2020 · 9 · 16

$(\Omega, \mathcal{F}, \mathbb{P})$

$\Omega = \{0,1\}^E$

$\mathcal{F}$: σ-algebra generated by events that depend only on finitely many edges.

$\mathbb{P} = \mathbb{P}_p$.

HW: $\{0 \leftrightarrow \infty\}$ is in the σ-algebra.

Thm. $\exists$ $p_c \in (0,1)$, such that

$$\begin{cases} \mathbb{P}_p[0 \leftrightarrow \infty] = 0 & \text{for } p < p_c; \\ \mathbb{P}_p[0 \leftrightarrow \infty] > 0 & \text{for } p > p_c. \end{cases}$$

Pf: ① existence of p_c.

② $p_c > 0$.

③ $p_c < 1$.

① $\theta(p) := \mathbb{P}_p[0 \leftrightarrow \infty]$.

$\{0 \leftrightarrow \infty\}$ increasing event

Mono. $\theta(p)$ increasing in p.

$p_c := \sup\{p : \theta(p) = 0\}$.

$p < p_c$, $\theta(p) = 0$.

$p > p_c$, $\theta(p) > 0$.

② $p_c > 0$. Goal: $\mathbb{P}_p[0 \leftrightarrow \infty] = 0$ for p small.

$$\mathbb{P}_p[0 \leftrightarrow \infty] \leq \mathbb{P}_p[\exists \text{ an open path of length } n \text{ starting from } 0]$$

$$\leq \sum_{\ell \in \mathcal{L}_n} \mathbb{P}_p[\text{all edges along } \ell \text{ are open}]$$

$$\leq p^n \cdot \# \mathcal{L}_n$$

$$\leq p^n \cdot 4^n$$

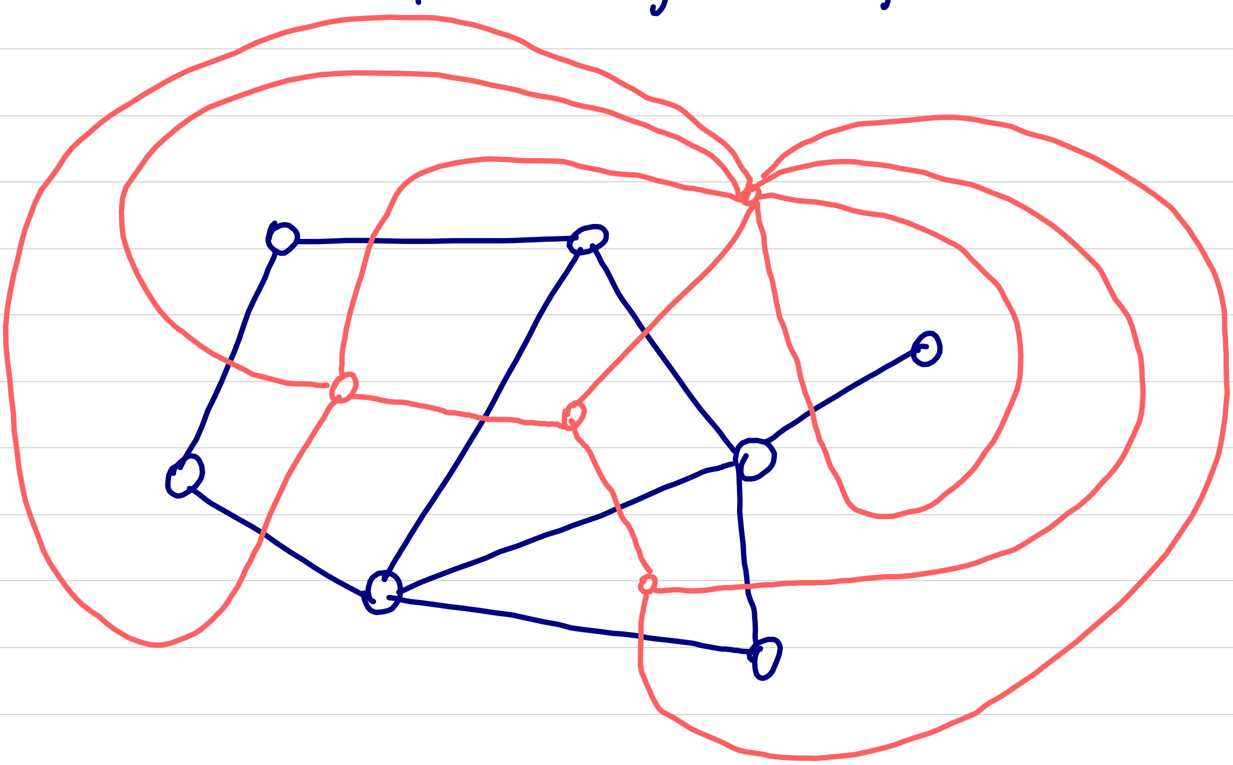
$p < \frac{1}{4}$, $\mathbb{P}_p[0 \leftrightarrow \infty] = 0$. $p_c \geq \frac{1}{4}$.

