Hydrodynamic Limit for the Facilitated Exclusion Process

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1 Introduction

2 Main Results

3 Formal Proof

4 Mapping

Hydrodynamic Limit

- In statistical physics, one of the main issues is to establish the partial differential equations that describe the evolution of the thermodynamic characteristics (the temperature, the density, the pressure) of a fluid.
- Example: Hamiltonian systems where particles evolve deterministically according to Newton's equations.
- A hard problem due to the lack of good ergodic properties of the system.
- One of the simplifications is to assume the evolution of the microscopic system to be stochastic, *e.g.* stochastic interacting particle systems. See [Liggett'85, 99, Kipnis and Landim'98].

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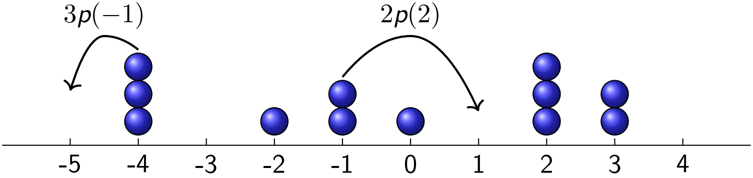
Independent Random Walks



- For $x \in \mathbb{Z}^d$, let η_x = number of particles at site x. Let $p(x)$ be a transition probability on $\mathbb{Z}^d$. Particles jump independently from x to y at rate $p(y-x)$.
- Assume $p(\cdot)$ has finite range: $p(x) = 0$ for $|x| > R$.
  Symmetric: $p(x) = p(-x)$ for all $x \in \mathbb{Z}^d$.
  Mean Zero: $\sum xp(x) = 0$.
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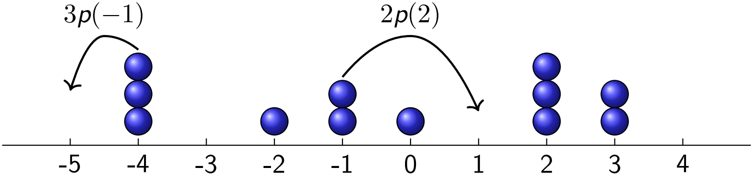

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Independent Random Walks

- For each $N \geq 1$, let η^N be a random configuration. Let $\rho : \mathbb{R}^d \to \mathbb{R}_+$ be some density profile. We say η^N is associated to the density profile ρ if for any smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}$ with compact support, for any $\varepsilon > 0$,

$$\lim_{N\to\infty} P\Big(\Big|\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_x^N\varphi\big(\tfrac{x}{N}\big) - \int_{\mathbb{R}^d}\rho(u)\varphi(u)du\Big| > \varepsilon\Big) = 0.$$

  In this case, we denote $\eta^N \sim \rho(u)$. Roughly speaking,

$$\eta_x^N \approx \rho(x/N).$$

- Assume the initial configuration $\eta(0) \sim \rho^{\text{ini}}$. Goal: prove for each t, $\eta(t\theta(N)) \sim \rho(t,u)$.
- The density profile $\rho(t,u)$ is called the hydrodynamic equation of the independent random walks.

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Independent Random Walks

Theorem [Kipnis and Landim'98]

(i) If $p(\cdot)$ has mean zero, then under diffusive scaling $(\theta(N) = N^2)$, the hydrodynamic equation is the heat equation

$$\begin{cases} \partial_t \rho(t,u) = \frac{1}{2}\sigma^2 \Delta \rho(t,u), \\ \rho(0,u) = \rho^{\mathrm{ini}}(u), \end{cases}$$

where $\sigma^2 = \sum x^2 p(x)$.

(ii) If $p(\cdot)$ is asymmetric, then under hyperbolic scaling $(\theta(N) = N)$, the hydrodynamic equation is the transport equation

$$\begin{cases} \partial_t \rho(t,u) + \frac{1}{2} m \cdot \nabla \rho(t,u) = 0, \\ \rho(0,u) = \rho^{\mathrm{ini}}(u). \end{cases}$$

where $m = \sum x p(x) \neq 0$.

Macroscopic/Microscopic Mapping

- Symmetric case: diffusive equation

	Macroscopic	Microscopic
Space	$u \approx \frac{x}{N}$	x
Time	t	tN^2

CLT: $\frac{S_N}{\sqrt{N}} \Rightarrow N(0,1)$.

