Applications of Quantum Cohomology to Birational Geometry, I

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Example: quantum cohomology of $\mathbb{C}P^n$

(Usual) cohomology ring $H^\bullet(\mathbb{C}P^n, \mathbb{Q}) \sim \mathbb{Q}[h]/(h^{n+1})$ where $h \in H^2(\mathbb{C}P^n) = c_1(\mathcal{O}(1))$ is dual to the hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Linear basis: $\Delta_0 = 1, \Delta_1 = h, \ldots, \Delta_n = h^n$.

Poincaré pairing: $\langle \Delta_i, \Delta_j \rangle = \begin{cases} 1 & \text{if } i+j=n \\ 0 & \text{otherwise} \end{cases}$

Multiplication table: $\Delta_i \cdot \Delta_j = \begin{cases} \Delta_{i+j} & \text{if } i+j \leq n \\ 0 & \text{otherwise} \end{cases}$

Quantum *cohomology ring*: the *same* space $H^\bullet(\mathbb{C}P^n)$, with the *same* Poincaré pairing, and with the ***new*** (*quantum*) multiplication $\star_q$ depending on parameter q:

$$\Delta_i \star_q \Delta_j := \begin{cases} \Delta_{i+j} & \text{if } i+j \leq n \\ q\Delta_{i+j-(n+1)} & \text{if } i+j > n \end{cases}$$

It is isomorphic as a ring to $\mathbb{Q}[h]/(h^{n+1} = q)$.

Geometric meaning (denote $X := \mathbb{C}P^n$):

$$\langle \Delta_i \star_q \Delta_j, \Delta_k \rangle = \int_{I_{3;0}} (\Delta_i \boxtimes \Delta_j \boxtimes \Delta_k) + q \int_{I_{3;1}} (\Delta_i \boxtimes \Delta_j \boxtimes \Delta_k)$$

where $I_{3;0} \in H_{2n}(X^3)$ is the class of the main diagonal, and $I_{3;1} \in H_{4n+2}(X^3)$ is the class of the cycle consisting of triples of points (x_1, x_2, x_3) lying on a line $\mathbb{C}P^1 \subset \mathbb{C}P^n = X$.

In general, for any integer $d \geq 0$ we may be tempted to define the homology class $I_{3;d}$ of $X^3 = X \times X \times X$ as follows: consider the variety $Maps_d(\mathbb{C}P^1, X)$ of polynomial maps $\varphi : \mathbb{C}P^1 \to X = \mathbb{C}P^n$ of degree d. This is a smooth algebraic variety of complex dimension $n + d(n+1)$, and we declare $I_{3;d}$ to be the image of its fundamental class (problem: noncompactness) under the evaluation map $Maps_d(\mathbb{C}P^1, X) \to X^3 : \ \varphi \mapsto (\varphi(0), \varphi(1), \varphi(\infty))$.
By degree reasons only $d = 0, 1$ terms survive in "correct formula"

$$\langle \Delta_i \star_q \Delta_j, \Delta_k \rangle = \sum_{d \geq 0} q^d \int_{I_{3;d}} \Delta_i \boxtimes \Delta_j \boxtimes \Delta_k$$

General theory of Gromov-Witten invariants

Input: X, a smooth projective algebraic variety over $\mathbb{C}$
$\rightsquigarrow$*Output*: for any $\beta \in H_2(X, \mathbb{Z})$ and any $k \geq 0$, a homology class

$$I_{k;\beta} \in H_{2\delta}(X^k, \mathbb{Q})$$

where $\delta = \delta_{k;\beta} := \dim_{\mathbb{C}} X + k - 3 + \int_\beta c_1(T_X)$ is the *virtual dimension* of certain compact possibly singular moduli orbi-space of so called *stable maps* $\overline{\mathcal{M}}_{genus=0,k}(X, \beta)$, whose open (but possibly not everywhere dense) part consists of isomorphism classes of tuples $(C, p_1, \ldots, p_k; \varphi)$ where $C \simeq \mathbb{C}P^1$ is a rational curve (i.e. of genus=0) with $k \geq 3$ distinct marked points $p_1, \ldots, p_k$, and $\varphi : C \to X$ is an algebraic map of degree $\beta = \varphi_*([C])$.
The class $I_{k;\beta}$ is defined as the image of virtual fundamental class $\in H_{2\delta_{k;\beta}}(\overline{\mathcal{M}}_{g=0,k}(X, \beta), \mathbb{Q})$ under the evaluation map.

We love Gromov-Witten invariants because their generating series satisfies a beautiful system of non-linear differential equations (WDVV equations) $\implies$ quantum deformation of $H^\bullet(X)$.

