Lecture 14. Khovanov Homology of Virtual Knots

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Contents

Introduction

In this lecture, we will construct the Khovanov homology for virtual knots. The main difficulty is algebraic: for virtual knots which do not admit source-sink structure ("orientable atoms"), the differential complex defined "in a natural way", does not satisfy $\partial^2 = 0$. To overcome this difficulty, we introduce twisted coefficients.The results of this lecture are due to the author [6, 7].

Since the Khovanov homology theory for virtual knots appeared, it was natural to look for Lee–Rasmussen's theory.

Note that when we restrict ourselves to virtual knots with oriented atoms and a special sort of cobordism where all sections are virtual knots with oriented atoms, the results of the previous lecture can be extended verbatim. For general virtual knots, there are two generalisations of Lee–Rasmussen theory which give bounds for slice genus estimates. The theory due to Dye, Kaestner and Kauffman [20] is based on the result of this lecture.

The theory due to William Rushworth relies on another complex called double Khovanov complex, see [37].

Recollections

Recall that the Khovanov chain complex (for classical knots) is defined by the axioms:

$$[[\emptyset]] = (0 \to \mathbb{Z} \to 0)\,, \qquad [[\bigcirc \mathrm{K}]] = \mathrm{V} \otimes [[\mathrm{K}]],$$

$$[[\times]] = \mathcal{F}\left(0 \to [[)(]] \xrightarrow{\mathrm{d}} [[\asymp]]\{1\} \to 0\right).$$

Here V is a vector space of graded dimension $\mathrm{q} + \mathrm{q}^{-1}$, the operator $\{1\}$ is the operation of grading shift by 1, $\mathcal{F}$ is the flatten operation which sets a double complex to a single complex by taking direct sums along diagonals, and d is a differential. The Khovanov invariant is the homology of a renormalization of the Khovanov complex. The Khovanov invariant is indeed a link invariant and its graded Euler characteristic is the unnormalised Jones polynomial.

This passage from polynomials to (bi)graded complexes is also called categorification: Complexes form a category in which there are natural morphisms generated, for example, by cobordisms.

This theory has many generalisations and led to solutions of many problems in classical knot theory (for example, a simple proof of Milnor's conjecture about the Seifert genus of torus links).

Recollections

An important generalisation in the theory of extraordinary homology of links was the construction of categorification for a set of polynomials of type HOMFLY, made by Khovanov and Rozansky [29, 30]. Polynomials of type HOMFLY have more complicated relations and the problem of categorification for them was elegantly solved by means of instruments of matrix factorisations and Koszul complex. Khovanov and Rozansky [31] devoted their paper to the categorification of the so(N)-type Kauffman polynomial in which virtual knots are also used besides matrix factorisations. The Khovanov homology possesses important properties coming from algebraic topology: the (projective) functoriality. In the given case, the morphisms are cobordisms of knots. Thus, the Khovanov homology is extended to invariants of knot cobordisms representing two-dimensional surfaces with boundary in $\mathbb{R}^3 \times \mathrm{I}$. The projective functoriality (i.e. functoriality up to the overall minus sign) was first established by Jacobsson [24], see also [17, 19, 34].

Introduction

The functoriality allows one to construct invariants of cobordisms of two-dimensional surfaces in $\mathbb{R}^4$ from the Khovanov complex; a particular case of cobordisms is the cobordism between two links consisting of an empty set of components. In the case of projective functoriality, a cobordism invariant is defined up to an inverse element of the ground ring. In this case the Khovanov construction gives an invariant of two-dimensional knots, and two-dimensional surfaces embedded in $\mathbb{R}^3 \times \mathrm{I} \subset \mathbb{R}^4$. The accurate functoriality was established by Clark, Morrison, and Walker [19], see also [14]. For Lee theory, such a functoriality is described explicitly in the previous lectures. One of the most natural problems in the theory of virtual knots is the problem of generalisation of the Khovanov complex for virtual knots. An immediate attempt to generalise the theory leads to an algebraic difficulty: By writing down all necessary equations for the Khovanov complex to be invariant, we conclude that the main ring of coefficients should be the two-element ring. The indicated generalisation was done in [3]. Some difficulties of the immediate approach can be avoided by using geometrical constructions related to atoms.

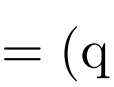
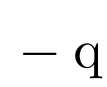
Introduction

The main goal of this lecture is the construction of a chain complex for a virtual diagram with the homology being invariant under the generalised Reidemeister moves.
Note that the Khovanov homology for knots in thickened surfaces and in bundles over surfaces S_g whose fiber is an interval (by using some additional gradings for curves in a given surface) was also constructed by Asaeda, Przytycki and Sikora [13], see also [12]. This homology does not lead to the Khovanov homology for virtual knots, since it depends on a concrete surface S_g and is not invariant under destabilisations and homeomorphisms of the surfaces onto itself.
A further development of the Khovanov homology theory for virtual knots representing a generalisation of the paper [13], and the results of this lecture, are given in [8, 9], see also [21, 12]. In these papers topological and combinatorial coefficients at terms in the Kauffman bracket polynomial are "lifted" to new gradings in the Khovanov homology.

Contents

In the sequel, we shall deal with bigraded complexes $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}^{i,j}$, where i is called the homological grading, and j is called the (quantum) grading. The differential in the complex does not change the quantum grading and increases the homological grading by one. As usual, we make the substitution $a = \sqrt{(-q^{-1})}$ in the Kauffman bracket. Then, instead of the Jones polynomial we shall get its modified version J. Let us consider the polynomial $\hat{J} = J \cdot (q + q^{-1})$. More precisely, $\hat{J}$ is defined as follows. Let K be an oriented virtual diagram, and let $|K|$ be the corresponding unoriented virtual diagram obtained from K by forgetting the orientation, let n_+ and n_- be the numbers of positive and negative classical crossings of K, and $n = n_+ + n_-$ be the total number of crossings. We set:

$$\hat{J}(K) = (-1)^{n_-} q^{n_+ - 2n_-} [K],$$

where $[K]$ is the modified Kauffman bracket defined according to the rule $[\bigcirc] = (q + q^{-1})$, $[K \sqcup \bigcirc] = (q + q^{-1}) \cdot [K]$, $[$  $] = [$  $] - q[$  $]$.

The polynomial $\hat{J}$ has the following conceptually important description in terms of the state cube. Taking away the normalising factor $(-1)^{n_-}q^{n_+-2n_-}$, we get a (slightly modified) Kauffman bracket $\sum_s(-q)^{\beta(s)}(q+q^{-1})^{\gamma(s)}$. This means that we take the sum over all vertices of the cube, of the following products $(-q)^h \times (q+q^{-1})^{\#\bigcirc}$, where h is the height of the vertex, and $\#\bigcirc$ is the number of circles in the state corresponding to the given vertex of the cube.
Thus, in order to compute the polynomial, one has to associate with every circle the Laurent polynomial $(q+q^{-1})$, and then multiply these polynomials taken with some coefficients of the form $\pm q^k$, and take the sum of the obtained polynomials over all vertices of the cube.
Consequently, the Jones polynomial can be restored from the information about the number of circles in each of the Kauffman states. If we also take into account how these circles interfere when passing from one state to another, we would be able to construct the Khovanov complex.

