Clarification about Lecture1:

W: a cobordism from M_0 to M_1

W is a cobordism if $\begin{array}{l} M_0 \hookrightarrow W \\ M_1 \hookrightarrow W \end{array}$ are both homotopy equivalences.

Exercise: Suppose $\pi_1(W) = \pi_1(M_0) = \pi_1(M_1) = 1$. Then

W is an h-cobordism $\Leftrightarrow$ $H_*(W, M_0) = 0$.

Hint: Whitehead's theorem + Lefschetz duality theorem.

Lecture 2: classical invariants of 4-manifolds

X: connected, closed, oriented topological 4-mfd.

$$H^2(X;\mathbb{Z})/\text{tors} \cong \mathbb{Z}^{b_2(X)}$$

Define:
$$\begin{aligned} Q_X : H^2(X;\mathbb{Z})/\text{tors} \times H^2(X;\mathbb{Z})/\text{tors} &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto (\alpha \cup \beta)[X] \end{aligned}$$

Q_X is called the "intersection form". It is a symmetric unimodular bi-linear form over $H^2(X;\mathbb{Z})/\text{tors}$.

(Unimodular: take a set of free generators $\{\alpha_1, \cdots, \alpha_{b_2(X)}\}$

Then Q_X is represented by the matrix $[Q_X(\alpha_i, \alpha_j)]_{i,j}$.

Unimodular means that $\det([Q_X(\alpha_i,\alpha_j)]) = \pm 1$)

$b_2^+(X) / b_2^-(X) :=$ number of $+/-$ eigenvalues of Q_X.

$$\sigma(X) := \text{signature of } Q_X = b_2^+(X) - b_2^-(X).$$

Q: Why do we call Q_X the intersection form?

Lemma: Given any $\alpha \in H_2(X;\mathbb{Z})$, α can be represented by a embeded surface $\Sigma \hookrightarrow X$.

Proof: Recall $\{\text{complex line bundles over } X\} \xleftrightarrow{1:1} H^2(X)$

$$L \longleftrightarrow c_1(L) = e(L)$$

Take L s.t. $e(L) = \text{P.D.}(\alpha)$. Take a generic section $s: X \to L$ that is transverse to the 0-section $s_0: X \to L$.

Then $\Sigma = \{x \in X \mid s(x) = s_0(x)\}$ $\square$

Remark: Given $\alpha \in H_2(X)$, it is an important question to understand $\min\{\text{genus of embeded } \Sigma \hookrightarrow X \text{ s.t. } [\Sigma] = \alpha\}$.

(Thom conjecture: $X = \mathbb{CP}^2$.)

Lemma: Let Σ_1, Σ_2 be embeded surfaces. Suppose $\Sigma_1 \pitchfork \Sigma_2$.

Then $Q_X(\text{P.D.}[\Sigma_1], \text{P.D.}[\Sigma_2]) = \Sigma_1 \cdot \Sigma_2$.

algebraic intersection number = # positive intersection points − # negative intersection points.

Proof: Exercise. $\square$

Example: $X = S^4$ $\quad Q_X \cong 0$

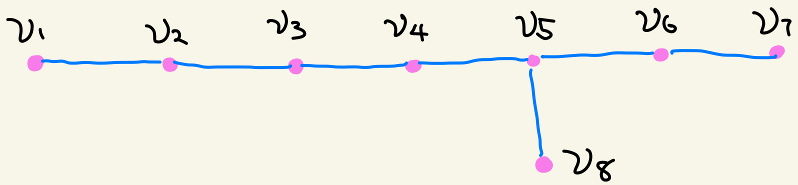
$X = \mathbb{CP}^2$ $\quad Q_X \cong [1]$

$X = S^2 \times S^2 \quad Q_X \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$X = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \quad Q_X \cong \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$X = K3 = \{[x_1, x_2, x_3, x_4] \in \mathbb{CP}^3 \mid \sum_{i=1}^{4} x_i^4 = 0\} \quad Q_X \cong 2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Here $E_8 = [a_{ij}]_{1 \le i,j \le 8} \qquad a_{ij} = \begin{cases} -2 & i = j \\ 1 & v_i \text{ and } v_j \text{ connected by edge} \\ 0 & \text{otherwise} \end{cases}$



Theorem (Whitehead) Suppose $\pi_1(X_1) = \pi_1(X_2) = 1$. Then

$Q_{X_0} \cong Q_{X_1} \Rightarrow X_0 \cong_{\text{homotopy}} X_1$.

Lemma (Hopf) $\pi_3(\underbrace{S^2 \vee \cdots \vee S^2}_{n\text{-copies}}) \cong \{n \times n$ symmetric matrices with integer coefficient$\}$.

