Lecture No 16 April 29, 2022 (Fri)

SDEs with random initial value

- We discussed the stochastic differential equation (SDE):
$$dX_t = \alpha(t, X_t)\, dB_t + b(t, X_t)\, dt \qquad (1)$$
with initial value (starting point) $x \in \mathbb{R}^d$.
- We can consider random initial value: For $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued square-integrable r.v. ξ, consider
$$X_t = \xi + \int_0^t \alpha(s, X_s)\, dB_s + \int_0^t b(s, X_s)\, ds. \qquad (2')$$
- As before, we assume the continuity in (t, x) and (global) Lipschitz continuity in x of the coefficients α and b.
- Then, in the successive approximation in the proof of Theorem 14.1, by the same argument, one can show that $X^{(n)} \in \mathcal{L}^2_T$ and $X^{(n)}$ is $(\mathcal{F}^B_t \vee \sigma(\xi))$-adapted, and prove the existence and uniqueness of the solution of (2').
- We call a strong solution of the SDE (1) with random initial value ξ, if it is $(\mathcal{F}^B_t \vee \sigma(\xi))$-adapted.

- The solution X_t of the SDE (1) starting at x is a function of x so that one can write it as

$$X_t = X(t, x, \omega).$$

- Then, the solution X_t of the SDE (1) with $\mathcal{F}_0$-measurable random initial value ξ is given by

$$X_t = X(t, \xi(\omega), \omega).$$

  ⨀ Indeed, $X^{(n)} = X^{(n)}(t, x, \omega)$ in the successive approximation satisfies this relation. Thus, by taking the limit, one can prove this relation also for X_t. □

[Remark] Actually $X(t, x)$ is continuous in (t, x) a.s. (or it has such modification), see Kunita "Stochastic flows and SDEs", Theorem 4.2.5. This is shown based on Kolmogorov's regularization theorem for random fields stated in Lect-12. □

§15 Markov property of the solution of SDE

- Consider the SDE (1) with coefficients $\alpha = \alpha(x)$ and $b = b(x)$ independent of t:

$$dX_t = \alpha(X_t)\, dB_t + b(X_t)\, dt. \tag{1'}$$

- SDE (1') is called temporally homogeneous.
- We assume the (global) Lipschitz continuity of α and b:

$$\|\alpha(x) - \alpha(y)\| + |b(x) - b(y)| \le K|x - y|, \quad x, y \in \mathbb{R}^d.$$

- Then, for each starting point $x \in \mathbb{R}^d$ (or for each $\mathcal{F}_0$-measurable L^2-integrable initial value ξ), the SDE (1') has a unique strong solution $X = (X_t)$ on $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$.
- For each $x \in \mathbb{R}^d$, set P_x the distribution of X starting at x on the path space $W^d = C([0, \infty), \mathbb{R}^d)$.
  $\to$ See the next page for the uniqueness of P_x.

- Note that P_x is uniquely determined independently of the choice of Brownian motion or the probability space on which the Brownian motion is realized.

  ☺ This is shown by observing the distribution of $X^{(n)}$ in the successive approximation is uniquely determined. For example, letting $b = 0$ for simplicity, we have

$$X_t^{(4)} = x + \int_0^t \alpha\left(s, x + \int_0^s \alpha\left(s_1, x + \int_0^{s_1} \alpha(s_2, x) dB_{s_2}\right) dB_{s_1}\right) dB_s$$

  The distribution of RHS is uniquely determined independently of the choice of Brownian motion.
  We may approximate the integrands by step processes.

  One can also apply Yamada-Watanabe's theorem stated below. (Pathwise uniqueness implies the uniqueness in law.)

  □

Markov property is formulated as follows.