- Asymmetric case: first order hyperbolic equation

	Macroscopic	Microscopic
Space	$u \approx \frac{x}{N}$	x
Time	t	tN

LLN: $\frac{S_N}{N} \to \mu$.

- Sub/super-diffusive scaling.

The Stefan Problem

- In one dimension, the Stefan problem reads

$$\begin{cases} \partial_t \rho = D\partial_{xx}^2 \rho, & \text{if } 0 < x < \Gamma(t), \\ \rho(x,t) = 0, & \text{if } x \geqslant \Gamma(t), \end{cases} \quad \text{and} \quad \frac{d\Gamma}{dt} = -\partial_x \rho(\Gamma(t), t)$$

- [Gravner and Quastel'00] for internal DLA
  [Funaki'99] for ice–water model
  [Blondel, Erignoux, Sasada and Simon'20, B., E. and Simon'21] for symmetric facilitated exclusion process.
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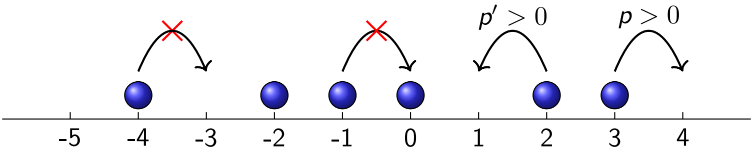
Facilitated Exclusion Process

State space $\Sigma_N := \{0,1\}^{\mathbb{L}_N}$, where $\mathbb{L}_N = \mathbb{Z}$ or $\mathbb{T}_N$. For a configuration $\eta \in \Sigma_N$, $\eta_x = 1$ if and only if x is occupied by a particle.



- Exclusion rule: there is at most one particle at each site.
- Facilitated rule: a particle has to be pushed forward in order to jump.
- Symmetric: $p = p'$, $\mathbb{L}_N = \mathbb{T}_N$. Asymmetric: $p > p'$, $\mathbb{L}_N = \mathbb{Z}$.

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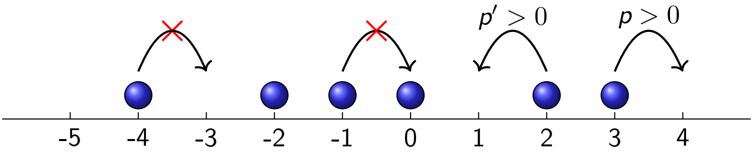

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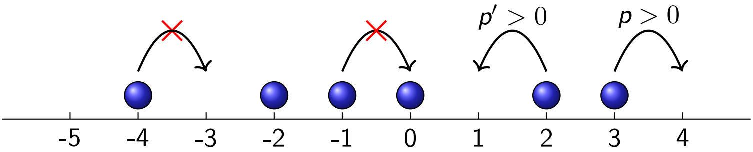

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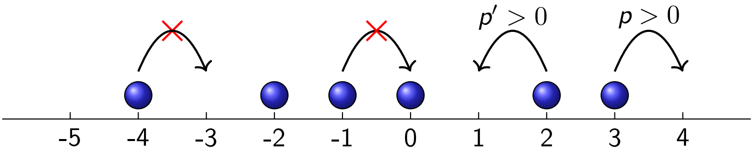

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Facilitated Exclusion Process

- For $x, y \in \mathbb{L}_N$, $\eta^{x,y}$denotes the configuration obtained from η by swapping the values at sites x and y.
- The jump rate is given by

$$c_{x,x+1}(\eta) = p\eta_{x-1}\eta_x\left(1-\eta_{x+1}\right) + p'\left(1-\eta_x\right)\eta_{x+1}\eta_{x+2}.$$

- For any local function $f\colon \Sigma_N \to \mathbb{R}$, the infinitesimal generator is

$$\mathscr{L}_N^{\mathrm{FEX}} f(\eta) := \sum_{x\in\mathbb{L}_N} c_{x,x+1}(\eta)\left(f\left(\eta^{x,x+1}\right) - f(\eta)\right).$$

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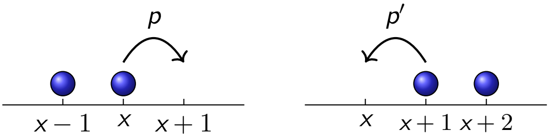

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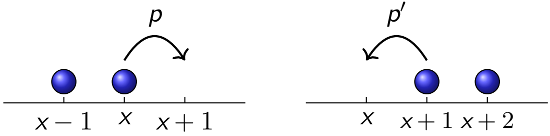

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Phase Transitions

The process displays a phase transition at the critical particle density $\rho_c = 1/2$.

