PART I. $SL_2(\mathbb{F}_q)$

Chapter 1. Structure

TODAY (Chapter 2. Harish-Chandra induction

Chapter 3. Introduction to ℓ-adic cohomology

Chapter 4. Deligne-Lusztig theory for $SL_2(\mathbb{F}_q)$

Group representations terminology / notation

Γ finite group

$\mathbb{K}$ alg. closed field of char. 0

often omitted

- A representation of Γ is a pair (M, ρ) where M is a finite dim. vector space and $\rho : \Gamma \to GL_{\mathbb{C}}(M)$

$$g \cdot m = \rho(g)(m)$$

- $\mathbb{K}\Gamma = \bigoplus_{\gamma \in \Gamma} \mathbb{K}\gamma$: group algebra

Representation $\Leftrightarrow$ fin. dim. $\mathbb{K}\Gamma$-module

- $\mathbb{K}\Gamma$-mod : category of fin. dim. $\mathbb{K}\Gamma$-modules
- $M \in \mathbb{K}\Gamma\text{-mod} \rightsquigarrow \chi_M : \Gamma \longrightarrow \mathbb{K}$, $\gamma \longmapsto \mathrm{Tr}(\gamma, M)$

Note that $\chi_M(1) = \dim M$.

- M is said irreducible if the only Γ-stable subspaces are 0 and M.

$$\mathrm{Irr}(\Gamma) = \{\text{irr. reps of } \Gamma\} / \text{isom}$$
$$\leftrightarrow \{\text{irr. char. of } \Gamma\}$$

- $\mathrm{Class}(\Gamma) := \{ f : \Gamma \longrightarrow \mathbb{K} \mid \forall \gamma, \gamma' \in \Gamma,\ f(\gamma\gamma') = f(\gamma'\gamma) \}$

$$\langle f, f' \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma)\, \overline{f'(\gamma)}$$

Theorem. (a) Every representation is a direct sum of irreducible ones

(b) $\mathrm{Irr}(\Gamma)$ is an orthonormal basis of $\mathrm{Class}(\Gamma)$

(c) $M \simeq M' \Leftrightarrow \chi_M = \chi_{M'}$.

Corollary. $M \simeq \bigoplus_{S \in \mathrm{Irr}\,\Gamma} S^{\oplus m_S}$

$$\iff \chi_M = \sum_{S \in \mathrm{Irr}\,\Gamma} m_S \chi_S$$

$$\iff \forall S \in \mathrm{Irr}\,\Gamma, \ \langle \chi_M, \chi_S \rangle = m_S$$

- $\mathbb{K}_\Gamma$: trivial representation of Γ:

$$1_\Gamma = \chi_{\mathbb{K}_\Gamma} : \gamma \longmapsto 1$$

- $\mathbb{K}\Gamma$: regular representation

$$\chi_{\mathbb{K}\Gamma} : \gamma \longmapsto \begin{cases} |\Gamma| & \text{if } \gamma = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{\mathbb{K}\Gamma} = \sum_{\chi \in \mathrm{Irr}\,\Gamma} \chi(1)\,\chi$$

$$\boxed{\mathbb{K} = K}$$

Example. Let X be a finite Γ-set. We denote by $K[X]$ the space of formal K-linear combinations of elts of X: it is $K\Gamma$-module.

$$\gamma \cdot \left(\sum_{x \in X} \lambda_x x \right) = \sum_{x \in X} \lambda_x (\gamma \cdot x)$$

It is called a permutation module.

$$\chi_{K[X]}(\gamma) = |X^\gamma|$$

where $X^\gamma = \{ x \in X \mid \gamma \cdot x = x \}$

- If $\Gamma' \subset \Gamma$:

$$\mathrm{Ind}_{\Gamma'}^{\Gamma} M' = K\Gamma \otimes_{K\Gamma'} M'$$

$\mathrm{Res}_{\Gamma'}^{\Gamma} M = M$ viewed as a $K\Gamma'$-module

Classical Mackey formula.

$$\langle \mathrm{Ind}_{\Gamma'}^{\Gamma} f', \mathrm{Ind}_{\Gamma''}^{\Gamma} f'' \rangle_\Gamma = \sum_{\gamma \in [\Gamma' \backslash \Gamma / \Gamma'']} \langle \mathrm{Res}^{\Gamma'}_{\Gamma' \cap {}^\gamma\Gamma''} f', \mathrm{Res}^{{}^\gamma\Gamma''}_{\Gamma' \cap {}^\gamma\Gamma''} {}^\gamma f'' \rangle_{\Gamma' \cap {}^\gamma\Gamma''}$$

(set of representatives)

Chapter 2. Harish-Chandra induction.