[Theorem 15.1] For ${}^\forall t_0 \geq 0$, ${}^\forall A \in \mathcal{B}(W^d)$, we have

$$P(X_{\cdot+t_0} \in A | \mathcal{F}_{t_0})(\omega) = P_{X_{t_0}(\omega)}(A), \quad P\text{-a.s.}\, \omega.$$

We consider the time shift $X_{\cdot+t_0}$ of X by t_0 under the conditional probability $P(\,\cdot\,|\mathcal{F}_{t_0})(\omega)$. □

[Remark] In the LHS, $P, \mathcal{F}_{t_0}, \omega \in \Omega$ are considered on the original probability space Ω, while, in the RHS, $P_{X_{t_0}(\omega)}$ means P_x with $x = X_{t_0}(\omega)$. Recall P_x is a measure on W^d. □

[Proof of Theorem 15.1] • Denote the solution X_t of (1') starting at x by $X_t = X(t, x, \omega)$.

• For fixed $t_0 \geq 0$, set

$$B_t^{t_0} := B_{t+t_0} - B_{t_0}, \quad t \geq 0.$$

Then, B^{t_0} is an $(\mathcal{F}_t^{t_0})$-Brownian motion, where $\mathcal{F}_t^{t_0} := \mathcal{F}_{t+t_0}$.

[Lemma 15.2] (1) Denote the solution of the SDE (1') taking B^{t_0} as the Brownian motion starting at x by $X^{t_0}(t, x) = X^{t_0}(t, x, \omega)$. Then, we have

$$X(t + t_0, x) = X^{t_0}(t, X(t_0, x)) \quad \text{a.s.}$$

(2) We have the independence: $\{X^{t_0}(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \perp\!\!\!\perp \mathcal{F}_{t_0}$.

□

[Remark] In (1), $X(t_0, x)$ is $\mathcal{F}_0^{t_0}$-measurable and L^2-integrable r.v. Therefore, noting that B^{t_0} is an $(\mathcal{F}_t^{t_0})$-Brownian motion, $X^{t_0}(t, X(t_0, x))$ is defined in the sense discussed at the end of §14 taking $\xi = X(t_0, x)$. □

[Proof of Lemma 15.2] • To show (1), observe for $X_t = X(t,x)$

$$\begin{aligned} X(t+t_0,x) &= x + \int_0^{t+t_0} \alpha(X(s,x))\,dB_s + \int_0^{t+t_0} b(X(s,x))\,ds \\ &= X(t_0,x) + \int_{t_0}^{t+t_0} \alpha(X(s,x))\,dB_s + \int_{t_0}^{t+t_0} b(X(s,x))\,ds \\ &= X(t_0,x) + \int_0^{t} \alpha(X(s+t_0,x))\,dB_s^{t_0} + \int_0^{t} b(X(s+t_0,x))\,ds. \end{aligned}$$

Note that $X(t_0,x)$ is $\mathcal{F}_0^{t_0}$-measurable. This implies that $X(\cdot + t_0, x)$ is a solution of the SDE (1') with initial value $X(t_0,x)$ and Brownian motion B^{t_0} and therefore, by uniqueness of the solution, we obtain

$$X(\cdot + t_0, x) = X^{t_0}(\cdot, X(t_0,x)) \text{ a.s.}$$

• To show (2), since $X^{t_0}(\cdot, x)$ is a strong solution of the SDE (1') with the Brownian motion B^{t_0}, it is $\sigma\{B_\cdot^{t_0}\}$-measurable. Therefore, it is $\perp\!\!\!\perp \mathcal{F}_{t_0}$. □

[We return to the proof of Theorem 15.1] • To show the conclusion, it is enough to prove

$$E[\Psi(X(\cdot + t_0, x))\Phi] = E[E_{X_{t_0}(\omega)}[\Psi(w(\cdot))]\Phi] \qquad (\star)$$

for $^\forall \Phi(\omega)$: $\mathcal{F}_{t_0}$-measurable bounded function on Ω and $^\forall \Psi \in C_b(W^d)$.

• In both sides, E is the expectation on Ω. In the RHS, $E_{X_{t_0}(\omega)}$ is that on W^d and $w(\cdot)$ means the canonical path of $w \in W^d$.

