

This appears in the 1973 Chern-Simons paper, here reinterpreted a bit

X	closed spin Riemannian 3-manifold
Θ_{LC}	Levi-Civita SO_3-connection on X
$\mathscr{S}_{\mathrm{SO}_3}$	secondary spin invariant for "$p_1/2$"

Application to conformal immersions (Chern-Simons)

This appears in the 1973 Chern-Simons paper, here reinterpreted a bit

X	closed spin Riemannian 3-manifold
Θ_{LC}	Levi-Civita SO_3-connection on X
$\mathscr{S}_{\mathrm{SO}_3}$	secondary spin invariant for "$p_1/2$"

Theorem: $\mathscr{S}_{\mathrm{SO}_3}(\mathbb{RP}^3;\Theta_{\mathrm{LC}}) = -1$

Corollary: $\mathscr{S}_{\mathrm{SO}_3}(S^3;\Theta_{\mathrm{LC}}) = 1$

Application to conformal immersions (Chern-Simons)

This appears in the 1973 Chern-Simons paper, here reinterpreted a bit

X	closed spin Riemannian 3-manifold
Θ_{LC}	Levi-Civita SO_3-connection on X
$\mathscr{S}_{\mathrm{SO}_3}$	secondary spin invariant for "$p_1/2$"

Theorem: $\mathscr{S}_{\mathrm{SO}_3}(\mathbb{RP}^3; \Theta_{\mathrm{LC}}) = -1$

Corollary: $\mathscr{S}_{\mathrm{SO}_3}(S^3; \Theta_{\mathrm{LC}}) = 1$

Theorem: If $X \to \mathbb{E}^4$ is a conformal immersion, then $\mathscr{S}_{\mathrm{SO}_3}(X; \Theta_{\mathrm{LC}}) = 1$

Application to conformal immersions (Chern-Simons)

This appears in the 1973 Chern-Simons paper, here reinterpreted a bit

X	closed spin Riemannian 3-manifold
Θ_{LC}	Levi-Civita SO_3-connection on X
$\mathscr{S}_{\mathrm{SO}_3}$	secondary spin invariant for "$p_1/2$"

Theorem: $\mathscr{S}_{\mathrm{SO}_3}(\mathbb{RP}^3;\Theta_{\mathrm{LC}}) = -1$

Corollary: $\mathscr{S}_{\mathrm{SO}_3}(S^3;\Theta_{\mathrm{LC}}) = 1$

Theorem: If $X \to \mathbb{E}^4$ is a conformal immersion, then $\mathscr{S}_{\mathrm{SO}_3}(X;\Theta_{\mathrm{LC}}) = 1$

Proof:

$$\begin{array}{ccc} TX & \xrightarrow{\tilde{\Gamma}} & TS^3 \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{Gauss}]{\Gamma} & S^3 \end{array}$$

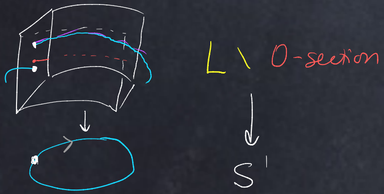
$$\Gamma^*(\Theta_{\mathrm{LC}}^{S^3}) = \Theta_{\mathrm{LC}}^{X}$$

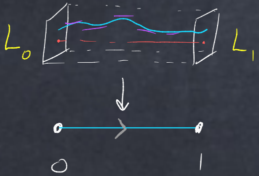
$$\mathscr{S}_{\mathrm{SO}_3}(X;\Theta_{\mathrm{LC}}^{X}) = \mathscr{S}_{\mathrm{SO}_3}(S^3;\Theta_{\mathrm{LC}}^{S^3})^{\deg(\Gamma)} = 1$$

Chern-Simons as an invertible field theory

Example:

$G = \mathbb{C}^\times$	
$p = 1$	
$\omega(\Theta)$	represents c_1
$\mathscr{F}(S^1;\Theta)$	holonomy of Θ around circle
$\mathscr{F}([0,1];\Theta)$	parallel transport
$\mathscr{F}(\mathrm{pt};\Theta)$	complex line





Chern-Simons as an invertible field theory

Example:

$G = \mathbb{C}^\times$	
$p = 1$	
$\omega(\Theta)$	represents c_1
$\mathscr{F}(S^1;\Theta)$	holonomy of Θ around circle
$\mathscr{F}([0,1];\Theta)$	parallel transport
$\mathscr{F}(\mathrm{pt};\Theta)$	complex line

- This generalizes to any (G,p)

Chern-Simons as an invertible field theory

Example:

$G = \mathbb{C}^\times$

$p = 1$

$\omega(\Theta)$	represents c_1
$\mathscr{F}(S^1;\Theta)$	holonomy of Θ around circle
$\mathscr{F}([0,1];\Theta)$	parallel transport
$\mathscr{F}(\mathrm{pt};\Theta)$	complex line