Quantum multiplication from GW invariants

Let us choose a linear basis (Δ_i) of $H^\bullet(X,\mathbb{Q})$ compatible with $\mathbb{Z}$-grading, and such that $\Delta_0 = 1 \in H^0(X,\mathbb{Q}) = \mathbb{Q}$ and $\Delta_1,\ldots,\Delta_r \in H^2(X,\mathbb{Q})$ are the first Chern classes of ample line bundles on X where $r = rk\, H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ (Hodge classes in degree 2). Let us introduce formal variables t_i (even or odd) corresponding to elements Δ_i of the basis, and also additional even formal variables $q_1,\ldots,q_r$.

The quantum multiplication is a linear map preserving $\mathbb{Z}/2$-grading:

$$\star = \star_{q,t} : H \otimes H \to H[[q_1,\ldots,q_r;\ \underbrace{t_0,t_1,\ldots}_{\substack{\text{exterior algebra}\\ \text{in odd } t_i}}\ ]], \quad H := H^\bullet(X,\mathbb{Q})$$

$$\langle \Delta_{i_1} \star \Delta_{i_2}, \Delta_{i_3}\rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \prod_{j=1}^{r} q_j^{\int_\beta \Delta_j} \sum_{\substack{k\geq 0,\\ i_4,\ldots,i_{k+3}}} \frac{\prod_{j\geq 4} t_{i_j}}{k!} \int_{I_{k+3,\beta}} \Delta_{i_1} \boxtimes \cdots \boxtimes \Delta_{i_{k+3}}$$

Quantum multiplication is ***associative*** $\iff$

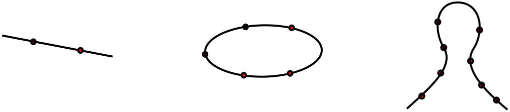
$$\langle \Delta_{i_1} \star \Delta_{i_2}, \Delta_{i_3} \star \Delta_4 \rangle = \langle \Delta_{i_1} \star \Delta_{i_3}, \Delta_{i_2} \star \Delta_4 \rangle,$$

(*WDVV equations*, or *associativity axiom*), does not depend on t_0 (corresponding to $1 \in H^0(X)$ (*unit axiom*), and depends on variables $q_1, \ldots, q_t, t_1, \ldots, t_r$ only via expressions $(q_i e_i^t)_{i=1,\ldots,r}$ (*divisor axiom*).

The origin of Witten-Dijkgraaf-Verlinde-Verlinde equation: consider the virtual fundamental cycle of stable maps from rational curves with $4+k$ points $p_1, \ldots, p_{k+4}$ such that the cross-ratio $\lambda = (p_1 : p_2 : p_3 : p_4) \in \mathbb{C}$ is fixed. This class does not depend on the value of λ, and in the limits $\lambda \to 0, \infty$ one obtains the matching of Taylor coefficients between the l.h.s. and the r.h.s. of WDVV equation.

Example: rational curves on $\mathbb{C}P^2$

By the unit and divisor axioms, GW invariants of $X = \mathbb{C}P^2$ are determined by a sequence of numbers $c_1 = 1, c_2 = 1, c_3 = 12, \ldots$ where $c_d \in \mathbb{Z}$ is the number of rational curves in $\mathbb{C}P^2$ of degree $d \geq 1$ passing through *generic* $(3d+2)$ points.



WDVV equtions $\iff$ recursion

$$\forall d \geq 2 : c_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2\geq 1}} c_{d_1} c_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

$\rightsquigarrow$ (using $c_1 = 1$) $c_2 = 1, c_3 = 12, c_4 = 620, c_4 = 87403, \ldots$
Generating series is related to a solution of Painlevé VI equation.

Theory of Gromov-Witten invariants is ~ 30 years old beautiful but isolated chapter of algebraic geometry, it was almost useless for "classical" questions.

GW invariants are invariant under continuous deformations of algebraic varieties, and in fact can be defined for arbitrary compact sympectic manifolds (following pioneering ideas of A.Floer and M.Gromov), via pseudo-holomorphic maps.

The only obvious relation between algebraic geometry and GW-invariants is that the latter are $\mathbb{Q}$-linear combinations of *algebraic* classes. It looks (in examples) like a weak constraint...

Not true any more!!!

