BV classical master equations, examples and applications.

$T^*[1]X$, φ^i as coord on X, and φ_i^* as coordinates on the fiber parity(φ_i^*) + parity$(\varphi^i) = 1$ (mod 2)

BV classical master equation is $$\frac{\partial S}{\partial \varphi^i} \frac{\partial S}{\partial \varphi_i^*} = 0$$

General theorem:

S - solution to cl. master equ on $X \times Y$: $T^*[1]X \times T^*[1]Y$

$\mathcal{L}$ - Lagrangian submanifold in $T^*[1]X$

then $$\int_{\mathcal{L} \subset T^*[1]X} \mu\, e^{\frac{1}{\hbar} S} = e^{\frac{1}{\hbar}(S^{ind} + O(\hbar))}$$ terms

and $S^{ind} \in \mathrm{Fun}(T^*[1]Y)$ also solves classical master equation.

Very powerfull principle - many examples of it.

Simplest case $S(\varphi, \varphi^*) = F(\varphi)$ - clearly a solution

take a vector field $v^i(\varphi)$, such that $$v^i(\varphi) \frac{\partial}{\partial \varphi^i} F(\varphi) = 0$$

$$\{\varphi_i^* v^i(\varphi), F(\varphi)\}_{BV} = 0$$

$$\{G_1(\varphi,\varphi^*), G_2(\varphi,\varphi^*)\}_{BV} =$$
$$= \frac{\partial G_1}{\partial \varphi_i}\frac{\partial G_2}{\partial \varphi_i^*} + \frac{\partial G_2}{\partial \varphi_i}\frac{\partial G_1}{\partial \varphi_i^*}$$ for even G_1 and G_2

Consider formal odd parameter c ← odd

Invariance of function F w.r. to a vector field v:

$$S = F(\varphi) + c\,\varphi_i^* v^i(\varphi)$$ — solves cl. m. equation

Generalize: $S =$

$$F(\varphi) + c^a \varphi_i^* v^i_a(\varphi) + \frac{1}{2} f^a_{be} c^b c^e c^*_a$$

classical action

(Here $\{v_a, v_b\}_{Lie} = f^e_{ab} v_e$

(f^e_{ab} are structure constants of the Lie algebra)

Here we compute cl. m. eq. on the space

$$T^*[1]X \times T^*[1](\mathfrak{g}[1])$$

Here c^a are coordinates on the Lie algebra $\mathfrak{g}$ (shifted by 1 in parity)

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d) EZ case X is a point $v=0$

$$S_{CE} = \frac{1}{2} f^a_{be} c^b c^e c^*_a \leftrightarrow \text{Ch.E. differential}$$

(rediscovered later as BRST differential in physics)

$$\{S_{CE}, S_{CE}\} = \{fccc^*, fccc^*\} = \underbrace{f \cdot f ccc}_{\text{Jacobi identity}} c^* =$$

$$\underbrace{f^a_{be} f^e_{ad}}\, c^b c^c c^d c^*_e$$

Jacobi since c's are anticommuting

β) EZ': $\quad c^a V_a^i(\varphi)\,\varphi_i^* + \frac{1}{2} f^c_{ab}\, c^a c^b c^*_c$

F(φ)

cl. action of matter fields

$$\{c V(\varphi)\varphi^* + \tfrac{1}{2} fccc^*,\ cV(\varphi)\varphi^* + \tfrac{1}{2} fccc^*\}$$

Terms $\sim c^2$ $\quad c^a c^b \{V_a, V_b\}^i_{Lie}\, \varphi_i^* \pm f^c_{ab}\, c^a c^b V_c^i \varphi_i^*$

f - should be structure constant of the vector fields that produce a symmetry

$\boxed{\text{System with symmetry may be reformulated as a solution to cl. master equation.}}$

cl. master equ's generalize concept of systems with symmetry.

Briefly what BRST is from BV perspective

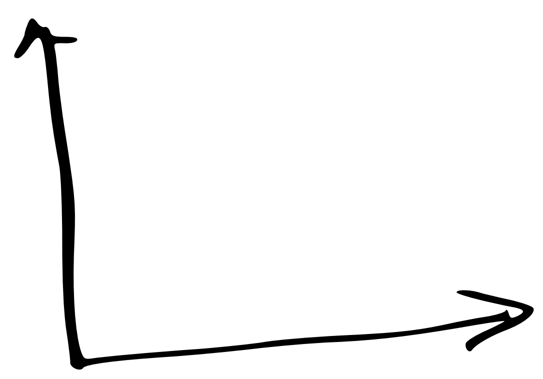
cl equ

Generalize: $S_{BV}' =$

$$F(\varphi) + c^a \varphi_i^* V_a^i(\varphi) + \frac{1}{2} f^a_{be}\, c^b c^e c^*_a$$

c

BRST : $S_{BV}\Big|_{\mathcal{L}_{Gauge}}$



$$\mathcal{E}_\alpha(\varphi) = 0$$

space $\mathcal{E}_\alpha(\varphi) = 0$

$\varphi^* = 0$ is no longer a Lagrange submanifold - it has dimension $< \frac{1}{2}$ dim of the space

of fields φ and φ^* –
– It is possible to construct conormal bundle

$$\varphi^*_i = b^\alpha \partial_i \mathcal{E}_\alpha(\varphi) \qquad \mathcal{E}_\alpha(\varphi) = 0$$