Proof:

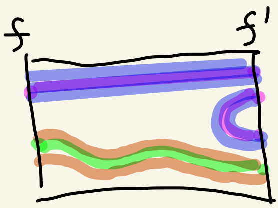
Given $f: S^3 \to S^2 \vee \cdots \vee S^2$, we form $C_f = (S^2 \vee \cdots \vee S^2) \cup_f D^4$

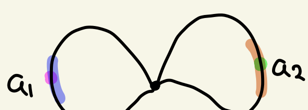
Then $H^4(C_f) = \mathbb{Z} \quad H^2(C_f) = \mathbb{Z}\langle \alpha_1, \cdots, \alpha_n \rangle$

Define $h(f) = [(\alpha_i \cup \alpha_j)]$, this is called the Hopf invariant

Suppose $h(f) = h(f')$, w.t.s f is homotopic to f'

$(\bigvee_n S^2) - \{a_i\}$ contractible





$\square$

Proof of Whitehead theorem:

Let $H_2(X_1) = \mathbb{Z}\langle \alpha_1, \cdots, \alpha_n \rangle$. Then Q_{X_1} is represented by the matrix

$A = [(\alpha_i \cup \alpha_j)[X_1]]$. Take $X_1' = X_1 - \mathring{D}^4$.

By Hurewicz theorem $\pi_2(X_1') = H_2(X_1') = H_2(X_1)$ so

α_i can be represented by map $f_i : S^2 \to X_1'$.

The map $f_1 \vee \cdots \vee f_n : \vee_n S^2 \to X_1'$ induces isomorphism on homology. By Whitehead's theorem, $f_1 \vee \cdots \vee f_n$ is a homotopy equiv.

So $X_1 = X_1' \cup_{S^3} D^4$ is homotopy equivalent to

$$C_f = (\vee_n S^2) \cup_f D^4 \quad \text{for some } f : S^3 \to \vee_n S^2$$

Similarly X_2 is homotopy equiv to $C_{f'}$ for some f'.

By definition of Hopf invariant. $h(f) = h(f') = A$. So

f is homotopic to f'.

So $X_1 \simeq C_f \simeq C_{f'} \simeq X_2$. $\square$

Conclusion: When $\pi_1(X) = 1$, Q_X completely determines the homotopy type of X.

Q: What about homeomorphism type?

Freedman: $Q_X + ks(X) \in \{0,1\} \overset{\text{determines}}{\rightsquigarrow}$ homeomorphism type.

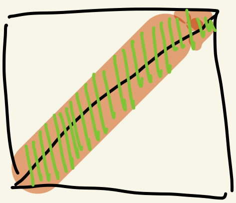
Kirby-Siebenman invariant

For $m \geq 1$, let $Top_m = \{$homeomorphism $f: \mathbb{R}^m \to \mathbb{R}^m$ s.t. $f(0)=0\}$

$PL_m = \{$piecewice linear homeo $f: \mathbb{R}^m \to \mathbb{R}^m$ s.t. $f(0)=0\}$

Theorem (Kirby-Siebenman) $m \geq 5$. $k < m$. Then $\pi_k(Top_m / PL_m) = \begin{cases} 0 & k \neq 3 \\ \mathbb{Z}/2 & k=3 \end{cases}$.

Now given a topological mfd X, we consider the diagonal map $X \xrightarrow{\Delta} X \times X$. A small neighborhood of $\Delta(X)$ is a fiber bundle over X with fiber $\mathbb{R}^n$. (This is how we define $T_* X$ for topological X)



We get a principal bundle $Top_m \hookrightarrow P_X \to X$

If we want to reduce the structure group to PL_m then by Kirby-Siebenman's theorem. The unique obstruction is a class $KS(X) \in H^4(X; \mathbb{Z}/2)$.

So X has a PL-structure $\Rightarrow KS(X) = 0$.

(when $\dim X > 4$, "$\Leftarrow$" is also true.)

when $\dim X = 4$, $KS(X) = 0 \Leftrightarrow X \times \mathbb{R}$ has a PL structure

(In $\dim \leq 7$, PL $\Leftrightarrow$ smooth)

Theorem (Freedman) (1) Let X_0, X_1 be two simply-connected topological 4-mfds. Then $X_0 \cong_{top} X_1 \Leftrightarrow Q_{X_0} \cong Q_{X_1}$ $KS(X_0) = KS(X_1)$

(2) Given any unimodular, symmetric bilinear form Q. Then

(i) Suppose Q is odd (i.e. $\exists v$ s.t. $Q(v,v)$ odd). Then Any combination of $(Q, \underset{\{0,1\}}{\underset{\frown}{i}})$ can be realized as $(Q_X, KS(X))$ for some simply-connected topological 4-mfd X.

(ii) Suppose Q is even. Then (Q, i) can be realized iff $i \equiv \frac{\sigma(Q)}{8} \mod 2$.