- If initially $\rho > 1/2$, then the system evolves until there are no longer two neighboring empty sites.

$t = 0$

$t > 0$:ergodic

- If initially $\rho < 1/2$, then the system evolves until all particles are isolated, and thus can no longer move.

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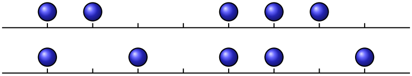

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  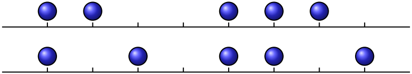

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  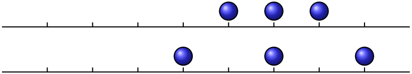

Invariant Measures on $\{0,1\}^{\mathbb{Z}}$

- For each $\rho > 1/2$, the process has a translation invariant and invariant measure π_ρ, which is not product. (exponentially decay)

$$\dots 01110101101 \dots$$

- Put a hole in some position with probability $1-\rho$, then put a random geometric number of parameter $\frac{1-\rho}{\rho}$ particles to its right, then a hole, starts again and so on.

$$\pi_\rho(11) = \pi_\rho(1) - \pi_\rho(01) = \rho - (1-\rho) \times 1 = 2\rho - 1.$$

$$\begin{aligned}\pi_\rho(111) &= \pi_\rho(11) - \pi_\rho(011) \\ &= (2\rho-1) - (1-\rho) \times 1 \times \frac{2\rho-1}{\rho} = \frac{(2\rho-1)^2}{\rho}.\end{aligned}$$

- For $\rho \le 1/2$, π_ρ is concentrated on frozen configurations.

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Facilitated Exclusion Process

- A new universality class (critical density, critical exponents) in the physics literature.
- Belongs to KPZ universality class [Baik, Barraquand, Corwin and Suidan'16].
- Stationary states of the model, see the work of Goldstein, Lebowitz, Speer etc. and [Chen and Z.'19].
- Open in higher dimensions.

Hydrodynamic Limit

Theorem [Blondel *et al.* '20, '21, Erignoux, Simon and Z. '22]

(i) In the symmetric case $p = p' = 1$, under diffusive scaling, the hydrodynamic equation is given by

$$\partial_t \rho(t,u) = \partial_u^2 \Big(\frac{2\rho(t,u)-1}{\rho(t,u)} \mathbf{1}_{\{\rho(t,u)>1/2\}} \Big)$$

with initial condition ρ^{ini}. (weak solution, uniqueness since $\frac{2\rho-1}{\rho}\mathbf{1}_{\rho>1/2}$ is nondecreasing)

(ii) In the asymmetric case, under hyperbolic scaling, the hydrodynamic equation is given by

$$\partial_t \rho(t,u) + (p-p')\partial_u \Big(\frac{(1-\rho(t,u))(2\rho(t,u)-1)}{\rho(t,u)} \mathbf{1}_{\{\rho(t,u)>1/2\}} \Big) = 0$$

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Entropy solution [Serre'99]

Consider

$$\begin{cases} \partial_t \rho(t,u) + \partial_u f(\rho(t,u)) = 0, & u \in \mathbb{R}, t > 0, \\ \rho(0,u) = \rho^{\mathrm{ini}}(u), & u \in \mathbb{R}. \end{cases}$$

The weak solution is not unique! Let (E, F) be an entropy-entropy-flux pair, i.e. $E'f' = F'$. Formally,

$$\partial_t E(\rho(t,u)) + \partial_u F(\rho(t,u)) = 0.$$

(Entropy inequality) For any E convex,

$$\partial_t E(\rho(t,u)) + \partial_u F(\rho(t,u)) \leq 0$$

in the sense of distributions.

The entropy solution could be obtained as $\lim_{\varepsilon \to 0} \rho^\varepsilon$, where

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$p = p' = 1$

- First recall

$$\rho(t, \tfrac{x}{N}) \approx \mathbb{E}[\eta_x(tN^2)], \quad x \in \mathbb{T}_N,$$

or equivalently,

$$\rho(t, u) \approx \mathbb{E}[\eta_{uN}(tN^2)], \quad u \in \mathbb{T}.$$

- Conservation law

$$\frac{d}{dt}\mathbb{E}[\eta_{uN}(tN^2)] = N^2\Big(\mathbb{E}[j_{uN-1,uN}(tN^2)] - \mathbb{E}[j_{uN,uN+1}(tN^2)]\Big),$$

where $j_{x,x+1}(t)$ is the instantaneous current from x to $x+1$,

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$$j_{x,x+1}(t) = \eta_{x-1}(t)\eta_x(t)(1-\eta_{x+1}(t)) - \eta_{x+2}(t)\eta_{x+1}(t)(1-\eta_x(t)).$$

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Remark: Most of the models are non-gradient and are more difficult!