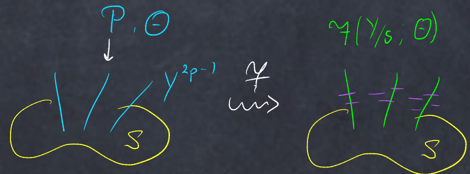
- This generalizes to any (G,p)
- Attach a complex line to a G-connection on a closed, oriented $(2p-2)$-manifold (Ramadas-Singer-Weitsman)

Chern-Simons as an invertible field theory

Example:

$G = \mathbb{C}^\times$	
$p = 1$	
$\omega(\Theta)$	represents c_1
$\mathscr{F}(S^1;\Theta)$	holonomy of Θ around circle
$\mathscr{F}([0,1];\Theta)$	parallel transport
$\mathscr{F}(\mathrm{pt};\Theta)$	complex line

- This generalizes to any (G, p)
- Attach a complex line to a G-connection on a closed, oriented $(2p-2)$-manifold (Ramadas-Singer-Weitsman)
- Constructions for smooth families of connections



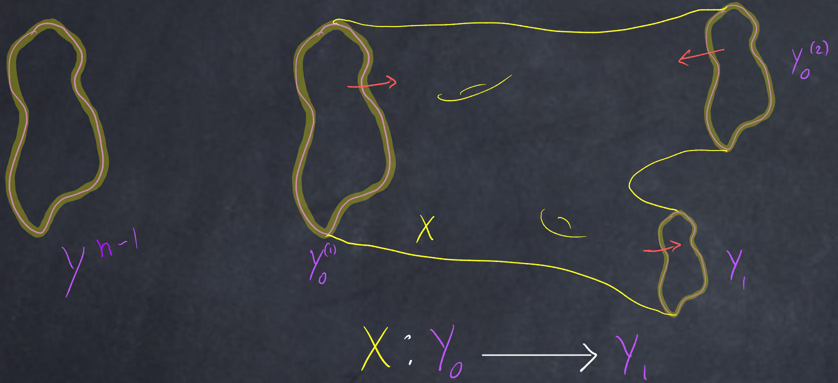
Chern-Simons as an invertible field theory

Example:

$G = \mathbb{C}^\times$	
$p = 1$	
$\omega(\Theta)$	represents c_1
$\mathscr{F}(S^1; \Theta)$	holonomy of Θ around circle
$\mathscr{F}([0,1]; \Theta)$	parallel transport
$\mathscr{F}(\mathrm{pt}; \Theta)$	complex line

- This generalizes to any (G, p)
- Attach a complex line to a G-connection on a closed, oriented $(2p-2)$-manifold (Ramadas-Singer-Weitsman)
- Constructions for smooth families of connections
- Express in language of field theory as a map out of a bordism category

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}$(orientation, G-connection) $(n = 2p-1)$ of manifolds equipped with orientation and G-bundle with connection



Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p - 1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p - 1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

"Classical" Chern-Simons theory: A homomorphism

$$\mathscr{F}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$$

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p-1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

"Classical" Chern-Simons theory: A homomorphism

$$\mathscr{F}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$$

- Includes evaluation in parametrized families and formulas for curvature and holonomy

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p-1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

"Classical" Chern-Simons theory: A homomorphism

$$\mathscr{F}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$$

- Includes evaluation in parametrized families and formulas for curvature and holonomy
- A *nonflat trivialization* of $\pi\colon P \to Y$ with connection η is a not-necessarily-flat section of π, and it produces a nonzero element of the complex line $\mathscr{F}(Y;\eta)$

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p-1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

"Classical" Chern-Simons theory: A homomorphism

$$\mathscr{F}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$$

- Includes evaluation in parametrized families and formulas for curvature and holonomy
- A *nonflat trivialization* of $\pi\colon P \to Y$ with connection η is a not-necessarily-flat section of π, and it produces a nonzero element of the complex line $\mathscr{F}(Y;\eta)$
- There is a spin refinement $\mathscr{S}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{spin structure}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$

Domain: The bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection})$ $(n = 2p - 1)$ of manifolds equipped with orientation and G-bundle with connection

Codomain: The linear category $\mathrm{Line}_{\mathbb{C}}$ of 1-dimensional complex vector spaces and invertible linear maps

"Classical" Chern-Simons theory: A homomorphism

$$\mathscr{F}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{orientation}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$$

- Includes evaluation in parametrized families and formulas for curvature and holonomy
- A *nonflat trivialization* of $\pi\colon P \to Y$ with connection η is a not-necessarily-flat section of π, and it produces a nonzero element of the complex line $\mathscr{F}(Y;\eta)$
- There is a spin refinement $\mathscr{S}\colon\ \mathrm{Bord}_{\langle n-1,n\rangle}(\text{spin structure}, G\text{-connection}) \longrightarrow \mathrm{Line}_{\mathbb{C}}$
- The theories $\mathscr{F}, \mathscr{S}$ are best viewed in the context of *generalized differential cohomology* (Cheeger-Simons, Hopkins-Singer, ...)