Cor: Let X_0, X_1 be two simply connected smooth 4-mfds. Then $Q_{X_0} \cong Q_{X_1} \Leftrightarrow X_0 \simeq_{top} X_1$

Example: $Q_X = [1] \Rightarrow X \cong_{top} \begin{cases} \mathbb{CP}^2 \\ \text{or} \\ \widetilde{\mathbb{CP}^2} \end{cases}$ ← fake projective plane a topological mfd with $KS = 1$

$$Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow X \cong_{top} S^2 \times S^2$$

There is a unique simply-connected topological mfd X with $Q_X = E8$. This is the famous Freedman's E8 manifold.

Q: Classify simply-connected smooth 4-mfd up to diffeomorphism.

Q1: Classify simply-connected smooth 4-mfds up to homeomorphism. (geography problem)

Q2: Classify all exotic smooth structures on a given smooth 4-mfd. (botany problem)

By Freedman's theorem, Q1 is equivalent to:

Q1': Given symmetric, unimodular bilinear form Q, when can we realize Q as Q_X for a smooth, s.c. X?

Algebraic classification of unimodular, symmetric bilinear forms.

Given Q, we say Q is even if $Q(v,v) \in 2\mathbb{Z}$ $\forall v$.

we say Q is positive definite if $Q(v,v) \geq 0$ $\forall v$.

--- negative -------- . --- ≤ 0 $\forall v$

Theorem (Hasse-Minkowski) Let $Q: \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ be a symmetric unimodular bilinear form Then

(1) If Q is definite, then for any m, there are only finitely manifold isomorphism class of Q.

(2) If Q is indefinite and odd. Then $Q \cong \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{bmatrix}$.

If Q is indefinit and even. Then $Q \cong mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $m \in \mathbb{Z}$, $n \geq 0$.

Q1': Given Q, when can we realize Q as Q_X for some smooth, simply connected 4-mfd X?

Definition: Let $SO(n) \hookrightarrow Fr \to X$ be the frame bundle of a smooth n-mfd. A spin structure is a lift of Fr to a $Spin(n)$ bundle. Here $Spin(n)$ is the nontrivial 2-fold cover of $SO(n)$.

Lemma: X has a spin structure $\Leftrightarrow$ $w_2(X)=0$. Furemore, when $\dim X=4$ and $\pi_1(X)=1$, X has spin str. $\Leftrightarrow$ Q_X is even. $\square$

Theorem (Rokhlin) Let X be a smooth spin 4-mfd. Then $16 \mid \sigma(X)$.
Various proof: Algebraic topology / index theory / Kirby calculas.

So for example $E8 \neq Q_X$ for any smooth s.c. X. But we can't rule out $2E_8$ by using this theorem.

Theorem (Donaldson) Let X be a smooth 4-mfd. Suppose Q_X is definite. Then $Q_X \cong \pm \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

So we see that although there are many crazy definite unimodular form, none of them arises as Q_X for smooth X.

Also note that $Q = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{pmatrix}$ is realized by $(\#^m \mathbb{CP}^2) \# (\#^n \overline{\mathbb{CP}^2})$

So we are left with the case Q is even and indefinite.

$Q = mE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By Rokhlin's theorem, m is even.

WLOG, assume $m = 2k$ $(k \geq 0)$.

Recall $Q_{K3} = 2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so if $n \geq 3k$. Then

$Q = Q_{(\#^k K3 \, \#^{n-3k}(S^2 \times S^2))}$.

<u>Conj (the $\frac{11}{8}$-conjecture)</u>: If X is spin, smooth and $Q_X = 2kE_8 \oplus n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $n \geq 3k$.

<u>Conj (the $\frac{11}{8}$-conjecture, refomulated)</u> If X is spin, smooth, then $b_2(X) \geq \frac{11}{8}|\sigma(X)|$.

The $\frac{11}{8}$-conjecture is the last open case of the geography problem of S.C. smooth 4-mfds.

- conjecture holds when $k \leq 1$ (i.e. $|\sigma(X)| \leq 16$) (Donaldson, Kronheimer)

Theorem (Furuta) If $\sigma(X) \neq 0$, then $n \geq 2k+1$.

(equivalently $b_2(X) \geq \frac{10}{8}|\sigma(X)| + 2$.)

4-mfd $X \xrightarrow{\text{Seiberg-Witten theory}}$ an equivariant map $(\mathbb{H}^k)^+ \to (\tilde{\mathbb{R}}^n)^+$

$\xrightarrow{\text{K-theory}}$ Such map exists only if $n \geq 2k+1$.

Theorem (Hopkins-L.-Shi-Xu) If $|\sigma(X)| > 16$, then

$$n \geq 2k + \begin{cases} 2 & k \equiv 1, 2, 5, 6 \\ 3 & \quad\;\; 3, 4, 7 \\ 4 & \quad\;\; 0 \end{cases} \mod 8.$$