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- Local ergodicity: the distribution of the process at time tN^2 around site uN is approximately $\pi_{\rho(t,u)}$. (replacement lemmas)
- For $\rho(t,u) > 1/2$,

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  The above term is zero if $\rho(t,u) \leq 1/2$.
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General methods [Kipnis and Landim'99]

- Entropy (GPV) method: uniqueness of solutions, explicit invariant measures.
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1 Introduction

2 Main Results

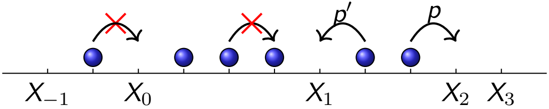
3 Formal Proof

4 Mapping

Mapping

- Label the empty sites from the left to the right in an increasing order. Let $X_y(t)$ be the position of the y–th empty site at time t.

EP:



- Let $\omega_y(t) \in \{0, 1, 2, \ldots\}$ be the number of particles between the y–th and $(y+1)$–th empty sites at time t.

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- A particle jumps to its right (resp. left) at rate $p\mathbf{1}_{\omega_y \geq 2}$ (resp. $p'\mathbf{1}_{\omega_y \geq 2}$).

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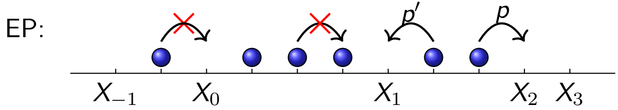

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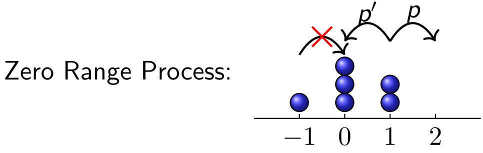

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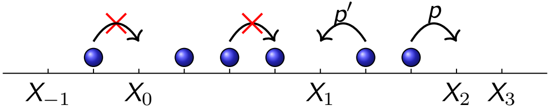

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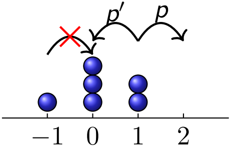

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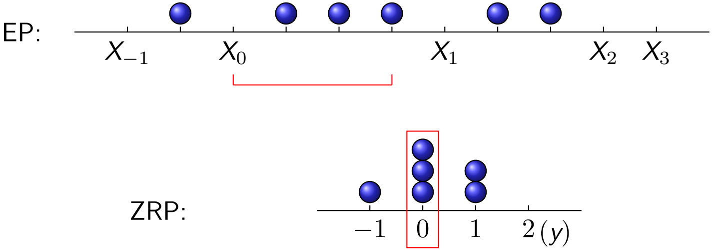
Mapping

Recall $\alpha(t,v)$: density profile for ZRP, $\rho(t,u)$ for EP.

$$\rho(t,u) = \frac{\alpha(t,v)}{1+\alpha(t,v)}, \quad \alpha(t,v) = \frac{\rho(t,u)}{1-\rho(t,u)}.$$

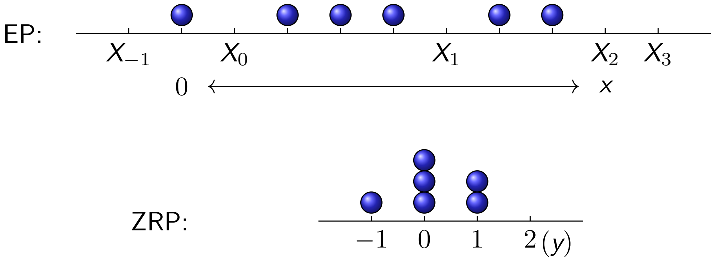


Mapping

Relation between $u = u(t, v)$ and $v = v(t, u)$: x, u for EP, y, v for ZRP.