The enhanced Rogers dilogarithm

Begin with power series defined for $|z| < 1$:
(analytically continue to $\mathbb{C} \backslash [1, \infty)$)

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{(Spence)}$$

The enhanced Rogers dilogarithm

Begin with power series defined for $|z| < 1$: $\qquad -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$

(analytically continue to $\mathbb{C}\backslash[1,\infty)$)

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{(Spence)}$$

$\mathcal{M}_T = (\mathbb{C}^\times)^2$

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ $\qquad$ universal $(\mathbb{Z}\times\mathbb{Z})$ cover of $\mathcal{M}_T$

$\mathcal{M}'_T = \{(\mu_1,\mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$ $\qquad$ (complex) curve in $\mathcal{M}_T$, $\quad \mathcal{M}_T \approx \mathbb{CP}^1\backslash\{0,1,\infty\}$

$\widehat{\mathcal{M}}'_T = \{(u_1,u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$ $\qquad$ $\mathbb{Z}\times\mathbb{Z}$ cover of $\mathcal{M}'_T$ $\qquad (u_1,u_2) \longmapsto (e^{u_1}, e^{u_2})$

$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$ $\qquad$ Tate twist

$\mathbb{Z}(2) = \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}$

The enhanced Rogers dilogarithm

Begin with power series defined for $|z| < 1$: $\quad -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$

(analytically continue to $\mathbb{C} \backslash [1, \infty)$)

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad \text{(Spence)}$$

$\mathcal{M}_T = (\mathbb{C}^\times)^2$

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ — universal $(\mathbb{Z} \times \mathbb{Z})$ cover of $\mathcal{M}_T$

$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$ — (complex) curve in $\mathcal{M}_T$, $\quad \mathcal{M}_T \approx \mathbb{CP}^1 \backslash \{0, 1, \infty\}$

$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$ — $\mathbb{Z} \times \mathbb{Z}$ cover of $\mathcal{M}'_T$ $\qquad (u_1, u_2) \longmapsto (e^{u_1}, e^{u_2})$

$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$ — Tate twist

$\mathbb{Z}(2) = \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}$

Differentiate the power series:

$(z = \mu_1)$

$$d\,\mathrm{Li}_2(z) = -\frac{\log(1-z)}{z}\,dz = -u_2\,du_1 = -\frac{u_2\,du_2}{1 - e^{-u_2}}$$

(meromorphic 1-form on the u_2-line with simple poles at $\mathbb{Z}(1) \subset \mathbb{C}$ and residues in $\mathbb{Z}(1)$)

$$\mathcal{M}_T = (\mathbb{C}^\times)^2$$

$$\widehat{\mathcal{M}}_T = \mathbb{C}^2$$

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

$$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$$

$$\mathbb{Z}(2) = \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}$$

$$\begin{aligned} F: \quad \widehat{\mathcal{M}}'_T &\longrightarrow \mathbb{C} \setminus \mathbb{Z}(1) \longrightarrow \mathbb{C}/\mathbb{Z}(2) \\ (u_1, u_2) &\longmapsto u_2 \longmapsto \mathrm{Li}_2(1 - e^{u_2}) \end{aligned}$$

$$\mathcal{M}_T = (\mathbb{C}^\times)^2$$

$$\widehat{\mathcal{M}}_T = \mathbb{C}^2$$

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

$$\mathbb{Z}(1) = 2\pi\sqrt{-1}\mathbb{Z}$$

$$\mathbb{Z}(2) = \mathbb{Z}(1)^{\otimes 2} = 4\pi^2\mathbb{Z}$$

$$\begin{array}{rcccc} F: & \widehat{\mathcal{M}}'_T & \longrightarrow \mathbb{C} \setminus \mathbb{Z}(1) & \longrightarrow & \mathbb{C}/\mathbb{Z}(2) \\ & (u_1, u_2) & \longmapsto u_2 & \longmapsto & \mathrm{Li}_2(1 - e^{u_2}) \end{array}$$

Rogers enhanced dilogarithm

$$\begin{array}{rcl} R: \ \widehat{\mathcal{M}}'_T & \longrightarrow & \mathbb{C}/\mathbb{Z}(2) \\ (u_1, u_2) & \longmapsto & F(u_2) + \dfrac{u_1 u_2}{2} \mod \mathbb{Z}(2) \end{array}$$

satisfies the differential equation

$$dR = \frac{u_1 du_2 - u_2 du_1}{2}$$

$$\widehat{\mathcal{M}}_T = \mathbb{C}^2 = \{(u_1, u_2)\}$$

$$\widehat{\mathcal{M}}'_T = \left\{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\right\}$$