Contents

Khovanov homology with Z_2-coefficients

From grading reasongings, it is easy to guess what the partial differentials can be in the case of $V \to V \otimes V$ and $V \otimes V \to V$.
They are just the same as in the case of classical Khovanov homology.
Moreover, it follows from the grading argument that the partial differential in the case $V \to V$ can be nothing by zero.
When considering all possible faces (atoms with 2 vertices) we will see one new relation: $m \circ \Delta = 0$ meaning that the composition of a comlutiplication followed by a multiplication should give zero. The standard Khovanov homology construction gives us $X \to X \otimes X \to 0$ which is fine but $1 \to 1 \otimes X + X \otimes 1 = 2X$ which is zero if we deal over the field of characteristic two.
This is why the Khovanov homology for virtual knots is easy to construct in the case of $\mathbb{Z}_2$-coefficients.

Khovanov homology with Z_2-coefficients

Let K be an oriented diagram of a virtual link with n classical crossings.
Consider the bifurcation cube of K. As usual, with each circle in each state of the cube we associate the linear space V over the field $\mathbb{Z}_2$ generated by two vectors v_+(also 1) and v_-(also X) having grading ± 1, resp. Thus, $\mathrm{qdim}\, V = (q + q^{-1})$. For each vertex $s = \{a_1, \ldots, a_n\}$ of the cube, we have a certain number of circles to be denoted by $\gamma(s)$. With such a vertex, we associate the vector space $V^{\otimes \gamma(s)}\{\sum_{i=1}^{n} a_i\}$ obtained from the tensor power of the space V by a grading shift.

Remark 3.1

In the sequel, we shall use the same notation V for the two-dimensional free module generated by the elements 1, X of grading ± 1 considered over an arbitrary ring of coefficients.

Remark 3.2

In this lecture, we consider the symmetric tensor product for which for elements $x_i \in V_i$, $i = 1, \ldots, n$, the following equality $x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} = x_1 \otimes \cdots \otimes x_n$ holds for any arbitrary permutation σ. We shall also call this product unordered. In Section 7, we shall consider the tensor product where the sign is the sign of the permutation when identifying products in different orders (this is also called signed tensor product).

We have defined the chain groups of our graded complex. This yields that whatever differentials we take for this complex (provided that $\partial^2 = 0$), the Euler characteristic of this complex will not depend on them. Namely, $\chi(\mathrm{Kh}(K)) = \hat{J}(K)$, where $\mathrm{Kh}(K)$ denote the bigraded homology of the complex we are going to construct.

As in the previous lectures, define the partial differentials between the chain groups, acting along the edges of the cube according to the edge directions; i.e. from a smoothing of type A to a smoothing of type B, in the following way. Let an edge of the bifurcation cube correspond to a passage from a state s to a state s' in such a way that l circles are not incident to the crossing in question. These circles do not change when passing from s to s'. At the crossing of $|K|$, corresponding to the edge either one circle splits into two circles or two circles merge into one. In the first two cases, we shall define the partial differential as it was defined in the case of classical knots [16], namely, on an edge increasing the number of circles we set $\Delta \otimes \mathrm{Id}^{\otimes l}\{1\}$ and on an edge decreasing the number of circles we set $m \otimes \mathrm{Id}^{\otimes l}\{1\}$.

Here the identical mapping Id is referred to the circles which are not incident to the crossing in question, and the maps $m\colon V \otimes V \to V$ and $\Delta\colon V \to V \otimes V$ are defined by formulas (1) and (2). The map m:

$$\begin{cases} 1 \otimes X \mapsto X, 1 \otimes 1 \mapsto 1, \\ X \otimes 1 \mapsto X, X \otimes X \mapsto 0 \end{cases} \tag{1}$$

The map Δ :

$$\begin{cases} 1 \mapsto 1 \otimes X + X \otimes 1 \\ X \mapsto X \otimes X. \end{cases} \tag{2}$$

For those chains corresponding to the fixed vertex of the cube, the differential ∂ is a sum of all partial differentials (each to be denoted by ∂', possibly, with an index indicating to the edge along which the partial differential acts) along all edges emanating from the given vertex of the cube (oriented in a way increasing the sum of coordinates).

In the general case, the main problem is to define the differential of type $(1 \to 1)$ in a way compatible with differentials of types $(1 \to 2)$ and $(2 \to 1)$ to make the cube anticommutative. For coefficients from $\mathbb{Z}_2$ this difficulty is easy to overcome.

Namely, in the case of bifurcation of type $(1 \to 1)$ we define the partial differential on the edge as the map taking the whole space to zero. Thus, we get the bifurcation cube, where in comparison with the state cube we additionally indicate how the partial differentials ∂' act. Denote the obtained set of the bigraded groups (the cube) by $[[K]]$. In order for the differential to be well defined, the cube has to be anticommutative, i.e. for every two-dimensional face of the cube, the composition of the maps corresponding to one pair of consecutive edges is equal to minus the composition of the maps corresponding to the other pair of consecutive edges connecting the same pair of points. Note that in this case (for the field $\mathbb{Z}_2$) the anticommutativity and commutativity are the same.
Let us define the differential ∂ as the sum of all differentials ∂'.

Lemma 3.3

The cube $[[K]]$ defined above is commutative.

This statement is verified by a routine check analogous to that from Lecture 12. It is left to as an exercise. Namely, we check the anticommutativity for every face of the cube.
Here we give an example; see Fig. 1.

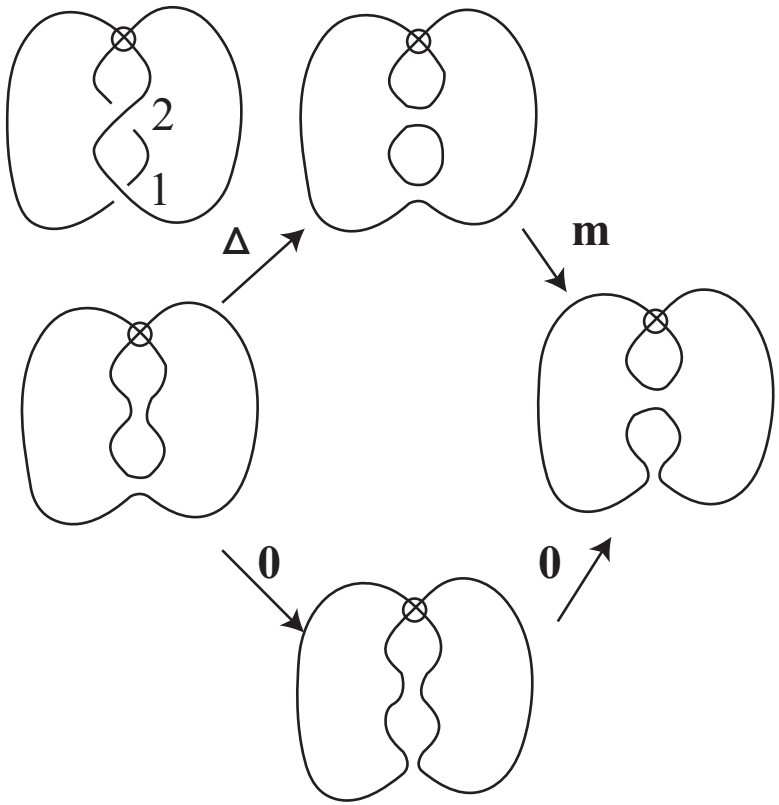


Figure 1: The commutativity check for a 2-face of the cube.

