2021-11-08 Kähler geometry

If we apply Kempf-Ness naively in this infinite dimensional setting, the problem can be regarded as a GIT problem. (8)

Remark K^c does not exist in this infinite dimensional setting.

But $k^c = \{ u + iv \mid u, v \in C_0^\infty(M) \}$ exists. This defines a foliation of Z. Each leaf plays the role of K^c-orbit P, and P/K is diffeomorphic to the space of Kähler forms in a fixed Kähler class.

$$L_{JX}\,\omega = i\partial\bar\partial v_X$$

X Hamiltonian $\to v_X$

$$\therefore \text{“}(\mathrm{Exp}(iv))^*\omega\text{''} = \omega + i\partial\bar\partial v_X$$

Thus

$$\text{“}K^c \cdot p\text{''} \;\boxed{\begin{array}{l} \cong \text{ the space of Kähler forms} \\ \text{ in a fixed Kähler class.}\end{array}}$$

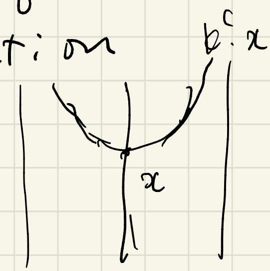
$\cap$

N.

Theorem (Kempf-Ness).

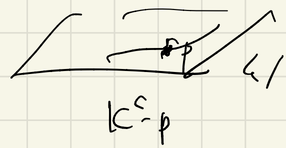
Let K^c be the complexification of K.

$p \in N$ is polystable w.r.t. K^c-action



$\iff$ $\mu^{-1}(0) \cap K^c \cdot p \neq \phi$.

"K^c-orbit of p contains a zero of the moment map".



We go back to finite dim moment map geometry.

Question: How can we check properness of $H = \log |r|^2$ $\quad r \in P \subset \Lambda^{-1}$.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mathbb{C}^*$-orbit

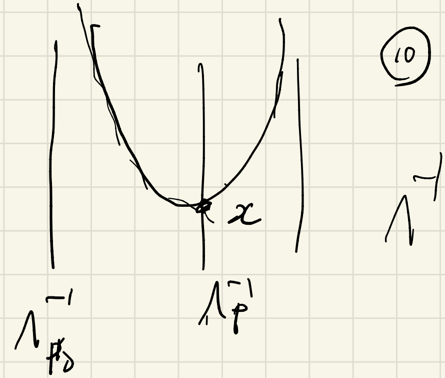
The answer (Mumford) is Hilbert-Mumford criterion.

$\Lambda^{-1} \longrightarrow N$ with $\mathbb{C}^*$-action.

$$\sigma : \mathbb{C}^* \longrightarrow K^c$$
$$t \longmapsto \sigma(t)$$

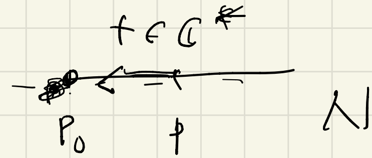
If $\lim_{t \to 0} (\sigma(t) p) = p_0$

then



$$\sigma(t) p_0 = p_0 \quad \text{for } \forall t$$

$$\sigma(t) \underset{\substack{\uparrow \\ \text{1-dim}}}{\Lambda^{-1}_{p_0}} = \Lambda^{-1}_{p_0}$$



$$\begin{array}{cccc} \sigma(t) : & \Lambda^{-1}_{p_0} & \longrightarrow & \Lambda^{-1}_{p_0} \\ & \rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ & z & \longmapsto & t^{-\alpha} z \end{array}$$

$$\left\{ \sigma(t) \underset{\substack{\Lambda^{-1}_p \\ \rotatebox[origin=c]{-90}{$\in$}}}{x} \;\middle|\; t \in \mathbb{C}^* \right\} \text{ is closed}$$

$\iff$ $H = \log |r|^2$ is proper.

$\iff$ $t^{-\alpha} z \to \infty$ as $t \to 0$

$\iff$ $\alpha > 0$.

<u>Def</u> α is called <u>Mumford weight</u>.