• Indeed, once $(\star)$ is shown, we have

$$E[\Psi(X(\cdot + t_0, x))|\mathcal{F}_{t_0}](\omega) = E_{X_{t_0}(\omega)}[\Psi(w(\cdot))], \quad P\text{-a.s.}$$

Theorem 15.1 follows by taking $\Psi = 1_A$ (by approximation).

• Recall that $X(\cdot + t_0, x) = X^{t_0}(\cdot, X_{t_0}(x))$.

• Thus, writing

$$Y = X_{t_0}(\omega),\ \mathcal{G} = \mathcal{F}_{t_0} \text{ and } Z(x, \omega) = \Psi(X^{t_0}(\cdot, x)),$$

the proof of $(\star)$ is reduced to show the following lemma.

• Note that $X^{t_0}(\cdot, x)$ is continuous in x a.s. by Remark in §14.

[Lemma 15.3] Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ a sub σ-field of $\mathcal{F}$ and Y a $\mathcal{G}$-measurable r.v. Let $Z(x, \omega)$ be $\mathcal{B}(\mathbb{R}^d) \times \mathcal{F}$-measurable bounded, continuous in x and $\{Z(x, \omega)\} \perp\!\!\!\perp \mathcal{G}$. Set $f(x) := E[Z(x, \cdot)]$. Then, we have

$$E[Z(Y)\Phi] = E[f(Y)\Phi]$$

for $^\forall\Phi$: $\mathcal{G}$-measurable bounded function. □

[Proof] [Step 1] If Z is a step function in x, i.e., if it has the form: $Z(x, \omega) = \sum_{i=1}^n 1_{A_i}(x)\tilde{Z}_i(\omega)$ with $A_i \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{Z}_i \perp\!\!\!\perp \mathcal{G}$. Then, $f(x) = \sum_{i=1}^n 1_{A_i}(x)E[\tilde{Z}_i]$. We also have

$$E[Z(Y)\Phi] = E\Big[\sum_{i=1}^n 1_{A_i}(Y)\Phi \cdot \tilde{Z}_i\Big] \underset{\perp\!\!\!\perp}{=} \sum_{i=1}^n E\big[1_{A_i}(Y)\Phi\big] E\big[\tilde{Z}_i\big]$$

$$= E\Big[\Big\{\sum_{i=1}^n 1_{A_i}(Y)E[\tilde{Z}_i]\Big\}\Phi\Big] = E[f(Y)\Phi].$$

Thus, the conclusion holds.

[Step 2] For general $Z(x,\omega)$, noting the continuity in x, we introduce an approximation by Z_n of the form of Step 1:

$$Z_n(x,\omega) := \sum_{k\in\frac{1}{n}\mathbb{Z}^d, |k|\le n} 1_{B(k,\frac{1}{n})}(x) Z(k,\omega),$$

where $B(k,\frac{1}{n}) = \prod_{i=1}^d [\frac{k_i}{n}, \frac{k_i+1}{n})(x)$ for $k=(k_i)_{i=1}^d$. Then, $Z_n(x,\omega)\to Z(x,\omega)$ as $n\to\infty$ for ${}^\forall x,\omega$. Thus, by Lebesgue's convergence theorem, we have as $n\to\infty$

$$E[Z_n(Y(\omega),\omega)\Phi] \to E[Z(Y(\omega),\omega)\Phi],$$
$$f_n(x) := E[Z_n(x,\omega)] \to E[Z(x,\omega)] = f(x), \quad {}^\forall x$$

and therefore, noting $|f_n|, |f| \le K$ (by the boundedness of Z), again by Lebesgue's convergence theorem,

$$E[f_n(Y(\omega))\Phi] \to E[f(Y(\omega))\Phi].$$

However, since Step 1 shows $E[Z_n(Y(\omega),\omega)\Phi] = E[f_n(Y(\omega))\Phi]$, we obtain the conclusion. □

§16 Martingale problem and weak solution of SDE

- For continuous coefficients

$$\alpha = \alpha(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times N}$$
$$b = b(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$$

  and N-dimensional $(\mathcal{F}_t)$-Brownian motion $B_t = (B_t^k)_{1 \le k \le N}$, we considered the SDE for X_t:

$$dX_t = \alpha(t, X_t)\, dB_t + b(t, X_t)\, dt. \tag{1}$$

- We also showed, additionally assuming the global Lipschitz continuity of α, b in x, the existence of a strong solution (i.e., $(\mathcal{F}_t^B)$-adapted solution) and uniqueness of the solution.