Later on, we shall see (Sections Khovanov homology of double knots and Khovanov complex for virtual knots) that every 2-face of the cube generates a certain atom.
In the present case (Fig. 1), it is necessary to check that the map $m \circ \Delta \colon V \to V$ takes the whole space V to zero. Indeed, for such a map we have: $X \mapsto X \otimes X \mapsto 0$, $1 \mapsto 1 \otimes X + X \otimes 1 \mapsto 2X = 0$ over $\mathbb{Z}_2$.
Note that this case is the only essential "non-classical" case where a bifurcation of type $1 \to 1$ takes place. Indeed, from the parity arguments it follows that on every 2-face of the cube the number of $1 \to 1$-bifurcations is either equal to zero or it is at least two. For more details see Section Khovanov complex for virtual knots. If there are no such bifurcations, then the problem is reduced to one of the classical cases (all such cases were considered in Lecture 12).
The most difficult part of our work is to construct a well-defined complex.
The proof of the invariance of its homology is standard; it is based on the fact that the map m is surjective and Δ is injective.

If we consider the case when there are two or four such bifurcations, then in the 2-face of the cube in question,

$$\begin{array}{ccc} V^{\otimes a}\{1\} & \xrightarrow{s} & V^{\otimes b}\{2\} \\ r\uparrow & & \uparrow t \\ V^{\otimes c} & \xrightarrow{p} & V^{\otimes d}\{1\}, \end{array}$$

either each of the compositions $t \circ p$ and $s \circ r$ contains a zero map corresponding to $1 \to 1$-bifurcation (for example, in the case $a = b$, $c = d$ the maps p and s are both zero) or the above case takes place.
We set (cf. [26]), $\mathcal{C}(K) = [[K]]\{n_+ - 2n_-\}[-n_-]$. In this case $\mathcal{C}(K)$ is a well-defined chain complex. Denote the homology groups of the complex $\mathcal{C}(K)$ by $Kh(K)$ (or by $Kh_{\mathbb{Z}_2}(K)$ in the case when we have to emphasise that the Khovanov complex is considered over the field $\mathbb{Z}_2$).

Theorem 3.4 ([3, 4])

The graded homology $Kh(K)$ is an invariant of the link K; the graded Euler characteristic $\chi(Kh(K))$ is equal to the Jones polynomial.

The second statement of the theorem follows from the fact that the Euler characteristic defined as the alternating sum of (graded) dimensions of homology groups is equal to the alternating sum of the graded dimensions of chain spaces.
The proof for the homology to be invariant under the Reidemeister moves just repeats the proof for the case of classical links (see the main theorem in Lecture 12).

Definition 3.5

Recall that the height h(Kh(K)) of the Khovanov homology of a virtual link K is the difference between the leading and lowest non-zero quantum gradings of non-zero Khovanov homology of K.

By construction it is clear that

$$h(Kh(K)) - 2 \geqslant \frac{\text{span}\langle K\rangle}{2}.$$

Note that the complex $\mathcal{C}(K)$ splits into the direct sum of two complexes: the complex with an even grading and the complex with an odd grading (recall that the differential preserves the grading). We get two types of the Khovanov homology: the even one Kh^e and the odd one Kh^o.
They correspond to monomials of the Jones polynomial, having degrees congruent to two modulo four (Kh^o), and monomials the degrees of which are divisible (congr$\equiv 0 \mod 4$) by four (Kh^e). A classical (or even virtual; i.e. virtual link having a diagram with orientable atom) link has only one of these two types, more precisely, the following theorem holds.

Theorem 3.6

For a classical (and even virtual) link with even number of components the isomorphism $\mathrm{Kh}^{\mathrm{o}} \cong 0$ holds. For a classical link with odd number of components the isomorphism $\mathrm{Kh}^{\mathrm{e}} \cong 0$ holds.

This theorem is completely analogous to Theorem 7.4 in [10] about degrees of monomials occurring in the Jones polynomial.
Moreover, it is easy to check that this theorem is true not only for classical links but also for virtual links having a diagram with orientable atom.

Example

Example 3.7

Let us consider the diagram K depicted in Fig. 2 (left).

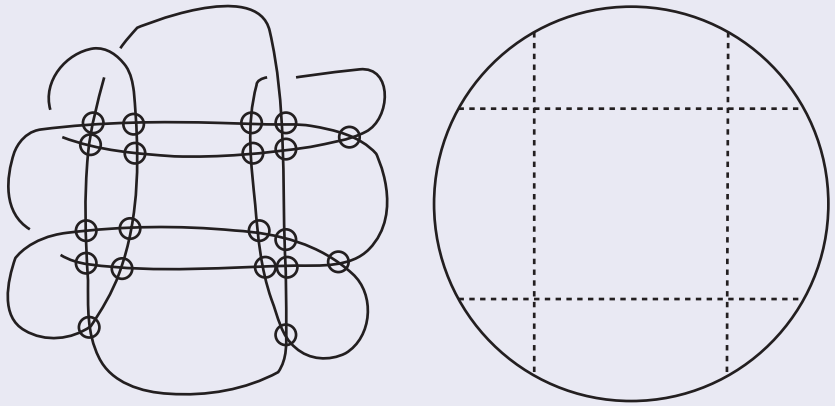


Figure 2: A virtual knot with orientable atom with genus 2.

The chord diagram corresponding to the leading state of the Kauffman bracket polynomial is depicted in the picture on the right. In this state there exists one circle, and in any of four crossings this circle can be transformed into one circle by using the corresponding dashed chord (with framing 1).

Example

We assert that this link has no diagrams with orientable atoms. Indeed, for the given diagram both complexes Kh^{o} and Kh^{e} (with coefficients in $\mathbb{Z}_2$) have non-trivial homology. Actually, the A-state of the diagram with one circle with a label 1 gives a non-trivial cycle (since all differentials coming from the A-state to neighboring states are zero). Further, in states where one crossing is B-smoothed and the other three crossings are A-smoothed there exists exactly one circle. Let us consider the chain equal to the sum of chains having label 1 at each of these four states. It is easy to check that this chain is a cycle. Further, it cannot be a boundary, since all chains in the A-state are cycles.

Thus, there are two homology groups, whose quantum gradings differ by 1, therefore, the link has no diagram with orientable atoms.

In particular, we have shown that the atom genus (the Turaev genus) of the link (see Chapter 16 in [10]) is equal to one.

Example

Note that this fact cannot be revealed by using the Kauffman bracket polynomial. Indeed, in the A-state (as well as in the B-state) there exists exactly one circle, at each state with one (or three) crossing A-smoothed we have one circle, and if we have two A-smoothed crossings and two B-smoothed crossings, then in two cases we shall have one circle and in the remaining four cases we shall have two circles. Therefore, the Kauffman bracket polynomial of K looks like:

$$\langle \mathrm{K} \rangle = \mathrm{a}^4 + 4\mathrm{a}^2 + 2 + 4(-\mathrm{a}^2 - \mathrm{a}^{-2}) + 4\mathrm{a}^{-2} + \mathrm{a}^{-4} = \mathrm{a}^4 + 2 + \mathrm{a}^{-4}.$$

All terms of this Kauffman bracket polynomial have degrees congruent to each other modulo four. Therefore, in the given case the Khovanov homology is more sensitive to non-orientability of atoms than the Kauffman bracket polynomial.