2.A. Bimodules.

Recall that p is odd.

Let Γ, Γ' be two finite groups

Let M be a fin. dim. $(K\Gamma, K\Gamma')$-bimodule

$$(\gamma\cdot m)\cdot\gamma' = \gamma\cdot(m\cdot\gamma')$$

$$\rightsquigarrow \mathcal{F}_M : K\Gamma'\text{-mod} \longrightarrow K\Gamma\text{-mod}$$

$$E' \longmapsto M \otimes_{K\Gamma'} E'$$

$$^*\mathcal{F}_M = \mathcal{F}_{M^*} : K\Gamma\text{-mod} \longrightarrow K\Gamma'\text{-mod}$$

$$E \longmapsto \underbrace{M^*}_{\substack{\downarrow \\ (K\Gamma', K\Gamma)\text{-bimodule}}} \otimes_{K\Gamma} E$$

$$(\gamma' m^* \gamma)(m) = m^*(\gamma\cdot m\cdot\gamma')$$

Proposition 2.1. They are both left and right adjoint:

$$\mathrm{Hom}_{K\Gamma}(E, \mathcal{F}_M E') \simeq \mathrm{Hom}_{K\Gamma'}({}^*\mathcal{F}_M E, E')$$

$$\mathrm{Hom}_{K\Gamma}({}^*\mathcal{F}_M E', E) \simeq \mathrm{Hom}_{K\Gamma'}(E', \mathcal{F}_M E)$$

Proof: • $M^* \otimes_{K\Gamma} M \xrightarrow{\alpha} K\Gamma'$

$$m^* \otimes m \longmapsto \frac{1}{|\Gamma'|}\sum_{\gamma'\in\Gamma'} \underbrace{m^*(m.\gamma')}_{=(\gamma'\cdot m^*)(m)}\gamma'^{-1}$$

is a morphism of $(K\Gamma', K\Gamma')$-bimodules

• $\beta : K\Gamma \longrightarrow M \otimes_{K\Gamma'} M^*$

$$\gamma \longmapsto \sum_{i=1}^{n} \gamma.e_i \otimes e_i^*$$

where $(e_1, \dots, e_n)$ is a basis of M

and $(e_1^*, \dots, e_n^*)$ is its dual basis

β is a morphism of $(K\Gamma, K\Gamma)$-bimodules

• $\varphi : E \longrightarrow M \otimes_{K\Gamma'} E'$

$$\rightsquigarrow \begin{tikzcd} M^* \otimes_{K\Gamma} E \arrow[r, "\mathrm{Id}_{\Gamma'} \otimes \varphi"] \arrow[d, bend right, "\alpha_\#(\varphi)"'] & M^* \otimes_{K\Gamma} M \otimes_{K\Gamma'} E' \arrow[d, "\alpha \otimes \mathrm{Id}_{E'}"] \\ E' \arrow[r, phantom, "\simeq"] & K\Gamma' \otimes_{K\Gamma'} E' \end{tikzcd}$$

• $\psi : M^* \otimes_{K\Gamma} E \longrightarrow E'$

$$\begin{tikzcd} E \simeq K\Gamma \otimes_{K\Gamma} E \arrow[r] \arrow[rr, bend right, "\beta_\#\psi"'] & M \otimes_{K\Gamma'} M^* \otimes E \arrow[r] & M \otimes_{K\Gamma'} E \end{tikzcd}$$

Exercise. $\alpha_\#$ and $\beta_\#$ are inverse to each other. $\blacksquare$

$$F_M : \mathrm{Class}(\Gamma') \longrightarrow \mathrm{Class}(\Gamma)$$

$$f' \longmapsto \left(\gamma \mapsto \frac{1}{|\Gamma'|} \sum_{\gamma' \in \Gamma'} \mathrm{Tr}((\gamma,\gamma'),M)\, f'(\gamma'^{-1})\right)$$

$${}^*F_M : \mathrm{Class}(\Gamma) \longrightarrow \mathrm{Class}(\Gamma')$$

Proposition 2.2.

$$\chi_{\mathcal{F}_M E'} = F_M(\chi_{E'})$$

$$\chi_{{}^*\mathcal{F}_M E} = {}^*F_M(\chi_E)$$

$$\langle f, F_M f' \rangle_\Gamma = \langle {}^*F_M f, f' \rangle_{\Gamma'}$$

Example 2.3. Assume that $\Gamma' \subset \Gamma$ and $M = K\Gamma$ viewed as a natural $(K\Gamma, K\Gamma')$-bimodule.