GW theory is a new powerful universal tool in birational geometry

F-manifolds with Euler fields

In general, the convergence of series in the definition of quantum product is not known. One of possible fixes is to work in an algebraically closed *non-archimedean* field $\mathbb{K} := \cup_{N\geq 1}\overline{\mathbb{Q}}(\!(y^{1/N})\!)$. Let us consider the $\mathbb{K}$-analytic super manifold $\mathcal{F}_X$ with coordinates $q_1, \ldots, q_r$ and t_i for $i \notin \{1, \ldots, r\}$ where

$$0 < |q_i| < 1, \quad 0 \leq |t_i| < 1 \text{ for j such that } \Delta_i \text{ is an even class}$$

Quantum multiplication gives an associative commutative product $\star$ on the tangent bundle $T_{\mathcal{F}_X}$ identified with $H^\bullet(X)$ via

$$\Delta_i \mapsto \begin{cases} q_i \partial_{q_i} & \text{if } i \in \{1, \ldots, r\} \\ \partial_{t_i} & \text{otherwise} \end{cases}$$

Another important structure is *Euler vector field* given by the cohomology class

$$Eu := c_1(T_X) + \sum_{i:\deg \Delta_i \neq 2} \frac{\deg \Delta_i - 2}{2} t_i \Delta_i$$

Decomposition theorem

Denote $\mathcal{M} := \mathcal{F}_X$. The multiplication $\star \in \Gamma(\mathcal{M}, (T^*_{\mathcal{M}})^{\otimes 2} \otimes T_{\mathcal{M}})$ and the Euler field $Eu \in \Gamma(\mathcal{M}, T_{\mathcal{M}})$ are related by

$$Lie_{Eu}(\star) = \star\,.$$

Let us consider a point $p \in \mathcal{M}^{even}$ and a finite collecton of disjoint open discs $(D_\alpha) \in \mathbb{K}$ such that the spectrum of the operator $Eu \star \cdot$ acting on $T_p\mathcal{M}$, is contained in the union $\sqcup_\alpha D_\alpha$. Then locally near p the same is true, and we get a decomposition of $T_{\mathcal{M}}$ in the vicinity of p into a direct sum of subspaces. The general result is that this decomposition comes from a canonical decomposition

$$(\mathcal{M}, \star, Eu) = \prod_\alpha (\mathcal{M}_\alpha, \star_\alpha, Eu_\alpha) \text{ near } p$$

of (quotient) varieties endowed with products and Euler fields.

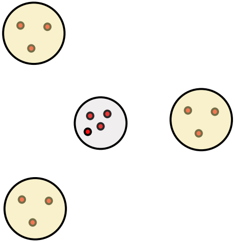
Blowup formula

Let $Z \subset X$ be a smooth closed subvariety of codimension $m \geq 2$.

By making blowup with center at Z we obtain a new smooth projective variety $\widetilde{X} = Bl_Z X$. It is well-known that there is a canonical identification of cohomology spaces (breaking $\mathbb{Z}$-grading and cup-product)

$$H^\bullet(\widetilde{X}) \simeq H^\bullet(X) \oplus \bigoplus_{(m-1) \text{ copies}} H^\bullet(Z)$$

If we consider spectrum of $(Eu \star \cdot)_{|T_p\mathcal{F}_{\widetilde{X}}}$ for a point $p \in \mathcal{F}^{even}_{\widetilde{X}}$ corresponding to an ample class on $\tilde{X}$ sufficiently close to the *semi-ample* class $[\tilde{X} \to X]^* \omega_X$ where ω_X is an ample class, we obtain a picture like this:



where eigenvalues close to 0 corresponding to classes in $H^\bullet(X)$, and eigenvalues close to the rescaled $(m-1)$-st roots of 1 corresponding to classes in $H^\bullet(Z)$.

The calculation is very easy, it is similar to the calculation of the quantum product for $\mathbb{C}P^n$ at the beginning of this lecture. The only relevant curves are constant maps and lines in the projectivization of the normal bundle to $Z \subset X$.

By the general decomposition theorem, we conclude that $\mathcal{M}_{\widetilde{X}}$ is locally isomorphic to the product of m different F-manifolds with Euler fields, which have the same dimensions as $\mathcal{M}_X$ and $(m-1)$ copies of $\mathcal{M}_Z$. In 2019 I conjectured that the factors are *canonically* isomorphic to open domains in $\mathcal{M}_X$ and $\mathcal{M}_Z$ respectively.

Last year Hiroshi Iritani (arXiv:2307.13555) proved this conjecture.

Iritani's result opens the gate to applications in birational geometry, and right now we (L.Katzarkov, T.Pantev, T. Yu and myself) are writing it up.

I'll show the force of the new theory, solving one of oldest open puzzles, which have resisted up to now all attempts based on the classical methods.