Note that the constructed Khovanov complex with coefficients in $\mathbb{Z}_2$ is completely defined by the structure of the bifurcation cube and the numbers n_+, n_-. Therefore, the Khovanov $\mathbb{Z}_2$-homology does not change under the virtualisation of the given link.
In the next section, we shall give another approach to the construction of the Khovanov complex (for framed links) which is sensitive to the virtualisation. The Khovanov complex given here coincides with the general Khovanov complex with coefficients in $\mathbb{Z}_2$ in the classical case; in this case it is easy to overcome the difficulty with bifurcations of type $1 \to 1$. Later on, we shall construct the Khovanov complex for not all diagrams of virtual links but only for "right" virtual diagrams, which have no partial differentials of type $1 \to 1$ on the cube. As we shall see later, "right" virtual diagrams are those diagrams which orientable atoms correspond to. Then we shall construct a "right" virtual diagram for each virtual diagram by some invariant way and see how the Khovanov homology of the corresponding "right" virtual diagram changes under the generalised Reidemeister moves applied to the initial diagram (not necessarily "right"). In the next section we shall construct the Khovanov complex for framed links where the double diagram plays the role of a "right" diagram.

Example

Example 3.8

Let us take the virtual knot diagram considered in the example on Page 54 of Lecture 13. (See Fig. 3.) This knot can be reduced to the unknot with virtualisations and generalised Reidemeister moves. Thus, the Khovanov $\mathbb{Z}_2$-homology of the knot depicted in Fig. 3 coincides with the Khovanov $\mathbb{Z}_2$-homology of the unknot.

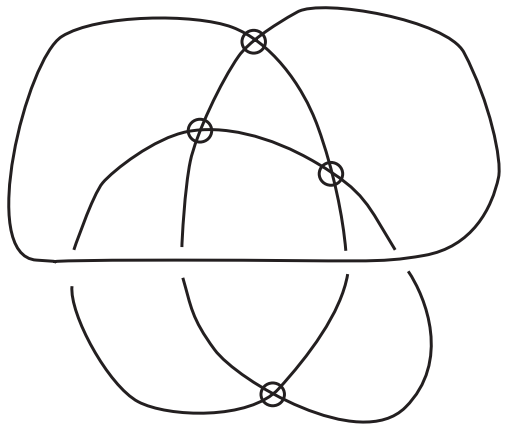


Figure 3: A virtual knot reduced to the unknot by the virtualisation and the generalised Reidemeister moves.

Contents

In the next three sections, we shall use the construction connecting atoms with virtual knots. Recall this construction given in Chapter 16 of [10] which assigns to a height atom a classical link. This construction is as follows. We embed the frame of an atom in the plane with its A-structure preserved, and each crossing is equipped with the over/undercrossing structure according to the B-structure of the atom.
Let an arbitrary atom be given. Let us immerse its frame in the plane with the A-structure preserved, construct a virtual diagram K from the atom in the way given in Chapter 16 of [10].
The equivalence class of K is well defined up to virtualisations.
Here we construct a map: (all links) —> (links with orientable atoms). For the latter, we have a well-defined Khovanov homology over $\mathbb{Z}$. Hence, we get an invariant of the former links.

Let K be a virtual diagram with an orientable atom.
Define the complex $\mathcal{C}(\mathrm{K})$ as follows. Fix a ring R of coefficients and two-dimensional free module V over this ring such that $\mathrm{qdim}\, \mathrm{V} = \mathrm{q} + \mathrm{q}^{-1}$.
The chain space of our complex is the same as in the case of coefficients from $\mathbb{Z}_2$. After that a differential is defined as the sum of partial differentials with signs, and partial differentials are defined with the maps m and Δ.
In the case of coefficients from the field $\mathbb{Z}_2$ the commutativity of each face is equivalent to its anticommutativity. In the case of coefficients from $\mathbb{Z}$ one can make an anticommutative cube from a commutative cube in the following way.
As in our last lecture, assign to all edges of the cube $\{0,1\}^{\mathrm{n}}$ sequences consisting of elements from $\{0, 1, *\}$ and having length n and one element $*$. Each such edge connects two vertices obtained by replacing $*$ by one and zero.
Thus, if we denote the map corresponding to an edge ξ by ∂'_{ξ}, then the differential looks like:

$$\partial^{\mathrm{r}} = \sum_{\{|\xi|=\mathrm{r}\}} (-1)^{\xi} \partial'_{\xi}.$$

Now we have to explain what the sign $(-1)^{\xi}$ means and define the map ∂_{ξ}. To well define the operator ∂ such that the property $\partial \circ \partial = 0$ holds, it is sufficient to show that partial differentials ∂'_{ξ} on two-dimensional faces of the cube are anticommutative diagrams.

Now, we slightly reformulate the conventions from Lecture 12 and 13. A commutative cube can be transformed to an anticommutative cube as follows. First, we have to construct maps on edges such that each two-dimensional face is a commutative diagram, and then we shall equip partial differentials ∂'_{ξ} with signs. A sign is defined by the following rule. Vertices of the cube are ordered (the homology will not depend on an order). To each vertex of the cube we assign the numbers of all its unit coordinates in the increasing order: $\mathrm{i}_1, \mathrm{i}_2, \ldots, \mathrm{i}_k$ and the formal exterior product $\mathrm{x}_{\mathrm{i}_1} \wedge \mathrm{x}_{\mathrm{i}_2} \wedge \cdots \wedge \mathrm{x}_{\mathrm{i}_k}$. For example, for $n = 3$ we assign to the vertex $\{1, 0, 1\}$ the exterior product $x_1 \wedge x_3$.

Each edge of the cube, increasing some jth coordinate, can be treated as the exterior multiplication on the right by x_j. If as a result of application of this exterior multiplication to a "lower" vertex we get an exterior product assigned to an "upper" vertex, we put the sign "plus" on the edge, and the sign "minus" otherwise. For example, for the edge $\{1, *, 1\}$ we have the sign minus since
$(x_1 \wedge x_3) \wedge x_2 = -x_1 \wedge x_2 \wedge x_3$.
Thus, we got a collection of chain groups [[K]] with the differential ∂.

Exercise 4.1

The frame of an atom admits a source–sink structure if and only if the atom is orientable.

From Exercise 4.1, it follows that if the atom corresponding to a virtual diagram is orientable, then there is no bifurcation of type $1 \to 1$ in the bifurcation cube corresponding to the diagram. Indeed, let us consider the frame Γ of the corresponding atom. Each state of the diagram is an atom having the frame Γ. Circles of the state serve for pasting black cells to the frame Γ. According to Exercise 4.1, the new atom is also orientable. Therefore, this atom cannot have a black cell approaching to itself in the non-orientable way (the way the smoothing at the crossing where this cell touches itself, does not change the number of circles).
Thus, bifurcation cubes are well defined for virtual diagrams with orientable atoms, namely, all bifurcations have the following types $1 \to 2$ and $2 \to 1$; partial differentials are defined by the maps m and Δ; the differential is defined as the sum of partial differentials with signs, and the statement that $\partial^2 = 0$ is checked analogously to the classical case.
Note the following two important lemmas.