Weak solution

- We call $X = (X_t)_{t \geq 0}$ a weak solution of SDE (1), if one can construct X on a certain probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$ and $(\mathcal{F}_t)$-Brownian motion $B = (B_t)_{t \geq 0}$ such that X satisfies (1).
- Namely, we may choose Ω, B as we like.
- It is easy to see that

  "Strong solution $\Longrightarrow$ Solution $\Longrightarrow$ Weak solution".
- We say that the weak solution is unique, if the distribution of the solution X on its path space $(W^d, \mathcal{W}^d)$ is uniquely determined.
- Recall that $W^d = C([0, \infty), \mathbb{R}^d)$, which is equipped with the topology determined by the uniform convergence on each bounded interval of $[0, \infty)$ and the Borel field $\mathcal{W}^d$ determined by this topology, i.e., $\mathcal{W}^d = \mathcal{B}(W^d)$.

Differential operator $\mathcal{L}_t$

- For $\varphi = \varphi(x) \in C^2(\mathbb{R}^d)$, set

$$\mathcal{L}_t\varphi(x) := \frac{1}{2}a^{ij}(t,x)\frac{\partial^2\varphi}{\partial x^i \partial x^j}(x) + b^i(t,x)\frac{\partial\varphi}{\partial x^i}(x).$$

(We use Einstein's convention so $\sum_{i,j=1}^{d}, \sum_{i=1}^{d}$ are omitted.)

- Here,

$$a^{ij}(t,x) := \sum_{k=1}^{N} \alpha^i{}_k(t,x)\alpha^j{}_k(t,x)$$

i.e. $a \equiv (a^{ij})_{ij} = \alpha\alpha^*$ and a is a $d \times d$ non-negative definite symmetric matrix.

$\because$ For $^\forall\xi \in \mathbb{R}^d$, the corresponding quadratic form satisfies

$$(\xi, a\xi) = \sum_{i,j=1}^{d} a^{ij}\xi_i\xi_j = \sum_{ijk} \alpha^i{}_k\alpha^j{}_k\xi_i\xi_j = \sum_{k=1}^{N}\Big(\sum_{i=1}^{d}\alpha^i{}_k\xi_i\Big)^2 \geq 0. \quad \square$$

[Proposition 16.1] If $X = (X_t)_{t \geq 0}$ is a solution of the SDE (1), then for $^\forall \varphi = \varphi(t,x) \in C_b^{1,2}([0,\infty) \times \mathbb{R}^d)$

$$\varphi(t, X_t) - \int_0^t \left\{ \frac{\partial \varphi}{\partial s} + \mathcal{L}_s \varphi \right\}(s, X_s)\, ds$$

is an $(\mathcal{F}_t)$-martingale. Note that the differential operator $\mathcal{L}_s$ acts on the variable x. Recall that, as we mentioned before, we consider the solution satisfying $\alpha^i_k(t, X_t) \in \mathcal{L}^2_T$ for $^\forall T > 0$.