- The Rogers enhanced dilogarithm $R\colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ is essentially the function

$$\mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \mod \mathbb{Z}(2)$$

$$\widehat{\mathcal{M}}_T = \mathbb{C}^2 = \{(u_1, u_2)\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

- The Rogers enhanced dilogarithm $R\colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ is essentially the function

$$\mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \mod \mathbb{Z}(2)$$

- It has known values at special z and satisfies several identities

$$\widehat{\mathcal{M}}_T = \mathbb{C}^2 = \{(u_1, u_2)\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

- The Rogers enhanced dilogarithm $R \colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ is essentially the function

$$\mathrm{Li}_2(z) + \frac{1}{2}\log(z)\log(1-z) \quad \mod \mathbb{Z}(2)$$

- It has known values at special z and satisfies several identities
- The most important is the 5-term identity

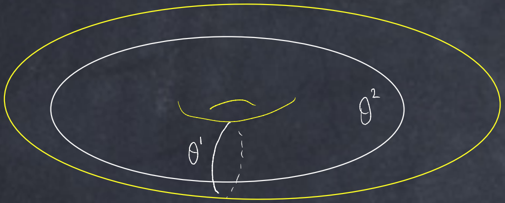
$$R(u_1, v_1) + \cdots + R(u_5, v_5) = \text{constant}, \qquad v_i = u_{i-1} + u_{i+1}$$

Interpretation in terms of flat connections

$\mathcal{M}_T = (\mathbb{C}^\times)^2$ moduli space of flat $\mathbb{C}^\times$-connections on 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ (θ^1, θ^2)

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ flat connections on trivial (product) $\mathbb{C}^\times$-bundle over T

moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over T/htpy



Interpretation in terms of flat connections

$\mathcal{M}_T = (\mathbb{C}^\times)^2$ moduli space of flat $\mathbb{C}^\times$-connections on 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ (θ^1, θ^2)

$\widehat{\mathcal{M}}_T = \mathbb{C}^2_{u_1, u_2}$ flat connections on trivial (product) $\mathbb{C}^\times$-bundle over T

moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over T/htpy

- Universal $\mathbb{C}^\times$-connection over $\widehat{\mathcal{M}}_T, \mathcal{M}_T$:

$$\widehat{\mathcal{M}}_T \times T \times \mathbb{C}^\times \underset{\hat{t}_0}{\overset{\hat{\rho}}{\rightleftarrows}} \widehat{\mathcal{M}}_T \times T$$

$$\hat{t}_0^*\hat{\eta} = -u_1\, d\theta^1 - u_2\, d\theta^2 \qquad \in \Omega^1_{\widehat{\mathcal{M}}_T \times T}(\mathbb{C})$$

Interpretation in terms of flat connections

$\mathcal{M}_T = (\mathbb{C}^\times)^2$ moduli space of flat $\mathbb{C}^\times$-connections on 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ (θ^1, θ^2)

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ flat connections on trivial (product) $\mathbb{C}^\times$-bundle over T

moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over T/htpy

- Universal $\mathbb{C}^\times$-connection over $\widehat{\mathcal{M}}_T, \mathcal{M}_T$:

$$\widehat{\mathcal{M}}_T \times T \times \mathbb{C}^\times \underset{\hat{t}_0}{\overset{\hat{\rho}}{\rightleftarrows}} \widehat{\mathcal{M}}_T \times T$$

$$\hat{t}_0^*\hat{\eta} = -u_1\, d\theta^1 - u_2\, d\theta^2 \qquad \in \Omega^1_{\widehat{\mathcal{M}}_T \times T}(\mathbb{C})$$

- Descend to $\rho\colon \mathcal{Q} \longrightarrow \mathcal{M}_T \times T$ with connection η

Interpretation in terms of flat connections

$\mathcal{M}_T = (\mathbb{C}^\times)^2_{\mu_1,\mu_2}$ moduli space of flat $\mathbb{C}^\times$-connections on 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ (θ^1, θ^2)

$\widehat{\mathcal{M}}_T = \mathbb{C}^2$ flat connections on trivial (product) $\mathbb{C}^\times$-bundle over T

moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over T/htpy

- Universal $\mathbb{C}^\times$-connection over $\widehat{\mathcal{M}}_T, \mathcal{M}_T$:

$$\widehat{\mathcal{M}}_T \times T \times \mathbb{C}^\times \underset{\hat{t}_0}{\overset{\hat{\rho}}{\rightleftarrows}} \widehat{\mathcal{M}}_T \times T$$

$$\hat{t}_0^* \hat{\eta} = -u_1\, d\theta^1 - u_2\, d\theta^2 \quad \in \Omega^1_{\widehat{\mathcal{M}}_T \times T}(\mathbb{C})$$