Lemma 4.2 and its proof

Lemma 4.2

Let K be a virtual diagram with an orientable atom. Then the collection of the groups [[K]] together with the differential ∂ gives a complex; i.e. $\partial^2 = 0$.

We have to check that each two-dimensional face of the cube [[K]] is anticommutative. This is equivalent to the verification of the commutativity of two-dimensional faces before putting the signs ± 1.
Each two-dimensional face of the cube [[K]] represents the atom with two vertices. Each two-dimensional face of the cube corresponds to a smoothing of some $(n-2)$ classical crossings of the diagram K; see Fig. 12. The remaining two crossings can be smoothed arbitrarily; four possibilities of such a smoothing correspond to vertices of the two-dimensional face.
In these four states there are some number of common circles not being incident to the two crossings under consideration. After deleting these circles, we get an atom with two vertices.
Thus, we have to check that each two-dimensional face which can correspond to some atom with two vertices represents an anticommutative diagram.

Since the atom corresponding to K is orientable, then the atom corresponding to any two-dimensional face of the corresponding complex is also orientable.
Let us now use the theorem from [1] which tells us that all orientable atoms with two vertices are height atoms.
This means that each atom corresponding to a two-dimensional face of the bifurcation cube corresponding to an orientable atom occurs in the classical case. All such two-dimensional faces are sorted out in [15] and for them the commutativity of the corresponding diagrams is proved (before placing signs in differentials).
After that the proof follows line-by-line the proof in the classical case (see, e.g. [15]) and from the verification of properties of the maps m and Δ.
Thus, we have shown that the collection of chains [[K]] with the differential ∂ represents an anticommutative cube. Therefore, the complex $\mathcal{C}(\mathrm{K})$ is well defined.
Denote the homology of the complex by Kh(K).

Lemma 4.3 and its proof

Lemma 4.3

Let K, K′ be two virtual diagrams with orientable atoms, herewith K′ differs from K by applying a detour move or one of the three classical Reidemeister moves. Then there exists an isomorphism of the Khovanov homology $\mathrm{Kh}(K) \cong \mathrm{Kh}(K')$.

Proof.

By applying the detour move, the structure of classical crossings does not change. Thus, the state cube does not change either, and, therefore, the complex does not change.
In the case of the classical Reidemeister moves we use the same proof based on the cancellation principle which was earlier used for the Khovanov homology of classical links. It is local; i.e. it uses only the local structure of Reidemeister moves (not depending on the fixed part of the link under the move). Therefore, the proof passes verbatim for virtual knots under the condition that all complexes are well defined. □

Exercise 4.4

Let K be a diagram of a virtual link. Then the atom corresponding to the double diagram $D_2(K)$ is orientable.

Taking into account Exercise 4.4 and Lemma 4.2 we conclude that the Khovanov complex for cables $D_{2n}(K)$ is well defined for any ring of coefficients. The map $K \mapsto D_{2n}(K)$ is almost invariant (it is invariant under all combinations of Reidemeister moves which do not change the writhe number). Therefore, it is natural to expect that the homology of the Khovanov complex for double diagrams of a knot is an invariant of framed links. Namely, the following statement is true.

Exercise 4.5

Let K, K′ be two diagrams of equivalent framed virtual links. Then there exists a collection of diagrams
$D_2(K) = K_0, K_1, \ldots, K_n = D_2(K')$ such that:

(1) all atoms corresponding to the diagrams K_i are orientable;

(2) for each $i = 0, \ldots, n-1$ the diagram K_{i+1} is obtained from the diagram K_i by applying one of the generalised Reidemeister moves.

Note that it suffices to consider only classical Reidemeister moves, since the detour move does not change an atom.
So, let us consider all classical Reidemeister moves.
If diagrams K and K′ differ by applying the first or third Reidemeister move, then the local source–sink structure for the diagram K is in one-to-one correspondence with the local source–sink structure for K′ such that outside the domain of the application of the move these diagrams coincide. Here the source–sink structure of lines depicted by dashed lines is defined as opposite to “thick” lines joining to them; see Fig. 4.

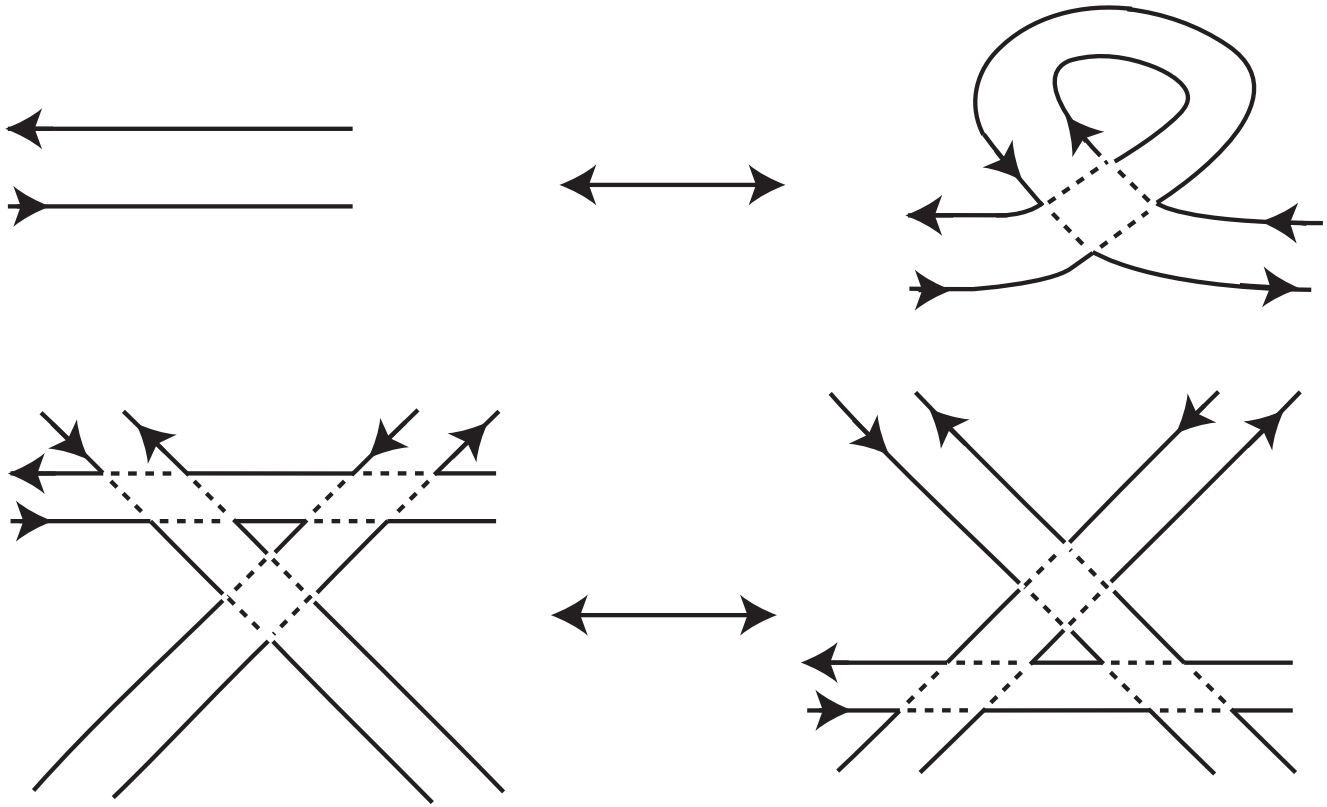
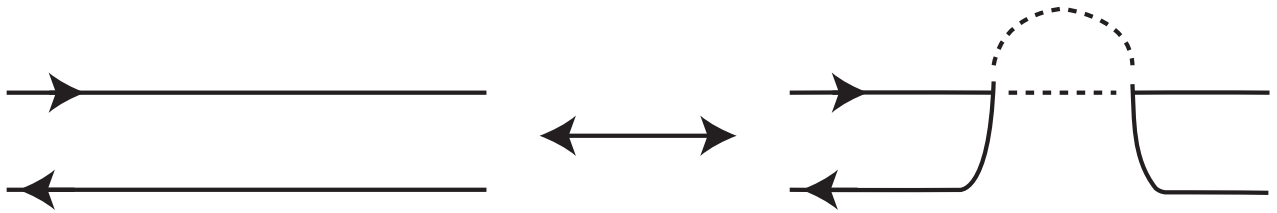
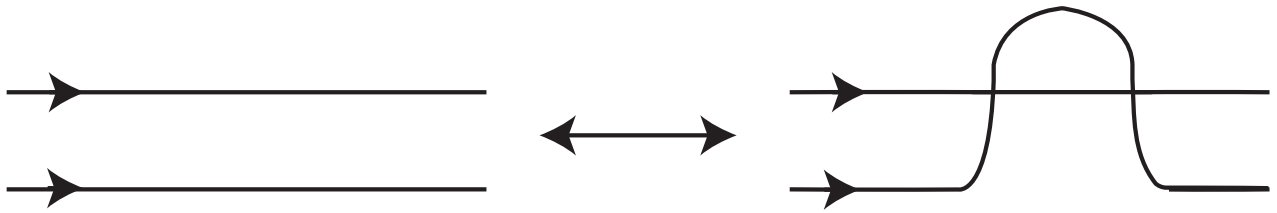