Atoms

Let X be a complex projective variety, consider the subspace of its even cohomology $H^{2\bullet}(X, \mathbb{Q})$ spanned by the Hodge classes:

$$H_{Hogde}(X) := \oplus_i \ (H^{i,i}(X) \cap H^{2i}(X, \mathbb{Q}))$$

This subspace gives a purely even submanifold $\mathcal{M}_{X,Hodge} \subset \mathcal{M}_X$ over $\mathbb{K}$, of dimension equal to the rank of $H_{Hogde}(X)$.
The spectrum of operator $Eu_p \star \cdot$ where $p \in \mathcal{M}_{X,Hodge}$ achieves certain maximal value μ at a dense open nonempty connected subset $\mathcal{M}^o_{X,Hodge} \subset \mathcal{M}_{X,Hodge}$. Eigenvalues of $Eu_p \star \cdot$ give a μ-fold *spectral cover* of $\mathcal{M}^o_{X,Hodge}$, possibly disconnected.

Definition: *the set of local atoms $Atoms_X$ is the set of connected components of the spectral cover described above.*

Important example: *if $K_X = \det T^*_X$ is numerically effective (has non-zero intersection with any curve), then $Atoms_X$ consists just of one point.* Reason: quantum product preserves filtration $H^{\geq \bullet}(X)$.

Now consider the following huge set:

$$\bigsqcup_{\text{iso classes of } X/\mathbb{C}} \left(Atoms_X / Aut\, X \right)$$

Iritani's theorem implies that one can relate certain elements of $\mathcal{M}_{\widetilde{X}}$ with some elements of $\mathcal{M}_X$ or $\mathcal{M}_Z$. This generates certain equivalence relation on the set above, and we denote by $Atoms_{\mathbb{C}}$ the set of equivalence classes. This set is naturally filtered by the *minimal* dimension of a variety in which an atom can appear.

Well-known fact: birational equivalences between smooth projective varieties are generated by blowups with smooth centers of codimension $\geq 2 \implies$ ***non-rationality*** criterion:

If for an N-dimensional variety X (here $N \geq 2$) at least one of atoms of X does not appear in varieties of dimension $\leq N - 2$, then X is not rational.

Some invariants of atoms

Our goal is to prove non-rationality of certain 4-dimensional varieties. Hence we have to study atoms coming from all ≤ 2-dimensional varieties, i.e. from points, curves and surfaces. Moreover, it is sufficient to consider only one representative in each birational class of surfaces.

For every atom α (in general) we have following invariants:

1. the rank ρ_α of the space of Hodge classes $H_{Hogde}(X) \otimes_{\mathbb{Q}} \mathbb{K}$ in the corresponding generalized eigenspace of $Eu \star \cdot$,
2. the Hodge polynomial $P_\alpha \in \mathbb{Z}[t, t^{-1}]$ whose coefficient at t^k is equal to the rank of the generalized α-eigenspace in $\oplus_{p,q:p-q=k} H^{p,q}(X)$.

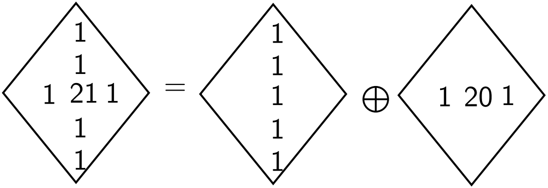
Using these two types of invariants we will be able to distinguish certain atom of the generic cubic 4-fold from those coming from points, curves and surfaces.

Atoms from ≤ 2-dimensional varieties

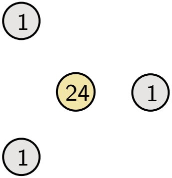
1. for any atom α coming from points or curves we obviously have $\boxed{Coeff_{t^2} P_\alpha = 0}$,
2. for minimal models X of all surfaces, *except* surfaces of general type and $K3$ surfaces, we have $\boxed{Coeff_{t^2} P_\alpha = 0}$ for any atom coming from X, because $H^{2,0}(X) = 0$,
3. for the minimal resolution X of ADE singularities of the minimal model of a K3 surface or a surface of general type, we have $K_X \geq 0$, hence only one atom α and then $\boxed{\rho_\alpha \geq 3}$, as X has two non-trivial algebraic cycles of dimensions $0, 2$ and *at least one* non-trivial algebraic cycle of dimension 1.

Non-rationality of the generic cubic 4-fold

Generic cubic 4-fold $X \subset \mathbb{C}P^5$ has the following Hodge diamond (and the decomposition into the sum of Hodge classes and the transcendental part):



Classical Givental's calculation: at a special (maybe non-generic) point of $\mathcal{M}_{X,Hogde}$ the spectrum of Eu is



hence the middle part has $\boxed{\rho = 2, Coeff_{t^2} P_\alpha = 1}$

$\Longrightarrow$ it **can not** come from ≤ 2 dimensions. ■