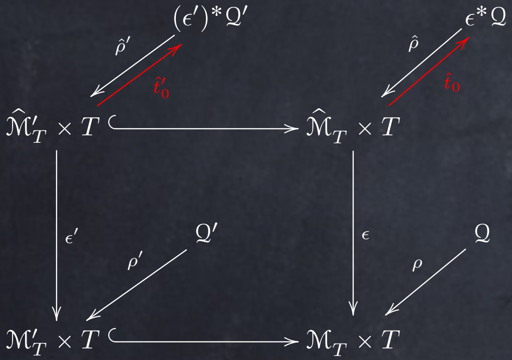
- Descend to $\rho\colon \mathcal{Q} \longrightarrow \mathcal{M}_T \times T$ with connection η

- Curvature: $\Omega(\eta) = -\dfrac{d\mu_1}{\mu_1} \wedge d\theta^1 - \dfrac{d\mu_2}{\mu_2} \wedge d\theta^2$

Application of spin Chern-Simons

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

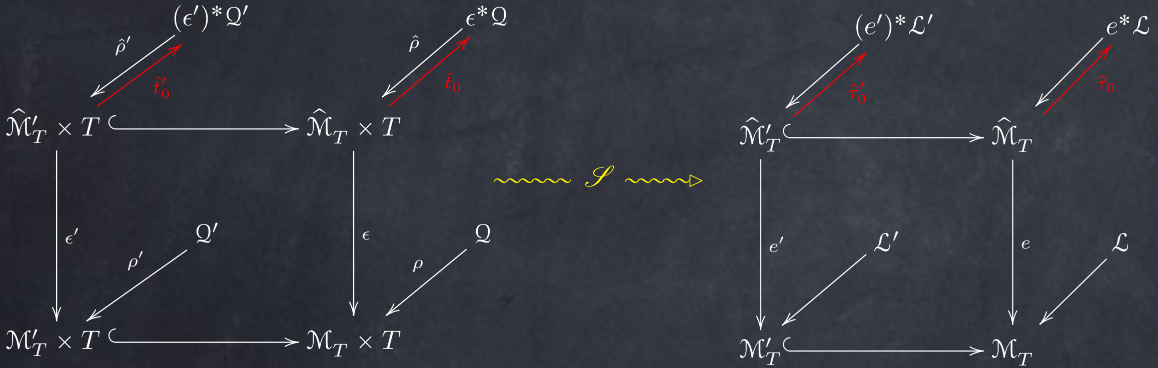
$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$



Application of spin Chern-Simons

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

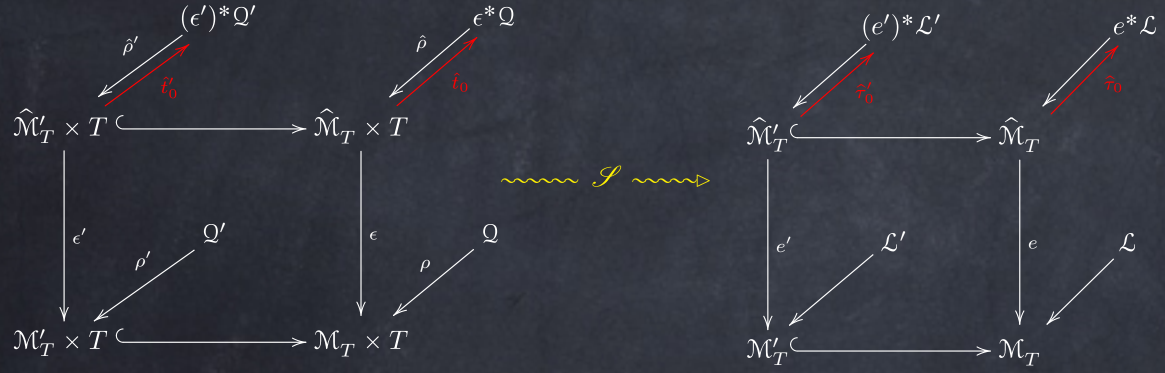
$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$