Figure 4: Labeling for the doubling moves Ω_1 and Ω_3.

The second Reidemeister move has two principal different cases, depicted in Fig. 5. In the first case (the upper picture), we have two opposite directed arcs (according to the orientation of the source–sink structure), and in the second case we have two arcs going in the same direction.



Admissible variant of the second Reidemeister move



Inadmissible variant of the second Reidemeister move

Figure 5: Labeling for the doubling move Ω_2.

In the first case, it is mentioned how the local labeling and the source–sink structure change.
The second case is not possible; i.e. it can lead to the fact that after applying the second Reidemeister move the atom becomes non-orientable.

Thus, the (increasing) second Reidemeister move is the only move from the classical Reidemeister moves which can violate the orientability of the atom. All moves from Exercise 4.5 do not violate the orientability.

Remark 4.6

One can consider the set of diagrams of virtual knots with orientable atoms and the set of moves on it consisting of all Reidemeister moves not violating the property of orientability (i.e. the detour move, the first and third classical Reidemeister moves and the "orientable" version of the second classical Reidemeister move).
This set was investigated by Kamada under the name alternating virtual links.
In particular, from the arguments given above (Lemma 4.3), it follows that the Khovanov complex is well-defined over any ring of coefficients and invariant in the category of orientable virtual links.

Theorem 4.7

Let n be a natural number. Then $Kh(D_{2n}(K))$ is an invariant of framed virtual links.

Proof.

According to Exercise 4.4, $\mathcal{C}(D_{2n}(K))$ is a well-defined complex. Let K, K′ be two diagrams of equivalent framed virtual links. Then by virtue of Exercise 4.5, there exists a collection of virtual diagrams $D_{2n}(K) = K_0, \ldots, K_m = D_{2n}(K')$ corresponding to orientable atoms such that the diagram K_{i+1} is obtained from the diagram K_i by applying generalised Reidemeister moves. By Lemma 4.2 for each of the diagrams K_j the homology $Kh(D_{2n}(K_j))$ is well defined. The invariance of the homology Kh under the detour move is obvious by construction. Thus, by virtue of Lemma 4.3 (which asserts the invariance under the classical Reidemeister moves), we get $Kh(D_{2n}(K)) = Kh(K_1) = \cdots = Kh(D_{2n}(K'))$. □

Note that the double diagram of K and the double diagram of K′ obtained from K by virtualizing one crossing, have different state cubes. Thus the complex constructed in the section can a priori distinguish framed virtual diagrams obtained from each other by virtualisation.
However, the "double" Khovanov complex constructed in this section essentially differs from the "general" Khovanov complex for classical knots. In the classical case as well as in the virtual case we have to double and after that we have to calculate the Khovanov homology. It is natural to raise the question whether the "general" Khovanov homology Kh(K) is invariant in the case of diagrams with orientable atoms. The positive answer to this question will be given (with some restrictions) in the next section and (completely) in the sections devoted to the Khovanov homology for virtual links.

Contents

Atoms

Definition 5.1

An atom is a pair: a connected 2–manifold M^2 without boundary and a graph $\Gamma \subset M^2$ such that $M^2 \backslash \Gamma$ is a disconnected union of cells that admit a chessboard colouring (with black and white colours).
The graph Γ is said to be the frame of the atom. The genus (respectively, Euler characteristic) of the atom is that of its first component.
The complexity of the atom is the number of vertices of its frame.

Atoms are considered up to natural isomorphism: two atoms are called isomorphic if there exists a one–to–one map of their first components taking frame to frame and black cells to black cells.
Atoms can be generated by Morse functions on 2–surfaces: an atom's frame is just the critical level with several critical points on it.

Atoms

Definition 5.2

An atom is called a height (or a vertical) atom if it is isomorphic to an atom obtained by the third projection function on some closed 2–manifold embedded in $\mathbb{R}^3$.

Each atom (more precisely, its equivalence class) can be completely restored from the following combinatorial structure:

1. the frame (four–valent graph);
2. the A–structure (dividing the outgoing half–edges into two pairs according to their disposition on the surface); and the
3. B–structure (for each vertex, we indicate some two pairs of adjacent half–edges (also: two angles) that constitute a part of the boundary of black cells).

Any orientable atom with two vertices is height (can be given by a height function of a surface embedded in $\mathbb{R}^3$).

The main goal of this section is the construction of the Khovanov homology by means of two-sheeted coverings. This will lead us to the following

Statement 5.3

Let F be a field, and let K, K′ be two equivalent virtual diagrams with orientable atoms. Then there exists an isomorphism of graded homology $\mathrm{Kh}_{\mathrm{F}}(\mathrm{K}) \cong \mathrm{Kh}_{\mathrm{F}}(\mathrm{K}')$.

Note that the general assertion about the invariance of this homology with arbitrary coefficients follows from the parity arguments (see below) and from the explicit construction of the Khovanov homology for virtual knots.

The main construction is as follows.
For each virtual diagram K one can consider the atom At(K) corresponding to it. Later on, we shall use the techniques of orientable covering. Namely, if the atom At(K) is orientable, we consider two copies of At(K); if it is not, then we consider the atom $\widetilde{\mathrm{At}}(\mathrm{K})$ which is the orientable two-sheeted covering over the atom At(K). It is defined as the two-sheeted covering over the corresponding surface; here, the preimage of the frame is a graph which we consider as the frame, the preimage of a black cell is a pair of black cells, and the preimage of a white cell is a pair of white cells. The atom obtained in such a way can be either two-component or one-component, and it depends on the orientation of the initial atom. Denote the virtual diagram corresponding to the atom $\widetilde{\mathrm{At}}(\mathrm{K})$ by $\widetilde{\mathrm{K}}$.