③ $p_c < 1$.

$G = (V, E)$ plane graph.

$G^* = (V^*, E^*)$.

- a vertex for each face of G.
- an edge whenever two faces of G are separated by an edge.



The dual graph of $\mathbb{Z}^2$ is $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$.

bond percolation. G: $\omega \in \{0,1\}^E$

G^*: $\omega^* \in \{0,1\}^{E^*}$

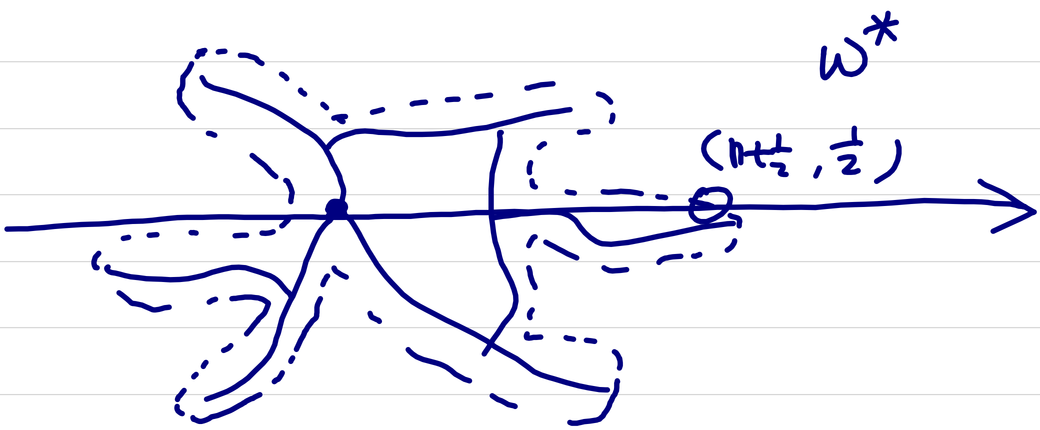
$\omega^*(e^*) = 1 - \omega(e)$

$\omega \sim \mathbb{P}_p$ on G

$\omega^* \sim \mathbb{P}_{1-p}$ on G^*

observation: $0 \not\leftrightarrow \infty$.

$\mathcal{C}$ = the open cluster containg 0.



$\mathbb{P}_p[0 \leftrightarrow\!\!\!\!/\ \infty]$

$\leq \mathbb{P}_p[\exists$ open circuit in ω^* surrounding $0]$

$\mathcal{L}_{m,n} = \{$ circuits of length m, passing through $(n+\frac{1}{2}, \frac{1}{2})\}$

$$\leq \sum_n \sum_{m \geq 2n} (1-p)^m \cdot \underline{\# \mathcal{L}_{m,n}}$$

$$\leq \sum_n \sum_{m \geq 2n} (4-4p)^m$$

$$= \sum_n \frac{(4-4p)^{2n}}{4p-3}$$

$$= \frac{4-4p}{(4p-3)^2}.$$

pick $p<1$, $\frac{4-4p}{(4p-3)^2} < 1$.