Application of spin Chern-Simons

$$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$$

$$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$$

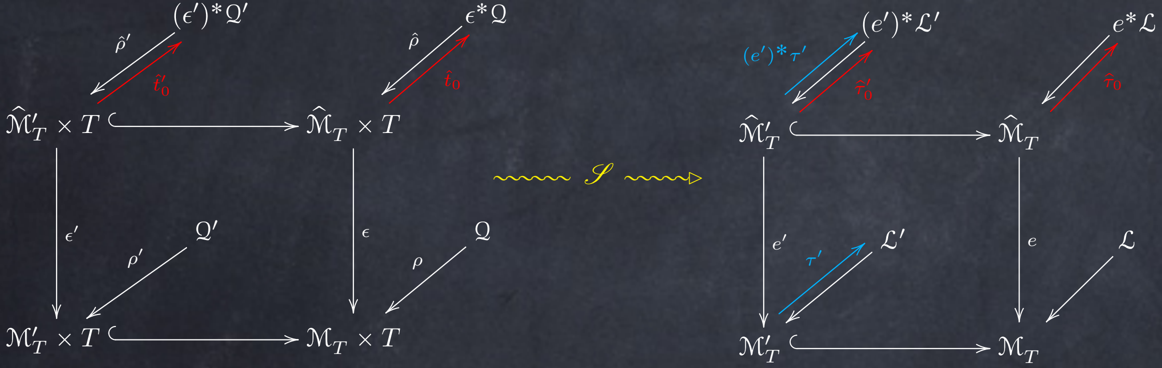


Proposition: : $\mathcal{L}' \longrightarrow \mathcal{M}'_T$ is flat with trivial holonomy

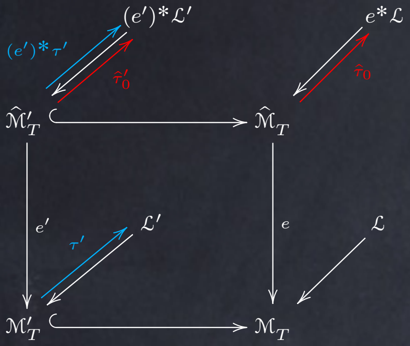
Application of spin Chern-Simons

$\mathcal{M}'_T = \{(\mu_1, \mu_2) \in \mathcal{M}_T : \mu_1 + \mu_2 = 1\}$

$\widehat{\mathcal{M}}'_T = \{(u_1, u_2) \in \widehat{\mathcal{M}}_T : e^{u_1} + e^{u_2} = 1\}$

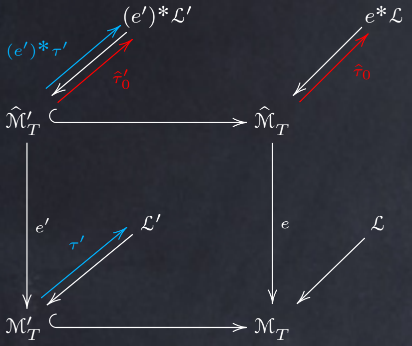


Proposition: : $\mathcal{L}' \longrightarrow \mathcal{M}'_T$ is flat with trivial holonomy



Define a function on $\widehat{\mathcal{M}}'_T$ as the ratio of two sections:

$$\varphi = \frac{\hat{\tau}'_0}{(e')^*\tau'} : \widehat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times$$

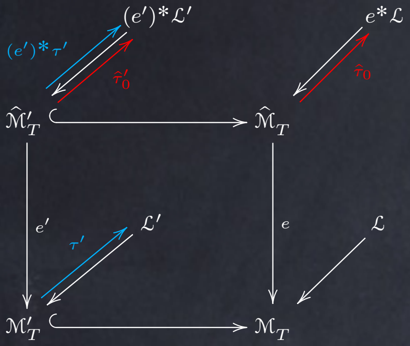


Define a function on $\widehat{\mathcal{M}}'_T$ as the ratio of two sections:

$$\varphi = \frac{\widehat{\tau}_0'}{(e')^*\tau'} : \widehat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times$$

The formula for the connection form implies

$$\frac{d\varphi}{\varphi} = \frac{1}{4\pi\sqrt{-1}}(u_1 du_2 - u_2 du_1)$$



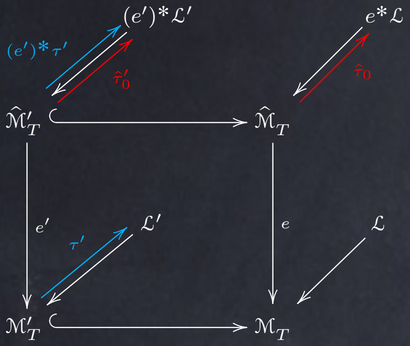
Define a function on $\widehat{\mathcal{M}}'_T$ as the ratio of two sections:

$$\varphi = \frac{\widehat{\tau}'_0}{(e')^*\tau'} : \widehat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times$$

The formula for the connection form implies

$$\frac{d\varphi}{\varphi} = \frac{1}{4\pi\sqrt{-1}}(u_1 du_2 - u_2 du_1)$$

Define $L\colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ by $\varphi = \exp\left(\frac{L}{2\pi\sqrt{-1}}\right)$, so $dL = \frac{u_1 du_2 - u_2 du_1}{2}$



