2021 - 9 - 18 Kähler geometry

If we set

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}}\right) \frac{\partial}{\partial x^l} = U^k \frac{\partial}{\partial x^k} =: U$$

then

$$R_{ijpl} = g\left(\frac{\partial}{\partial x^p}, U\right) = U^k g_{pk}$$

$$\therefore \quad U^k = g^{kp} R_{ijpl} = R_{ij}{}^k{}_l$$

$$\therefore \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^l} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^l} = R_{ij}{}^k{}_l \frac{\partial}{\partial x^k}$$

For $s = s^l \frac{\partial}{\partial x^l}$

excercise

$$\left(\nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \right) s = s^l R_{ij}{}^k{}_l \frac{\partial}{\partial x^k}$$

$$\left(= \frac{\partial}{\partial x^k} \cdot R_{ij}{}^k{}_l s^l \right.$$

$$\left(\begin{array}{l} T(e_1 \cdots e_r) = (e_1 \cdots e_r) A \\ T(x^i e_i) = e_i a^i{}_j x^j \end{array} \right)$$

This means

$$\nabla_i \nabla_j s^k - \nabla_j \nabla_i s^k = R_{ij}{}^k{}_l s^l$$

<u>Ricci identity</u>

equivalent to the definition of curvature

$$\left[\nabla_i \nabla_j \bigcirc = \nabla_j \nabla_i \bigcirc + \text{curvature terms} \right.$$

Curvature measures to what extent covariant derivatives commute.]

In the same way

$$\nabla_i \nabla_j s_k - \nabla_j \nabla_i s_k = (\nabla_i \nabla_j - \nabla_j \nabla_i)(g_{kp} s^p)$$

$$= g_{kp} (\nabla_i \nabla_j - \nabla_j \nabla_i) s^p = g_{kp} R_{ij}{}^p{}_l s^l$$

$$= R_{ijkl} s^l = R_{ijk}{}^p s_p$$

$= - R_{ij}{}^{k}{}_{l}\, s_k$ $\leftarrow$ Ricci identity for 1-forms.

By the following homework.

(1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$

(2) $R(X, Y, Z, W) = R(Z, W, X, Y)$

(3) $R(X, Y, Z, W) + R(Y, W, Z, X) + R(W, X, Z, Y) = 0.$ (Jacobi identity) the first Bianchi identity

(4) $\nabla_X R(Y, Z, W, V) + \nabla_Y R(Z, X, W, V) + \nabla_Z R(X, Y, W, V) = 0$ (Second Bianchi identity)

<u>Def</u> $R(X, Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$

$R(X, Y, Z, W) = g(Z, R(X, Y)W)$

Both R are called the <u>Riemannian curvature tensor</u>.

(3) $\Longleftrightarrow$ $R(X, Y)W + R(Y, W)X + R(W, X)Y = 0.$

$\Updownarrow$ exercise

$[[X, Y], W] + [[Y, W], X] + [[W, X], Y] = 0$

Jacobi identity

<u>Def</u> $J \in C^\infty(\mathrm{End}(TM)) \cong C^\infty(TM \otimes T^*M)$
is called an almost complex structure if
$J^2 = -\mathrm{id}$.

<u>Def</u> (M, J) is called an almost complex manifold.

When is (M, J) a complex manifold?

Example: Let M be a complex manifold.
$(z^1, \dots, z^m)$ local holo coordinates
$$z^i = x^i + \sqrt{-1}\, y^i$$
$(x^1, y^1, \dots, x^m, y^m)$ real local coordinates

$$z^i = x^i + \sqrt{-1}\, y^i \qquad i = 1, \dots, m$$

$m = 1 \qquad z = x + \sqrt{-1}\, y$

$$TM \otimes \mathbb{C} \ni \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

