2021-11-17 Kähler geometry

(6)

Lemma For $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ we put $\omega_\beta = \omega_0 + i\partial\bar\partial\varphi_\beta$.

If we define h_β by

$$\mathrm{Ric}(\omega_\beta) = \mu\,\omega_\beta + (1-\beta)2\pi[D] + \sqrt{-1}\partial\bar\partial h_\beta.$$

then

$$h_\beta = h_{\omega_0} - (1-\beta)\log\|S\|^2_{h_0^\lambda} - \log\frac{\omega_\beta^m}{\omega_0^m} - \mu\varphi_\beta$$

h_0 metric $K_M^{-1} \rightsquigarrow h_0^\lambda$ metric for $K_M^{-\lambda} = [D]$.

where $c_1(h_0) = \omega_0$, $\mathrm{Ric}(\omega_0) - \omega_0 = i\partial\bar\partial h_{\omega_0}$

$\because$

$i\partial\bar\partial(\mathrm{RHS})$

$$= \cancel{\mathrm{Ric}(\omega_0)} - \omega_0 + (1-\beta)\lambda\omega_0 - (1-\beta)[D].$$

$$+ \mathrm{Ric}(\omega_\beta) - \cancel{\mathrm{Ric}(\omega_0)} - \mu(\omega_\beta - \omega_0)$$

$$-1 + (1-\beta)\lambda + \mu = 0$$

$$= \mathrm{Ric}(\omega_\beta) - (1-\beta)[D] - \mu\,\omega_\beta$$

$$= \sqrt{-1}\partial\bar\partial h_\beta.$$

$\therefore$

Cor $\varphi_\beta \in \hat{\mathcal{H}}_\beta$ is a solution to $(*_\beta)$

$$\Longleftrightarrow \omega_{\varphi_\beta}^m = e^{-\mu\varphi_\beta + h_{\omega_0}} \frac{\omega_0^m}{\|S\|_h^{2(1-\beta)}} \qquad (*_\beta)$$

Test config $\mathcal{M} \longrightarrow \mathbb{C}$ can be extended to (7)

$(\mathcal{M}, \mathcal{D}) \longrightarrow \mathbb{C}$ naturally $\qquad M \supset D$

(M_0, D_0) the central fiber.

<u>Def-Lemma</u> (Chi Li). Write a cone metric as $\omega = \omega_0 + i\partial\bar\partial\varphi_\beta$.

Let X be a holomorphic vector field on M_0 such that $\exp(tX)$ leaves D_0 invariant

$$\left(\begin{array}{l} X \text{ is tangent to } D_0 \\ X = Y z_1 \frac{\partial}{\partial z_1} + \sum X^j \frac{\partial}{\partial z_j} \\ \quad (Y, X^j \text{ holomorphic}) \end{array} \right)$$

$$i(X)\,\omega = -\bar\partial u_X$$

$$f_\beta(X) = \int_{M_0} u_X\,(\mathrm{Ric}(\omega) - \omega) \wedge \omega^{m-1}$$

$$+ (1-\beta)\left(\lambda \int_{M_0} u_X\,\omega^m - 2\pi \int_{D_0} u_X\,\omega^{m-1}\right)$$

"log Futaki invariant"

$$-1 + (1-\beta)\lambda = -\mu$$

(1) $f_\beta(X)$ is indep of $\varphi_\beta \in \hat{\mathcal{H}}_\beta$.

(2) If $\exists$ K-E with cone angle $2\pi\beta$ then $f_\beta = 0$.

$(\mathcal{M}, \mathcal{D})$

log DF is defined as follows.

$$\frac{w_k}{k d_k} = F_0 + F_1 k^{-1} + \cdots \quad \longrightarrow \begin{cases} F_j(M_0) \\ F_j(D_0) \end{cases}$$

$$\log DF_\beta(\mathcal{M}, \mathcal{D}) =: f_\beta(M_0, D_0)$$

$$= -F_1(M_0) - (1-\beta)\frac{m\lambda}{2}\left(F_0(M_0) - F_0(D_0)\right)$$

(Chi Li).

Proof of closedness of E

Step 1 (Sun) For any test configuration

$f_0(M_0, D_0) \geq 0$.

[That is, for cone angle 0, always K-semi stable].

$\boxed{\text{Step 2}}$ $\beta_i \xrightarrow{\in E} \beta_\infty$. Suppose $\beta_\infty \notin E$.