Define a function on $\widehat{\mathcal{M}}'_T$ as the ratio of two sections:

$$\varphi = \frac{\hat{\tau}'_0}{(e')^*\tau'} : \widehat{\mathcal{M}}'_T \longrightarrow \mathbb{C}^\times$$

The formula for the connection form implies

$$\frac{d\varphi}{\varphi} = \frac{1}{4\pi\sqrt{-1}}(u_1 du_2 - u_2 du_1)$$

Define $L\colon \widehat{\mathcal{M}}'_T \to \mathbb{C}/\mathbb{Z}(2)$ by $\varphi = \exp\left(\frac{L}{2\pi\sqrt{-1}}\right)$, so $dL = \frac{u_1 du_2 - u_2 du_1}{2}$

Theorem: $L = R$ up to a constant

Application of spin Chern-Simons to prove 5-term identity

Theorem: The sum $L(u_1, v_1) + \cdots + L(u_5, v_5)$ is independent of $(u_i, v_i) \in \widehat{\mathcal{M}}'_T$, $i \in \mathbb{Z}/5\mathbb{Z}$, which satisfy

$$v_i = u_{i-1} + u_{i+1} \qquad \text{for all } i$$

Application of spin Chern-Simons to prove 5-term identity

Theorem: The sum $L(u_1, v_1) + \cdots + L(u_5, v_5)$ is independent of $(u_i, v_i) \in \widehat{\mathcal{M}}'_T$, $i \in \mathbb{Z}/5\mathbb{Z}$, which satisfy

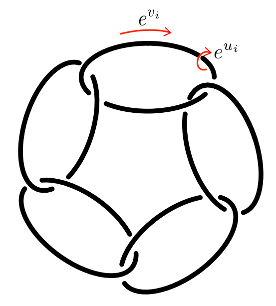
$$v_i = u_{i-1} + u_{i+1} \qquad \text{for all } i$$

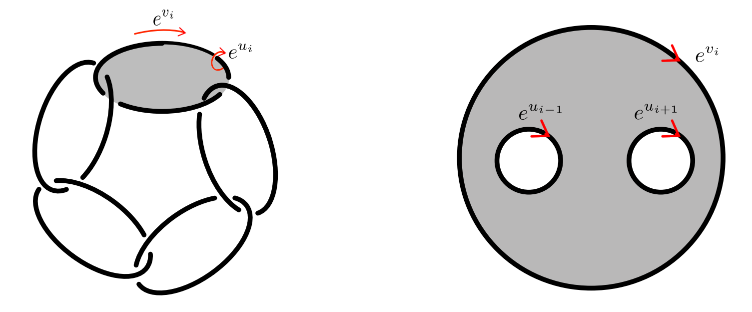
Remark: Write $z_i = e^{u_i}$, so $1 - z_i = z_{i-1} z_{i+1}$. There is a connected complex 2-manifold $\mathcal{M}'_X$ of solutions

$$x \ , \ \frac{1-x}{1-xy} \ , \ \frac{1-y}{1-xy} \ , \ y \ , \ 1-xy$$

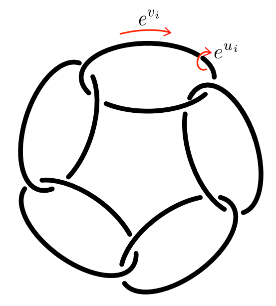
parametrized by $x, y \in \mathbb{C}$ satisfying $xy \neq 0$, $x \neq 1$, $y \neq 1$, and $xy \neq 1$.

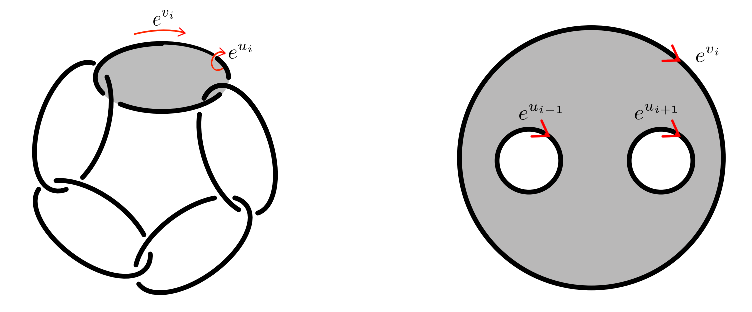
Proof sketch: Let X be the compact spin 3-manifold with boundary formed from S^3 by removing a tubular neighborhood of the 5-component link





Proof sketch: Let X be the compact spin 3-manifold with boundary formed from S^3 by removing a tubular neighborhood of the 5-component link