Let us define J a follows

$$J \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \qquad J \frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$$

$$J^2 = -\mathrm{id}$$

Then

$$J \frac{\partial}{\partial z} = \frac{1}{2} J\left(\frac{\partial}{\partial y} + \sqrt{-1}\,\frac{\partial}{\partial x}\right)$$

$$= \sqrt{-1}\,\frac{1}{2} J\left(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\right)$$

$$= \sqrt{-1}\,\frac{\partial}{\partial z}$$

$$J \frac{\partial}{\partial \bar{z}} = -\sqrt{-1}\,\frac{\partial}{\partial \bar{z}}$$

Recall

Frobenius Thm (involutive $\Leftrightarrow$ integrable)

Prop In order for an almost mfd (M, J) comes from true complex manifold only if

(1) $[C^\infty(T'M), C^\infty(T'M)] \subset C^\infty(T'M)$.

Theorem (Newlander-Nirenberg)

Conversely if (1) is satisfied (M, J) comes from true complex manifold.

(See Donaldson-Kronheimer.)

If X, Y are real vector fields and $X + \sqrt{-1}Y \in T'M$ then

$$J(X + \sqrt{-1}Y) = \sqrt{-1}(X + \sqrt{-1}Y)$$

$$JX = -Y \Leftrightarrow JY = X \qquad J^2 = -id$$

$$\therefore Y = -JX$$

So $T'M = \{X - \sqrt{-1}JX \mid X \in C^\infty(TM)\}$

Similarly $T''M = \{X + \sqrt{-1}JX \mid X \in C^\infty(TM)\}$

(1) $\Longleftrightarrow [X-\sqrt{-1}JX,\ Y-\sqrt{-1}JY] \in T'M$

$\Longleftrightarrow J[X-\sqrt{-1}JX,\ Y-\sqrt{-1}JY] = \sqrt{-1}[X-\sqrt{-1}JX,\ Y-iJY]$

$\Longleftrightarrow J[X,Y] - J[JX,JY] = [X,JY] + [JX,Y]$

$-J[JX,Y] - J[X,JY] = [X,Y] - [JX,JY]$

$N(X,Y) = [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY]$

N is called the Nijenhuis tensor

$$\left(\begin{array}{l} N(fX, gY) = fg\,N(X,Y) \qquad f,g \in C^\infty(M) \\ \qquad\qquad \Downarrow \\ X_p = X'_p \Rightarrow N(X_p, Y) = N(X'_p, Y) \end{array} \right.$$

(1) $\Longleftrightarrow N \equiv 0$

Theorem (Newlander–Nirenberg)

(M,J) is a complex mfd $\Longleftrightarrow N \equiv 0$.

<u>Def</u> J is integrable $\overset{\text{def}}{\Longleftrightarrow} N = 0$

We consider <u>integrable J</u>. (an)

<u>Def</u> Riemannian metric g is called a <u>Hermitian metric</u> if g is J-invariant

i.e. $g(JX,JY) = g(X,Y)$ for all X, Y.

Def $\omega(X,Y) = g(JX,Y)$

Lemma ω is a 2-form, i.e. ω is skew symm.

:) $\omega(Y,X) = g(JY,X) \overset{J\text{-invariant}}{=} g(\underbrace{JJY}_{-Y}, JX)$

$= -g(JX,Y) = -\omega(X,Y)$:)

g symmetric

Def g is a Kähler metric $\overset{\text{def}}{\Longleftrightarrow}$ $d\omega = 0$.

Def (M,g,J) with $d\omega = 0$ is called a <u>Kähler manifold</u>.

ω is called the Kähler form.

$[\omega] \in H^2_{DR}(M)$ is called the Kähler class

Def Sometimes, a complex manifold admitting a Kähler metric is called a Kähler manifold.

Def M is non-Kähler $\overset{\text{def}}{\Longleftrightarrow}$ M does not admit a Kähler metric

Ex. If $H^2_{DR}(M,\mathbb{R}) = 0$ then M is non-Kähler.
(non-symplectic)