$\mathbb{P}_p[0 \leftrightarrow\!\!\!\!/\ \infty] < 1$, $\mathbb{P}_p[0 \leftrightarrow \infty] > 0$.

$p_c \leq p$.

Thm. For Bernoulli bond perco. on $\mathbb{Z}^2$,

$\mathbb{P}_p[\ \exists\ \infty\text{-cluster}\ ] = 0$

$\mathbb{P}_p[\ \exists!\ \infty\text{-cluster}\ ] = 1.$

① $\theta(p) = 0.$ $\mathbb{P}_p[\ \exists\ \infty\text{-cluster}\ ] = 0$

② $\theta(p) > 0.$ $\mathbb{P}_p[\ \exists\ \infty\text{-cluster}\ ] = 1$

③ $\theta(p) > 0.$ $\mathbb{P}_p[\ \exists!\ \infty\text{-cluster}\ ] = 1.$

① $\mathbb{P}_p[\ 0 \leftrightarrow \infty\ ] = 0.$ $\mathbb{P}_p[\ x \leftrightarrow \infty\ ] = 0.$

$$\mathbb{P}_p[\ \exists\ \infty\text{-cluster}\ ] = \mathbb{P}_p[\ \exists\, x \leftrightarrow \infty\ ]$$

$$\leq \sum_{x \in \mathbb{Z}^2} \mathbb{P}_p[\ x \leftrightarrow \infty\ ] = 0.$$

② ergodic.

$x \in \mathbb{Z}^2$, $\quad \tau_x : \{0,1\}^{E(\mathbb{Z}^2)} \to \{0,1\}^{E(\mathbb{Z}^2)}$

$$(\tau_x \omega)(\{a,b\}) = \omega(a+x, b+x), \quad \forall (a,b) \in E(\mathbb{Z}^2).$$

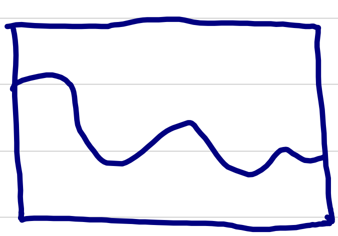
for an event A, define

$$\tau_x A = \{\omega : \tau_x \omega \in A\}.$$

Example:

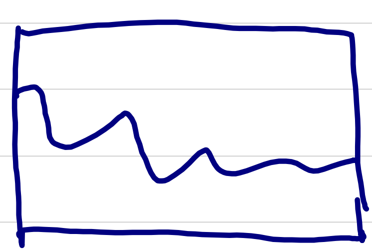
R

A



$\tau_x R$

$\tau_x A$:



An event A is invariant under translation if

$$\tau_x A = A, \quad \forall x \in \mathbb{Z}^2.$$

Example: $A = \{\exists\ \infty\text{-cluster}\}$.

Lemma: Bernoulli bond perco on $\mathbb{Z}^2$ is ergodic,

i.e. any translation-invariant event A

satisfies $\mathbb{P}_p[A] = 0$ or 1.

Pf: A.

$\forall \varepsilon > 0$, $\exists$ event B depending only on finitely many edges s.t. $\mathbb{P}_p[A \triangle B] \leq \varepsilon$.

B depends only on edges in $\Lambda_N \doteq [-N, N]^2$

$\tau_x A = A$. $\quad \mathbb{P}[A] = \mathbb{P}[A \cap \tau_x A]$.

$\mathbb{P}[B] + O(\varepsilon) = \mathbb{P}[\underline{B \cap \tau_x B}] + O(\varepsilon)$.