- Gromov-Hausdorff limit

$$(M, \omega_{\beta_i}, D) \longrightarrow (M_\infty, D_\infty)$$

is in fact an algebraic limit.

$M_\infty \neq M$.

- $\exists$ test configuration $(\mathcal{X}, \mathcal{D}) \to \mathbb{C}$ with central fiber (M_∞, D_∞).

(Cheeger-Colding-Tian cone angle version)

- (M_∞, D_∞) has a weak solution of KE cone angle β_∞, so that

$$f_{\beta_\infty}(M_\infty, D_\infty) = 0.$$

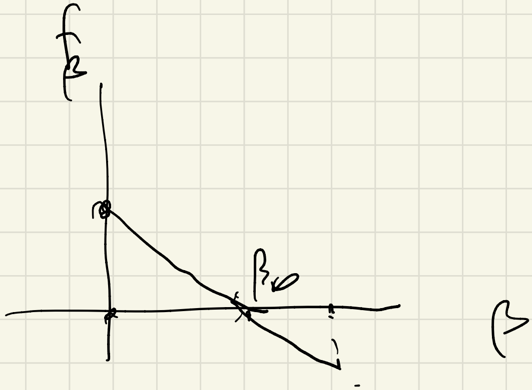
Step 3 As a function β

- $f_\beta(M_\infty, D_\infty) = f(M_\infty) + (1-\beta)\frac{m\lambda}{2}\left(F_0(M_\infty) - F_0(D_\infty)\right)$

  "linear function in β."

- $f_0(M_\infty, D_\infty) \geq 0$, $f_{\beta_\infty}(M_\infty, D_\infty) = 0$

  Step 1 (Sun)



$$\therefore\ f_1(M_\infty, D_\infty) \le 0.$$

$$\shortparallel$$

$$0 \le \quad \sim F_1(M_\infty)$$

$\uparrow$ K-polystability.

$$DF = -F_1(M_\infty) = 0$$

$M \ncong M_\infty$ not product configuration

$\Longrightarrow$ contradiction.

Def (S, D) contact mfd, $\dim S = 2n+1$

- D is a $2n$-dim distribution
- $\eta_D : TS \to TS/D$ projection
- $L_D : D \times D \to TS/D$

$$L_D(X,Y) = -\eta_D([X,Y])$$

is non-degenerate. (L_D is called the Levi form.)

D is called the contact structure.

Suppose TS/D is an oriented line bundle, and if τ is a positive section, $\eta_\tau = \tau^{-1}\eta_D$ is a contact form, i.e. $d\eta_\tau|_D$ is non-degenerate.

There is a unique vector field ξ called the Reeb vector field s.t. $i(\xi)\eta_\tau = 1$, $i(\xi)d\eta_\tau = 0$.

Thus $\eta_D(\xi) = \tau$.

Def $J \in \mathrm{End}(D)$, $J^2 = -id$,

(D, J) is called a CR structure

$\iff$ $D^{1,0} = \{X - iJX \mid X \in D\}$ $\quad X' = X - iJX$

sections of $D^{1,0}$ is closed under bracket

$$[X', Y'] \in D^{1,0}$$

Def (D, J) is a strictly pseudo-convex CR structure

$\iff$ $d\eta_\tau(\cdot, J\cdot)|_D$ is positive definite.

$\exists \tau$ positive section.

(Rem $f>0$ $(d\eta_{f\tau})(X, JY) = (d(f^{-1}\eta_\tau))(X, JY)$

$= f^{-1}\, d\eta_\tau(X, JY)$

So the strictly pseudo-convex does not depend on τ.)

<u>Def.</u> A vector field X on S is said to be a <u>contact vector field</u> if $L_X C^\infty(D) \subset C^\infty(D)$

A contact vector field X is said to be a <u>CR-vector field</u> if $L_X J = 0$.

<u>Def</u> If X is a CR vector field on a strictly pseudo-convex CR manifold s.t. $\tau(X)$ gives a positive section, then we call (S, D, J, X) a Sasakian (Sasaki) manifold.

Define a Riemannian metric g so that

$$g(X, X) = 1, \quad g(X, D) = 0.$$

$$g|_D = d\eta_\tau(\cdot, J\cdot), \quad \tau = \eta_D(X).$$

So local orbit spaces of the flow generated by X is Kähler.

Fact (S, g) Einstein $\Longleftrightarrow$ $d\eta_\tau$ is a K-E on local orbit space.

(Sasaki-Einstein).