Claim – this is also a Lagrangian submanifold

$\delta\varphi^*_i \wedge \delta\varphi^i = \ldots \rightarrow$ we will compute it later.

$$F(\varphi) + c\, V\, \varphi^* + \frac{1}{2} c\, c\, c^* f$$

would be restricted to a submanifold $c^* = 0$

$$\underbrace{F(\varphi)}_{\text{Gauge fixed action}} + \underbrace{b^\alpha \mathcal{L}_{V_a} \mathcal{E}_\alpha\, c^a}_{b\text{-}c \text{ ghosts}} + \Big|_{\mathcal{E}_\alpha = 0}$$

Solutions to cl. master equation generalize the concept of symmetry in several ways.

$$F(\varphi) + c^a V^i_a \varphi^*_i + c^a c^b \pi^{ij}_{ab} \varphi^*_i \varphi^*_j + \frac{1}{2} f^e_{ab} c^a c^b c^*_e = S_{BV}$$

modifications of eqn's

$L_{V_a} F = 0$ (comes from terms linear in c)

$\{V_a, V_b\}^i_{Lie} = f^c_{ab} V^i_c + \pi^{ij}_{ab} \frac{\partial F}{\partial \varphi^j}$. (from terms quadratic in c)

It is called
<u>on-shell</u> action of a symmetry

We understand $F(\varphi)$ as action in classical field theory; $\frac{\partial F}{\partial \varphi^j} = 0$ are classical equations of motion

Algebra is represented moduli Ideal generated by cl. E. of motion.

Example from ordinary geometry:

X is $\mathbb{R}^N$, $\mathcal{Y} = SO(N)$

$$S_{BV} = \underbrace{(\varphi^1)^2 + \ldots + (\varphi^N)^2 +}_{F(\varphi)}, \quad V_a^i(\varphi) = T_{a\,j}^{\;i}\, \varphi^j$$

$$a = 1, \ldots, \frac{N(N-1)}{2}$$

I will consider a BV integral in the following setup:

$$X_1 = \mathbb{R} \qquad X_2 = \mathbb{R}^{N-1} \times \mathcal{Y}[1]$$

S_{BV} is a function on $T^*[1]\mathbb{R} \times T^*[1]\mathbb{R}^{N-1} \times T^*\mathcal{Y}[1]$

$$\cup$$

$\mathcal{L}$: φ^1 - arbitrary, $\varphi_1^* = 0$

After the integration I will get S^{ind} on $T^*[1]\mathbb{R}^{N-1} \times T^*\mathcal{Y}[1]$ that solves class. master equation.

The integral would be gaussian and could be easily taken:

$$S_{BV} = (\varphi^1)^2 + (\varphi^2)^2 + \ldots + (\varphi^N)^2 + \underbrace{C^a T_{a\,j}^{\;i}\, \varphi^j \varphi_i^*}_{\boxed{C^a T_{a\,1}^{\;i}\, \varphi^1 \varphi_i^*}} + \frac{1}{2} f_{ab}^{\;c}\, C^a C^b C_c^*$$

red terms $\to$

$$C^a T_{a\,1}^{\;i}\, \varphi_i^*\, C^b T_{b\,1}^{\;j}\, \varphi_j^*$$

This phenomena happens in SUSY that we studied ($N=1$ and $N=2$ theories)

Consider a dimensional reduction of $N=1$ theory to dimension 0, i.e. where all fields do not depend on time:

$$S = \underbrace{\int d^2\theta \, W(\hat{\Phi})}_{S_1} + F^2 \leftarrow \text{term}$$

Consider it from the BV point of view

There where 2 supersymmetries

Q and $\bar{Q}$

$$Q = \frac{\partial}{\partial \theta} + \bar{\theta}\frac{\partial}{\partial t}$$

$$\bar{Q} = \frac{\partial}{\partial \bar{\theta}} + \theta\frac{\partial}{\partial t}$$

Fields that do not depent on t

$$Q = \frac{\partial}{\partial \theta} \quad \bar{Q} = \frac{\partial}{\partial \bar{\theta}}$$

$$\hat{\Phi} = \Phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F$$

$$S_1 = W'(\Phi)\cdot F + W''(\Phi)\psi\bar{\psi}$$

$D\Phi =$
$= \psi +$
$+ \bar{\theta}F$

$\bar{D}\Phi =$
$= \bar{\psi} + \theta F$

we got F^2 term from

$$\int d^2\theta \, \bar{D}\hat{\Phi} \, D\hat{\Phi} =$$
$$= F^2$$

$$D = \frac{\partial}{\partial \theta}$$
$$\bar{D} = \frac{\partial}{\partial \bar{\theta}}$$

$$S_{matter} = F^2 + W'(\Phi)F + W''(\Phi)\psi\bar{\psi}$$

we may integrate out F field like we integrated out Φ^1 field in the previous example

And we will get susy realized on-shell

Q - supersymmetry $\bar{\Phi} \to \Psi \quad \bar{\Psi} \to F$

corresponding vector field is

$$\epsilon \Psi \Phi^* + \epsilon F \bar{\Psi}^*$$

ghost for supersym. transformation

$$(W'(\Phi) + \epsilon \bar{\Psi}^*)^2 : W'(\Phi)^2 + W''(\Phi)\Psi\bar{\Psi}$$

$$+ \epsilon \Psi \Phi^* + \epsilon W'(\Phi) \bar{\Psi}^* +$$