$M^* \simeq K\Gamma$ as a $(K\Gamma', K\Gamma)$-bimodule

$$K\Gamma \times K\Gamma \longrightarrow K$$

$$(\gamma_1, \gamma_2) \longmapsto \begin{cases} 1 & \text{if } \gamma_1 = \gamma_2^{-1} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}_M E' = K\Gamma \otimes_{K\Gamma'} E' = \mathrm{Ind}_{\Gamma'}^{\Gamma} E'$$

$${}^*\mathcal{F}_M E = K\Gamma \otimes_{K\Gamma} E = \mathrm{Res}_{\Gamma'}^{\Gamma} E. \quad \blacksquare$$

2.B. Harish-Chandra induction.

$$G = SL_2(\mathbb{F}_q)$$

$$G/U \text{ is a } \underset{\substack{\uparrow \\ \text{acts on} \\ \text{the left}}}{G}\text{-set-}\underset{\substack{\uparrow \\ \text{acts on} \\ \text{the right}}}{T}$$

(because T normalizes U)

$$gU \cdot t = gUt = gtU \in G/U$$

So $K[G/U]$ becomes a (KG, KT)-bimodule

$$\mathcal{R} := \mathcal{F}_{K[G/U]} : KT\text{-mod} \longrightarrow KG\text{-mod}$$

$${}^*\mathcal{R} := {}^*\mathcal{F}_{K[G/U]} : KG\text{-mod} \longrightarrow KT\text{-mod}$$

So $K[G/U]$ becomes a

(KG, KT) - bimodule

$$\mathcal{R} := \mathcal{F}_{K[G/U]} : KT\text{-mod} \longrightarrow KG\text{-mod}$$

$\searrow$ Harish-Chancha induction

$$^*\mathcal{R} := {}^*\mathcal{F}_{K[G/U]} : KG\text{-mod} \longrightarrow KT\text{-mod}$$

$\searrow$ Harish-Chandra restriction

$$R : \mathrm{Class}(T) \longrightarrow \mathrm{Class}(G)$$

$$^*R : \mathrm{Class}(G) \longrightarrow \mathrm{Class}(T)$$

Proposition 2.4.

(a) $\mathcal{R} L = \mathrm{Ind}_B^G \tilde{L}$ $\nwarrow$ KB-module equal to L by making act U trivially

$(B = T \ltimes U)$

(b) $^*\mathcal{R}\, M = M^U := \{ m \in M \mid \forall g \in U, g\cdot m = m \}$

In particular

$\dim \mathcal{R} L = (q+1) \dim L$

$(|G/B| = q+1)$

Proof.

(a)
$$KG \otimes_{KB} \tilde{L} \longrightarrow K[G/U] \otimes_{KL} L$$
$$g \otimes_{KB} \ell \longmapsto gU \otimes_{KT} \ell$$
$$g \otimes_{KB} \ell \longleftarrow\!\!| \; gU \otimes_{KT} \ell$$

(b) $K[G/U]^* \simeq K[U\backslash G]$ as a (KT, KG)-bimodule

$$(gU, Uh) \longmapsto \begin{cases} 1 & \text{if } gU = h^{-1}U \\ 0 & \text{otherwise} \end{cases}$$

$$K[U\backslash G] \otimes_{KG} M \longrightarrow M^U$$
$$Ug \otimes m \longmapsto \frac{1}{|U|} \sum_{h \in U} hg \cdot m$$
$$U \otimes m' \longleftarrow\!\!| \; m' \,. \blacksquare$$

Mackey formula 2.5. Let $\alpha, \beta \in \mathrm{Class}(T)$.

$$\langle R(\alpha), R(\beta) \rangle_G = \langle \alpha, \beta \rangle_T + \langle \alpha, {}^s\beta \rangle_T$$

Recall that $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ normalizes T.

Mackey formula 2.5. Let $\alpha, \beta \in \mathrm{Class}(T)$.

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Recall that $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ normalizes T.