If we apply a classical Reidemeister move Ω_i to the initial diagram K, then the move Ω_i will be applied to the diagram $\widetilde{K}$ in two places; here in the case of the move Ω_2, the admissible variant of the second Reidemeister move will be applied to $\widetilde{K}$ twice.
This construction can be treated as follows: We consider two sets of vertices of the atom with the A-structure at them and connect vertices by edges.
Thus, for each virtual knot we can consider its "covered version":

$$K \to At(K) \mapsto \widetilde{At}(K) \mapsto Kh_F(\widetilde{K}).$$

In terms of a knot diagram, this construction is described as follows. Let a virtual diagram K be given; this diagram has n classical crossings $v_1, \ldots, v_n$. These crossings are connected with each other in some way. Thus, we have a graph Γ immersed in the plane. Each crossing v_i has four (adjacent) ends v_{i1}, v_{i2}, v_{i3}, v_{i4} enumerated, for example, in clockwise manner, with crossings connected by branches of the diagram which edges of the atom correspond to. Let an edge e_j connect the ends $v_{j_1 j_2}$ and $v_{j_3 j_4}$, where $j_2, j_4 \in \{1, 2, 3, 4\}$.

The diagram $\widetilde{K}$ is constructed as follows. It contains 2n crossings $v'_1, \ldots, v'_n$, $v''_1, \ldots, v''_n$, which are connected by edges. Each edge e_j of the initial diagram has two preimages: e^1_j and e^2_j. Each of two edges e^i_j connects an end $v'_{j_1j_2}$ or $v''_{j_1j_2}$ with an end $v'_{j_3j_4}$ or $v''_{j_3j_4}$. For each edge e^1_j we have to choose which ends are connected (v' or v''). Here we have an ambiguity. The matter is that before describing edges we have not had a natural ordering of vertices: Which of the vertices v'_i or v''_i is the "first" and which one is the "second"? To overcome this difficulty let us choose some spanning tree T for Γ and say that all edges e^1_j corresponding to edges of this graph connect ends $v'_{j_1j_2}$ with $v'_{j_3j_4}$ (thereby edges e^2_j connect ends $v''_{j_1j_2}$ and $v''_{j_3j_4}$).

Another choice of the tree will correspond to some change of notation: v'_j and v''_j swap places in some pairs. After that the rule for connecting the remaining ends by edges e^1_i and e^2_i follows. In order to indicate which pairs of ends are connected by an edge e^α_i, we shall either connect them by the edge e^1_i or e^2_i: the "symmetric" pair of ends corresponding to it obtained by swapping $v' \longleftrightarrow v''$ will also be connected by an edge. Henceforth, for constructing a virtual diagram it is not important for us to remember the notation for these edges. We shall not pay attention to how we place edges e^α_i on the plane. The resulting class of virtual link will not depend on it (by construction, diagrams will differ from each other by applying a finite sequence of detour moves).

So, we have fixed a maximal tree $T \subset \Gamma$. Each edge e_j not belonging to this tree represents the minimal cycle on the subgraph $T \cup e_j \subset \Gamma$. In the case when this cycle is good (see below), we connect the ends $v'_{j_1j_2}$ and $v'_{j_3j_4}$ by the edge e^1_j, and the ends $v''_{j_1j_2}$ and $v''_{j_3j_4}$ by the edge e^2_j. In the case of a bad cycle we connect the ends $v'_{j_1j_2}$ and $v''_{j_3j_4}$ by the edge e^1_j, and the ends $v''_{j_1j_2}$ and $v'_{j_3j_4}$ by the edge e^2_j. The notion of good and bad edges goes back to orientable and non-orientable cycles on the corresponding atom.

An edge is called good if the corresponding cycle is orientable. Under the covering of the atom, orientable cycles are taken to cycles, and non-orientable cycles are sent to paths (with some ends in v'_k, v''_k). Let us define the notion of a good edge (for edges not belonging to T), and the notion of a good (orientable) cycle in terms of a diagram of the virtual link. For this we consider all edges of the given cycle $e_{j_1}, e_{j_2}, \ldots, e_{j_k}, e_{j_{k+1}} = e_{j_1}$, where edges $e_{j_i}, e_{j_{i+1}}$ meet at a vertex (indices i are taken modulo k) and let us try to define locally the source–sink structure along them. Let us orient the edge e_{j_1} in some way. Further, if the edge e_{j_2} is opposite to the edge e_{j_1} at a vertex, then we orient e_{j_2} such that either both edges e_{j_1} and e_{j_2} come into the vertex, or both edges emanate from it; in the case when the edges are not opposite, we shall make one of them come into the vertex and the other emanate from it. Further, we do the same for the orientation of $e_{j_3}, e_{j_4}, \ldots$. If the process converges; i.e. we have the orientation of $e_{j_{i+1}} = e_{j_1}$ coincides with the initial one, we call the cycle good, and bad otherwise. Namely, a cycle is called good (orientable) if the number of its transversal passages through classical crossings, vertices of the atom, is even.

Remark 5.4

For a plane diagram the parity of the number of transverse passages through classical crossings coincides with the parity of passages through virtual crossings (all these passages are transverse).

It is easy to check that this definition of a good cycle coincides with the definition of an orientable cycle on the atom defined by the A-structure. Setting successively orientations of edges according to the source–sink structure, we define orientations of black cells approaching (locally) to these edges. The first vector of the basis is directed along the orientation of the edge, and the second one is directed inward the black cell. If we return to the initial edge with the same orientation, then this means that we have traveled along an orientable cycle, and a non-orientable cycle otherwise. Indeed, if we pass through a classical crossing, then orientations of neighboring cells defined in such a way, are opposite to each other. Thus, getting a compatible orientation means precisely that our path goes transversely evenly many times.

So, we have defined the notion of a good (orientable) cycle and a good edge (for edges not belonging to the tree T). Therefore, we have completely constructed the virtual diagram $\widetilde{K}$. Note that the definition of a good cycle does not depend (up to detour moves) on the choice of the tree T.

Moreover, from the atom $\widetilde{\mathrm{At}}(\mathrm{K})$ the knot corresponding to the two-sheeted covering is restored up to virtualisations; we have already mentioned the explicit way of constructing the diagram $\widetilde{\mathrm{K}}$ with the diagram K; it corresponds to some immersion of the frame of the atom $\widetilde{\mathrm{At}}(\mathrm{K})$ (with preserving the A-structure).
It is easy to see that the detour move in the initial diagram K of the link induces some combinations of the detour moves on the diagram $\widetilde{\mathrm{K}}$. Moreover, the following lemma takes place.

Lemma 5.5 ([11])

By applying one of the classical Reidemeister moves to a diagram K the diagram $\widetilde{\mathrm{K}}$ will change in the following way: The same Reidemeister move is applied to it in two places. Herewith, the atom corresponding to the "middle" diagram obtained from $\widetilde{\mathrm{K}}$ by applying the second Reidemeister move in one place (any of two places) is orientable.

The proof of the lemma is left as an exercise.