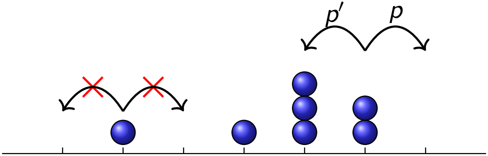
$$x = X_0(t) + \sum_{y'=0}^{y} [1 + \omega_{y'}(t)] \quad \Rightarrow \quad u = v_0(t) + \int_0^v [1 + \alpha(t, v')] dv'.$$

$$y = \sum_{x'=X_0(t)}^{x} [1 - \eta_{x'}(t)] \quad \Rightarrow \quad v = \int_{v_0(t)}^{u} [1 - \rho(t, u')] du.$$



Facilitated Zero Range Process

- If there are at least two particles at a site, then one of them jumps to the right (resp. to the left) at rate p (resp. p').



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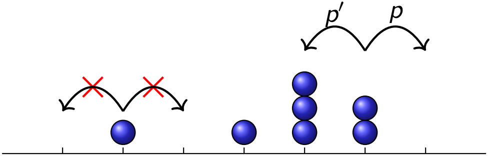

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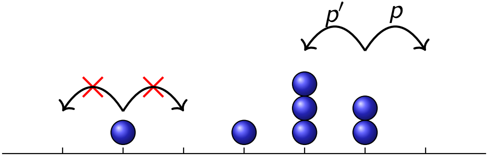

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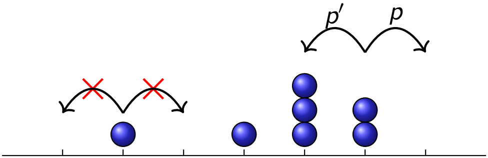

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Attractiveness

Attractiveness: if $\omega_y(0) \leq \widetilde{\omega}_y(0)$ for all y, then there exists a coupling such that $\omega_y(t) \leq \widetilde{\omega}_y(t)$ for all y and $t > 0$.

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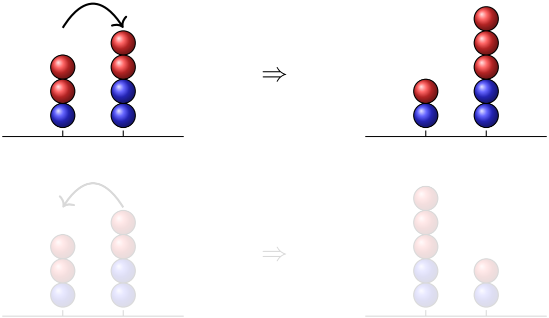

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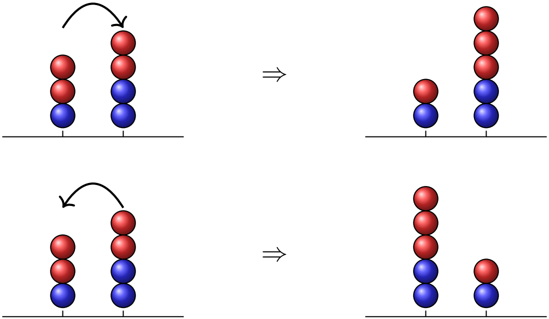

Proof Outline

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Facilitated Exclusion} \arrow[r, leftrightarrow, "\text{microscopic mapping}"] \arrow[d, "\begin{array}{r}\text{HL???}\\ \text{non-ergodic.}\\ \text{non-attractive.}\\ \text{invariant measures}\\ \text{not product.}\end{array}"'] & \text{Facilitated Zero Range} \arrow[d, "\begin{array}{l}\text{HL } \checkmark\\ \text{by attractiveness}\end{array}"] \\
\begin{array}{r}\text{(S) } \partial_t\rho = \partial_u^2\mathcal{H}(\rho)\\ \text{(A) } \partial_t\rho + (p-p')\partial_u\mathfrak{H}(\rho)\\ = 0\end{array} \arrow[r, leftrightarrow, "\text{macroscopic mapping}"'] & \begin{array}{l}\text{(S) } \partial_t\alpha = \partial_v^2\mathcal{G}(\alpha)\\ \text{(A) } \partial_t\alpha + (p-p')\partial_v\mathcal{G}(\alpha)\\ = 0\end{array}
\end{tikzcd}
$$

$$\mathcal{G}(r) = \frac{r-1}{r}\mathbf{1}_{\{r>1\}}$$
$$\mathcal{H}(r) = \frac{2r-1}{r}\mathbf{1}_{\{r>1/2\}}$$
$$\mathfrak{H}(r) = (1-r)\mathcal{H}(r)$$

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Thanks!

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