Proof. By prop. 2.4. (a), we have:

$$\langle R(\alpha), R(\beta)\rangle_G = \langle \mathrm{Ind}_B^G \tilde{\alpha}, \mathrm{Ind}_B^G \tilde{\beta}\rangle_G$$

(where $\tilde{\alpha}(tu) = \alpha(t)$)

$$\underset{\text{(classical)}}{=} \sum_{g \in [B\backslash G/B]} \left\langle \mathrm{Res}^{B}_{B\cap {}^gB} \tilde{\alpha}, \mathrm{Res}^{{}^gB}_{B\cap {}^gB} {}^g\tilde{\beta} \right\rangle_{B\cap {}^gB}$$

$\searrow$ Bruhat dec. 1.3 (a)

two $\{1, s\}$: $G = B \cup BsB$

$$= \langle \tilde{\alpha}, \tilde{\beta}\rangle_B + \langle \alpha, {}^s\beta\rangle_T$$

$$= \langle \alpha, \beta\rangle_T + \langle \alpha, {}^s\beta\rangle_T . \blacksquare$$

Since T is abelian,

$$\mathrm{Irr}(T) = \mathrm{Hom}_{gp}(T, K^\times)$$
$$\rotatebox{90}{$\in$}$$
$$\alpha, \beta$$

$${}^s\alpha = \alpha^{-1}$$

(because ${}^st = t^{-1}$)

$\mathrm{Hom}(T, K^\times)$ is cyclic of order $q-1$:

It has one element of order 2, which will be denoted by α_0:

$$\alpha_0(d(a)) = \begin{cases} 1 & \text{if } a \text{ is a square} \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{array}{c} \| \\ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \end{array}$$

Theorem 2.6. Let $\alpha, \beta \in \mathrm{Irr}(T)$.

(a) $R(\alpha) = R(\alpha^{-1})$

(b) $\langle R(\alpha), R(\beta)\rangle = 0$ if $\alpha \neq \beta^{\pm 1}$

(c) If $\alpha^2 \neq 1$ (i.e. $\alpha \neq 1_T, \alpha_0$) then $R(\alpha)$ is irreducible

(d) If $\alpha^2 = 1$, then $R(\alpha)$ is the sum of two distinct irreducible characters.

Proof. (a) $\left\langle R(\alpha) - R(\alpha^{-1}) , R(\alpha) - R(\alpha^{-1}) \right\rangle_G = 0$

by Mackey formula 2.5.

(b) is easy.

(c) and (d):

$$\left\langle R(\alpha) , R(\alpha) \right\rangle_G = \langle \alpha, \alpha \rangle + \langle \alpha, \alpha^{-1} \rangle$$

(Mackey formula 2.5)

$$= \begin{cases} 1 & \text{if } \alpha \neq \alpha^{-1} \\ 2 & \text{if } \alpha = \alpha^{-1} . \end{cases} \quad \blacksquare$$

Example 2.7.

$$\mathcal{R} K_T = \mathrm{Ind}_B^G K_B$$
$$= K[G/B]$$
$$= K[\mathbb{P}^1(\mathbb{F}_q)]$$

Let $v = \sum_{\ell \in \mathbb{P}^1(\mathbb{F}_q)} \ell$ "Steinberg rep"

$$St = \left\{ \sum_{\ell \in \mathbb{P}^1(\mathbb{F}_q)} \lambda_\ell \ell \;\middle|\; \sum \lambda_\ell = 0 \right\}$$

$$\mathcal{R} K_T = \underbrace{Kv}_{\text{G-stable}} \oplus \underbrace{St}_{\text{G-stable}}$$

$Kv \simeq K_G$

St is irreducible (by 2.6 (d)). $\blacksquare$

Exercise. Compute χ_{St} ; $\dim St = q$

Example 2.8.

$$R(\alpha_0) = R(\alpha_0)^+ + R(\alpha_0)^-$$

irreducible

We will see later that

$$\dim R(\alpha_0)^\pm = \frac{q+1}{2}$$

(2.9) $$R(\chi_{KT}) = \sum_{\alpha \in \mathrm{Irr}\, T} R(\alpha) = \chi_{K[G/B]}$$

$$= \underbrace{1_G}_{\dim 1} + \underbrace{St}_{\dim q} + \overbrace{R(\alpha_0)^+ + R(\alpha_0)^{-1}}^{\dim \frac{q\pm1}{2}} + \overbrace{2 \sum_{\substack{\alpha \in \mathrm{Irr}(T)/\mathrm{INV} \\ \alpha \neq 1, \alpha_0}} R(\alpha)}^{\dim q+1}$$

Conclusion. We have built $\frac{q+5}{2}$ irreducible representations of G.

Recall that $|\operatorname{Irr} G| = q+4$.

It remains to build $\frac{q+3}{2}$ representations.

Exercise. What happens if $p = 2$?