Hilbert-Mumford criterion

$\mathbb{C}^*$-orbit is stable $\Longleftrightarrow$ Mumford weight > 0.

Space of Kähler forms ((Calabi style)

$\xrightarrow[\text{Moser}]{}$ space of almost complex strs.

$\xrightarrow[\text{Hilbert-Mumford criterion}]{}$ "test configuration" "DF inv"

<u>Def</u> A pair of (M, L) of compact complex manifold M and an ample line bdle L on M is called a polarized manifold.

$c_1(L) > 0$

$$\Omega = \{ \underbrace{\omega}_{\text{Kähler form}} \mid [\omega] \in c_1(L) \}$$

space of Kähler forms.

- Kodaira embedding

$$\imath : M \hookrightarrow \mathbb{P}^N(\mathbb{C}) \quad \mathcal{O}(1)$$

$$L = \imath^* \mathcal{O}(1).$$

- Kodaira vanishing

If $L > 0$ then $\exists n_0$ s.t.

for $\forall n \geq n_0$, $H^i(M, L^n) = 0$

for $\forall i > 0$.

(So only $H^0(M, L^n)$ remain)

Riemann-Roch $\longrightarrow$ $\dim H^0(M, L^k)$

$$= \frac{c_1(L^k)^m}{m!} + \cdots$$

polynomial on k of degree m.

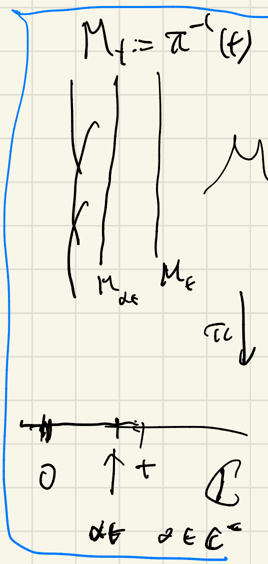
Definition (test configuration)

A test configuration of a polarized manifold (M, L) with exponent r consists of the following.

1. a scheme $\mathcal{M}$ with a $\mathbb{C}^*$-action.
2. a $\mathbb{C}^*$-equivariant line bundle
   $$\mathcal{L} \to \mathcal{M}$$
3. a flat $\mathbb{C}^*$-equivariant map
   $$\pi : \mathcal{M} \longrightarrow \mathbb{C}$$
   where $\mathbb{C}^*$ acts on $\mathbb{C}$ by standard multiplication such that

   (a) $M_t = \pi^{-1}(t) \cong M$ for $\forall t \neq 0$ in $\mathbb{C}$ ($\exists t \neq 0$)

   (b)
   $$(M_t, \mathcal{L}|_{M_t}) \cong (M, L^r)$$
   $$(t \neq 0)$$



Point here is that M_0 (the <u>central fiber</u>) could be singular, but admits a $\mathbb{C}^*$-action.

Remark Because of $\mathbb{C}^*$-action

$$(M_t, \mathcal{L}|_{M_t}) \cong (M_s, \mathcal{L}|_{M_s})$$

for all $t \neq 0$ and $s \neq 0$.

Remark Flatness implies that for r large

$\underbrace{\pi_* \mathcal{L}}$ $\longrightarrow \mathbb{C}$ is a vector bundle.

(sheaf of $H^0(U, \mathcal{L}|_U)$, $U \subset \mathbb{C}$.)

of rank $h^0(M, L^r) := \dim H^0(M, L^r)$

$$\begin{array}{ccl} \mathcal{M} & \longrightarrow & \mathbb{P}^{h_0 - 1}(\mathbb{C}) \quad \mathbb{C}^*\text{-equiv.} \\ \cup & \nearrow & \\ M_t & \text{embedding} & \end{array}$$

Def (product configuration).

Suppose (M, L) admits a $\mathbb{C}^*$-action. Then $(M \times \mathbb{C}, L^r \times \mathbb{C})$ is naturally a test configuration.

This is called the product configuration.

Remark Given (M, L), there are many test configurations.