$|x| \geq 2N$, $\quad B \sslash \underline{\Lambda_N}$, $\tau_x B \sslash \underline{\tau_x \Lambda_N}$

$B \perp \tau_x B$.

$$\mathbb{P}[B] + O(\varepsilon) = \mathbb{P}[B] \cdot \mathbb{P}[\tau_x B] + O(\varepsilon)$$

$$\mathbb{P}[A] + O(\varepsilon) = \mathbb{P}[A] \cdot \mathbb{P}[\underbrace{\tau_x A}_{A}] + O(\varepsilon)$$

$$\mathbb{P}[A] = \mathbb{P}[A]^2 + O(\varepsilon).$$

$\varepsilon \to 0$. $$\mathbb{P}[A] = \mathbb{P}[A]^2.$$

$$\mathbb{P}[A] = 0 \text{ or } 1.$$

② $$\mathbb{P}_p[\exists\ \infty\text{-cluster}] \geq \theta(p) > 0.$$

$$\mathbb{P}_p[\exists\ \infty\text{-cluster}] = 1.$$

③ $\theta(p) > 0$. $\mathbb{P}_p[\exists!\ \infty\text{-cluster}] = 1$.

$k \geq 1$, $\mathcal{A}_k = \{\exists\ k\ \infty\text{-clusters}\}$.

$k = \infty$. $\mathcal{A}_k$ is translation-inv.

$$\mathbb{P}_p[\mathcal{A}_k] = 0 \text{ or } 1.$$

$\mathbb{P}_p[\mathcal{A}_k] = 0$ or 1.

$\sum_k \mathbb{P}_p[\mathcal{A}_k] = 1$.

$\exists\, k_0$ s.t. $\mathbb{P}_p[\mathcal{A}_{k_0}] = 1$, $\mathbb{P}_p[\mathcal{A}_k] = 0$ for $k \neq k_0$.

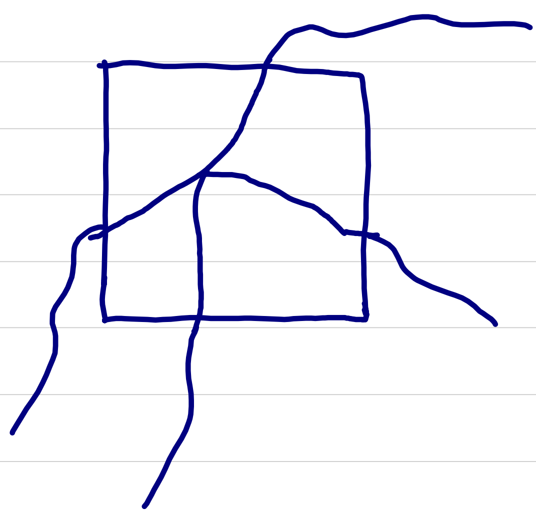
Goal: $k_0 = 1$.

Assume $\infty > k_0 \geq 2$. $\mathbb{P}_p[\mathcal{A}_{k_0}] = 1$.

$\exists\, n$ large enough,

$\mathbb{P}_p[$all k_0 ∞-clusters intersect $\Lambda_n] \geq \frac{1}{2}$.

observation:



$\mathcal{E}_n = \{$all edges in Λ_n are open$\}$

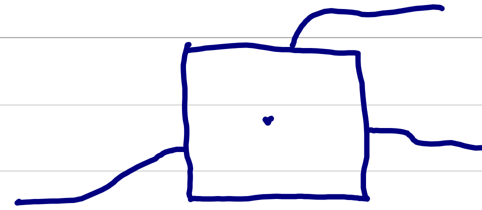
$\mathbb{P}_p[\mathcal{A}_1] \geq$

$\mathbb{P}_p\left[\mathcal{E}_n \cap \{\text{all } k_0\ \infty\text{-clusters intersect } \Lambda_n\}\right]$

$= \mathbb{P}_p[\mathcal{E}_n] \cdot \mathbb{P}_p[\cdots]$

$\geq p^{n^2} \cdot \frac{1}{2} > 0$. $\mathbb{P}_p[\mathcal{A}_1] > 0$. contradiction.