□

⚆ By Itô's formula, we have

$$\begin{aligned} d\varphi(t, X_t) =& \frac{\partial \varphi}{\partial t}(t, X_t) dt + \frac{\partial \varphi}{\partial x^i}(t, X_t)\, dX^i_t + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(t, X_t)\, dX^i_t dX^j_t \\ =& \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}_t \varphi \right\}(t, X_t)\, dt + \frac{\partial \varphi}{\partial x^i}(t, X_t)\, {\alpha^i}_k(t, X_t)\, dB^k_t . \end{aligned}$$

We used $dX^i_t = \alpha^i_k(t, X_t)\, dB^k_t + b^i(t, X_t)\, dt$ and $dX^i_t dX^j_t$ $= a^{ij}(t, X_t) dt$. However, since the term of the stochastic integral is $(\mathcal{F}_t)$-martingale, we obtain the conclusion. □

[Definition 16.1] Let $x \in \mathbb{R}^d$. We call a probability measure P on $(W^d, \mathcal{W}^d)$ a solution of the $\mathcal{L}_t$-martingale problem starting at x, if P satisfies the following two conditions:

(1) $P(w_0 = x) = 1$, and

(2) For $^\forall \varphi \in C_b^{1,2}([0,\infty) \times \mathbb{R}^d)$,

$$\varphi(t, w_t) - \int_0^t \left\{ \frac{\partial \varphi}{\partial s} + \mathcal{L}_s \varphi \right\}(s, w_s)\, ds$$

is an $(\mathcal{F}_t)$-martingale under the measure P,

where $w_t, t \geq 0$ is the value of $w \in W^d \equiv C([0,\infty), \mathbb{R}^d)$ at time t and $\mathcal{F}_t := \sigma\{w_s; s \leq t\}$. We call w_t the canonical coordinate function. □

By Proposition 16.1, if a weak solution $X = (X_t)$ of the SDE (1) starting at x exists, its distribution P_x on $(W^d, \mathcal{W}^d)$ is a solution of the $\mathcal{L}_t$-martingale problem.

Another consequence of Proposition 16.1.

[Corollary 16.2] Let $X = (X_t)_{t \geq 0}$ be a solution of the SDE (1). We denote the distribution of X_t on $\mathbb{R}^d$ by $\mu(t)\,(\in \mathcal{P}(\mathbb{R}^d))$. Then, for ${}^\forall \varphi = \varphi(x) \in C_b^2(\mathbb{R}^d)$, we have

$$\frac{d}{dt}\langle \mu(t), \varphi \rangle = \langle \mu(t), \mathcal{L}_t \varphi \rangle, \quad t \geq 0,$$

where $\langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x)\mu(dx)$. □

[Proof] By Proposition 16.1 and taking the expectation, we have for any $\varphi \in C_b^2(\mathbb{R}^d)$

$$E[\varphi(X_t)] - \int_0^t E[\mathcal{L}_s \varphi(X_s)]ds = E[\varphi(X_0)]\,(= \text{constant in } t).$$

Noting that $E[\varphi(X_t)] = \langle \mu(t), \varphi \rangle$, $E[\mathcal{L}_s \varphi(X_s)] = \langle \mu(s), \mathcal{L}_s \varphi \rangle$ and differentiating the above in t, we obtain the conclusion. Note that $\langle \mu(t), \mathcal{L}_t \varphi \rangle$ is continuous in t so that the identity holds for ${}^\forall t \geq 0$ (at $t = 0$, we take right-derivative). □

In particular, if $\mu(t)$ has a density $u(t,x)$ with respect to the Lebesgue measure dx so that $\mu(t,dx) = u(t,x)dx$, and if $u(t,x) \in C^{1,2}((0,\infty) \times \mathbb{R}^d)$, then $u(t,x)$ satisfies Kolmogorov's forward equation:

$$\frac{\partial u}{\partial t} = \mathcal{L}_t^* u,$$

where $\mathcal{L}_t^*$ is the adjoint of $\mathcal{L}_t$ given by

$$\mathcal{L}_t{}^* u(x) := \frac{1}{2} \frac{\partial^2}{\partial x^i \partial x^j}\Big(a^{ij}(t,x)u(x)\Big) - \frac{\partial}{\partial x^i}\Big(b^i(t,x)u(x)\Big).$$

We may apply the integration by parts formula.

The differential operator $\mathcal{L}_t$ or $\mathcal{L}$ (in temporally homogeneous case) are called the generator of $X = (X_t)_{t \geq 0}$.