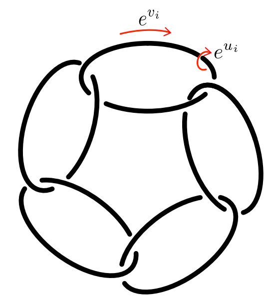


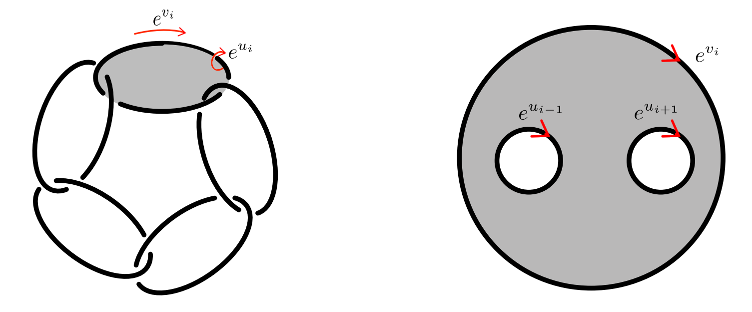


$\partial X \approx T^{\sqcup 5}$ and the restriction of a flat connnection on X to the boundary satisfies

$$v_i = u_{i-1} + u_{i+1} \qquad \text{for all } i$$

Proof sketch: Let X be the compact spin 3-manifold with boundary formed from S^3 by removing a tubular neighborhood of the 5-component link



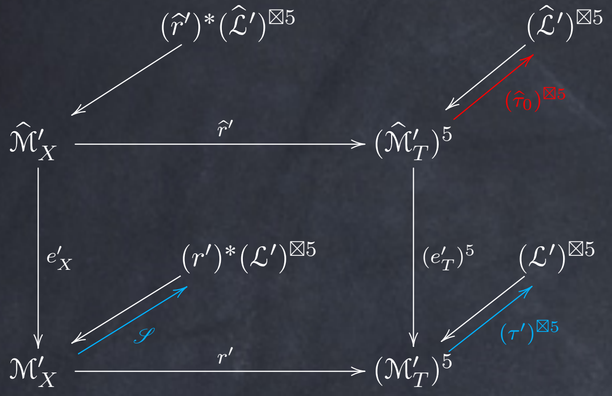


$\partial X \approx T^{\sqcup 5}$ and the restriction of a flat connnection on X to the boundary satisfies

$$v_i = u_{i-1} + u_{i+1} \qquad \text{for all } i$$

$\mathcal{M}_X$ moduli space of flat $\mathbb{C}^\times$-connections on X

$\widehat{\mathcal{M}}_X$ moduli space of flat $\mathbb{C}^\times$-connections with nonflat trivialization over X/htpy



- The section $\mathscr{S}$ is flat (covariant derivative = integral of $\frac{1}{2}\omega \wedge \omega$ vanishes)
- Choose flat section τ' so $\mathscr{S} = (r')^*\left[(\tau')^{\boxtimes 5}\right]$ (unique up to 5th root of unity)
- Pullbacks of $\mathscr{S}$ and $(\widehat{\tau}_0)^{\boxtimes 5}$ to $\widehat{\mathcal{M}}'_X$ agree $\Longrightarrow (\widehat{\tau}_0/\tau')^{\boxtimes 5} = 1$ (exp of 5-term identity) //

Application to 3-manifolds

X	closed oriented 3-manifold
Θ	*flat* connection on principal $\mathrm{SL}_2\,\mathbb{C}$-bundle $P \longrightarrow M$
$\mathscr{F}(X;\Theta) \in \mathbb{C}^\times$	Chern-Simons invariant

Well-known that if X is triangulated, then $\mathscr{F}(X;\Theta)$ can be expressed in terms of dilogs

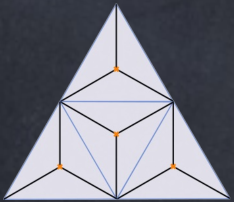
Application to 3-manifolds

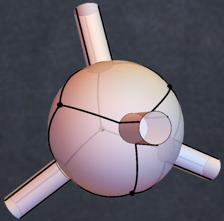
X	closed oriented 3-manifold
Θ	*flat* connection on principal $\mathrm{SL}_2\,\mathbb{C}$-bundle $P \longrightarrow M$
$\mathscr{F}(X;\Theta) \in \mathbb{C}^\times$	Chern-Simons invariant

Well-known that if X is triangulated, then $\mathscr{F}(X;\Theta)$ can be expressed in terms of dilogs

New idea: Apply *stratified abelianization* via a *spectral network* (Gaiotto-Moore-Neitzke) to identify $\mathscr{F}(X;\Theta)$ as the spin Chern-Simons invariant of a flat $\mathbb{C}^\times$-connection on the total space of a branched double cover $\widetilde{X} \longrightarrow X$

To appear. . .





Happy Birthday and Thank You