Assume $\mathbb{P}_p[\not\exists\infty] = 1.$

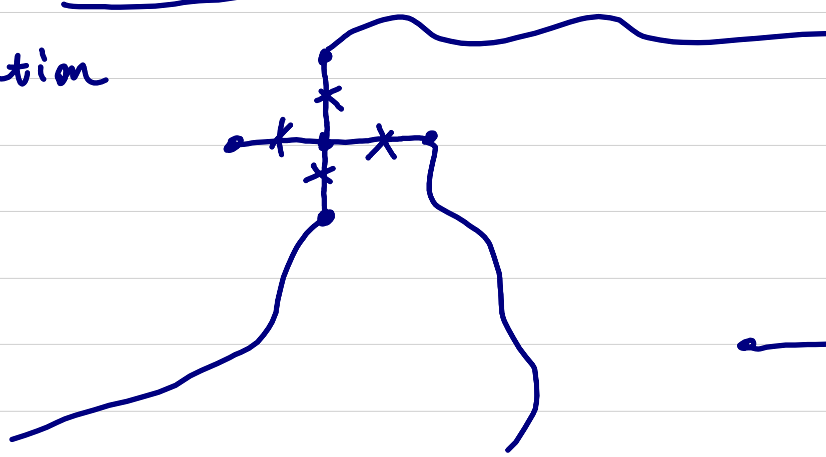


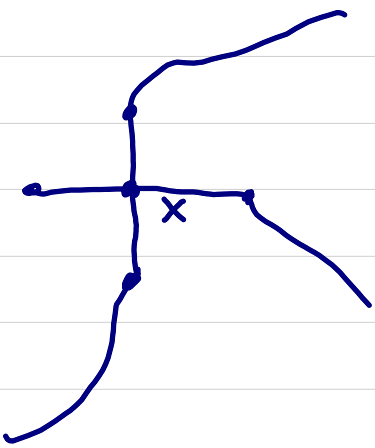
$\exists$ n large enough

$$\mathbb{P}_p[\ \exists\ 3\ \infty\text{-clusters intersecting } \Lambda_n\ ] \geq \tfrac{1}{2}.$$

$$\mathbb{P}_p[T_0] > 0.$$

T_0 : trifurcation





$T_x = \tau_x T_0.$

$\Upsilon := \{$ all trifurcation points in $\Lambda_n \}.$

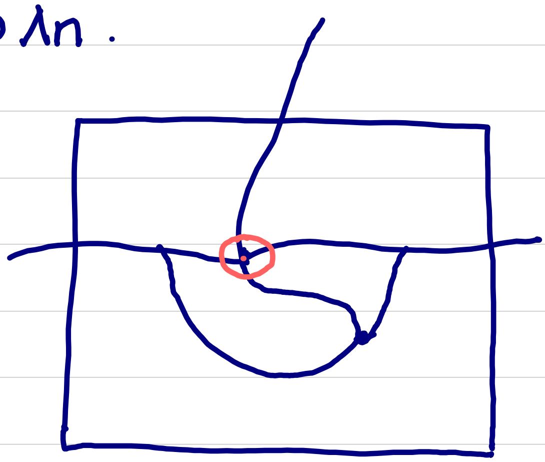
$$\mathbb{E}_p[\#\Upsilon] = \sum_{x\in\Lambda_n} \mathbb{P}_p[x \text{ trifurcation}]$$

$$= \mathbb{P}_p[T_0]\cdot\#V(\Lambda_n).$$

trifurcation points.

Claim: $\# \Upsilon \le \# \partial \Lambda_n$.

$\mathbb{E}_p[\# \Upsilon] \le \# \partial \Lambda_n$.



$$\underbrace{\mathbb{P}_p[T_0] \cdot \# V(\Lambda_n)}_{O(n^2)} \le \underbrace{\# \partial \Lambda_n}_{O(n)}.$$

$\mathbb{P}_p[T_0] = 0$. contradiction.

$\mathbb{P}_p[A_1] = 1$. $\square$