According to Lemma 4.2, the homology $\mathrm{Kh}(\widetilde{\mathrm{K}})$ is well defined. Therefore, by Lemma 4.3 the Khovanov homology of a "covered" link does not change under applying the Reidemeister move to the initial knot. This leads us to the following

Theorem 5.6

The map $\mathrm{K} \mapsto \mathrm{Kh}(\widetilde{\mathrm{K}})$ gives a well-defined invariant of virtual links.

Remark 5.7

Note that only the second Reidemeister move Ω_2 can change the type of the corresponding atom (i.e. it can convert a non-orientable atom to an orientable one and vice versa). If, for example, we have an orientable atom $\mathrm{At}(\mathrm{K})$ and two components of the atom $\widetilde{\mathrm{At}}(\mathrm{K})$, then the application of the second Reidemeister move (non-admissible version) to K can "connect" these components into one (this corresponds to the fact that after applying the second Reidemeister move, the atom may become non-orientable).
Herewith the moves Ω_1, Ω_3 preserve the orientability of the atom.

Now let the atom corresponding to a diagram K be orientable. Then $\widetilde{\mathrm{K}}$ consists of two copies of the atom corresponding to K. Since F is a field, we have $\mathrm{Kh_F}(\widetilde{\mathrm{K}}) = \mathrm{Kh_F}(\mathrm{K})^{\otimes 2}$.
Therefore, the homology Kh(K) is obtained from the invariant homology $\mathrm{Kh}(\widetilde{\mathrm{K}})$ by "extracting of the tensor square root". In the case when the ring of coefficients is a field, we have the Poincaré polynomial $\mathfrak{P}$ in two variables with all integer non-negative coefficients. From this polynomial we have to extract the "square root"; i.e. to find the Laurent polynomial $\mathfrak{Q}$ in the same two variables with integer non-negative coefficients (coefficients are non-negative since they are the ranks of Khovanov homology groups) such that the equality $\mathfrak{Q}^2 = \mathfrak{P}$ holds. It is obvious that if we can do this, then it can be done uniquely. Since this operation is unique, if it exists, we get the claim of Theorem 5.3.

Moreover, from these discussions we get the following

Theorem 5.8

Let F be a field, and let for a virtual diagram K the graded homology $Kh_F(\widetilde{K})$ cannot be represented as the tensor square. Then K has no diagram with an orientable atom. In particular, the virtual link generated by K is not classical.

It is natural that the Khovanov complex constructed in this section cannot detect non-triviality of the virtual knot depicted in Fig. 3, since this knot is obtained from the unknot by generalised Reidemeister moves and virtualisations.

The question about whether two non-isotopic classical links can be obtained from each other by a finite sequence of generalised Reidemeister moves and virtualisations is an important and interesting conjecture (virtualisation conjecture). Note that the virtualisation conjecture is true for the unknot (i.e. if a classical diagram of a knot is obtained from a diagram of the unknot by applying a finite sequence of the generalised Reidemeister moves and the virtualisation, then the classical diagram represents the unknot), since the Khovanov homology detects the unknot, see [33]. The Khovanov complex gives a partial answer to this question.

From Theorem 5.3 and the invariance of the Khovanov homology under virtualisation, we have the following theorem.

Theorem 5.9

If a classical link is obtained from a classical link by applying generalised Reidemeister moves and virtualisations, then these links have the same Khovanov homology with coefficients from any preassigned field.

Later we shall show that this theorem is true for arbitrary coefficients (e.g. from the ring $\mathbb{Z}$), see Theorems 7.8 and 7.12.

Contents

Parity

Assume there exists a map $\widetilde{f}$ sending the set of diagrams of virtual knots into itself and having the following properties:

1. for each virtual diagram K the diagram $\widetilde{f}(K)$ is a virtual diagram with an orientable atom;
2. if a diagram K has an orientable atom, then $\widetilde{f}(K) = K$;
3. if two diagrams K and K' are equivalent by means of Reidemeister moves, then $\widetilde{f}(K)$ and $\widetilde{f}(K')$ are equivalent by means of Reidemeister moves, where all intermediate diagrams connecting the diagrams $\widetilde{f}(K)$ and $\widetilde{f}(K')$ have orientable atoms.

Theorem 6.1

The map $K \mapsto Kh(\widetilde{f}(K))$ is an invariant of virtual links.

Contents

Bifurcations of types $2 \to 1$ and $1 \to 2$ in the Khovanov complex will (see Section 2) correspond to partial differentials ∂'; the differential ∂ consists of (see below); the bifurcation of type $2 \to 1$ corresponds to the multiplication m, and the bifurcation of type $1 \to 2$ corresponds to the comultiplication Δ.

The complete information about the number of circles in states of the diagram can be extracted from the corresponding atom. In other words, the state cube can be completely restored from the atom.

An actual problem is the problem of finding the the minimal genus of atoms corresponding to diagrams of the virtual link. Classical link diagrams of genus zero are the connected sums of alternating diagrams.

This genus is called the virtual link genus or the Turaev genus due to [38], cf. Definiton 5.1. It turned out [32] that this genus had an important significance in studying Heegaard–Floer homology of classical knots.

We shall construct the Khovanov complex starting with a given virtual link diagram. As we shall see, the homology of the complex constructed in this way really depends only on the corresponding atom. Thus the homology will be invariant under virtualisation. This supports the virtualisation conjecture mentioned above.

Twisted virtual knots [18, 40] are close relatives of virtual knots. They are represented by knots in oriented thickenings of not necessarily orientable surfaces modulo stabilisation/destabilisation. A particular case of the theory of twisted virtual knots is the theory of knots in $\mathbb{R}P^3$ (cf. Chapter 23, [10]).

Definition 7.1

An orientable thickening of a two-dimensional surface M is an orientable three-dimensional I-bundle over M, where I is a segment.

Let us consider a non-orientable surface S and construct the canonical oriented I-bundled over it. It represents a three-dimensional manifold $S\widetilde{\times}I$ with boundary.
A nice example of such a thickened surface is $\mathbb{R}P^2\widetilde{\times}I$, which is homeomorphic to $\mathbb{R}P^3\backslash\{*\}$. Thus, by constructing the Khovanov homology for such knots, we shall get the Khovanov homology theory for knots in $\mathbb{R}P^3$.
Given a surface M and its thickening $M\widetilde{\times}I$, then links in $M\widetilde{\times}I$ can be considered by means of their diagrams: projections on M.

There are two types of stabilisation/destabilisation of such thickening surfaces: along orienting cycles and along non-orienting cycles. In the second case, we add/remove a thickened Möbius band.
In general position, a projection is a framed 4-graph. In order to restore the link, one should indicate for each crossing how the two branches behave in a neighbourhood of this crossing. In the orientable case, one just indicates which branch should be over, and which branch should be under. However, in the non-orientable case this indication is relative. While walking along a non-orienting circuit, the direction upwards changes to the direction downwards. So, for example, knots in $\mathbb{R}P^3 \backslash * = \mathbb{R}P^2 \tilde{\times} I$ can be represented by diagrams in $\mathbb{R}P^2$ such that all crossings lie inside the disc $D^2 \subset \mathbb{R}P^2$; when passing the boundary of the disc the direction changes; see Fig. 6. To handle this, we choose an affine chart such that the complement to this chart in S is one-dimensional. For this chart we have a well-defined notion of an over/undercrossing.

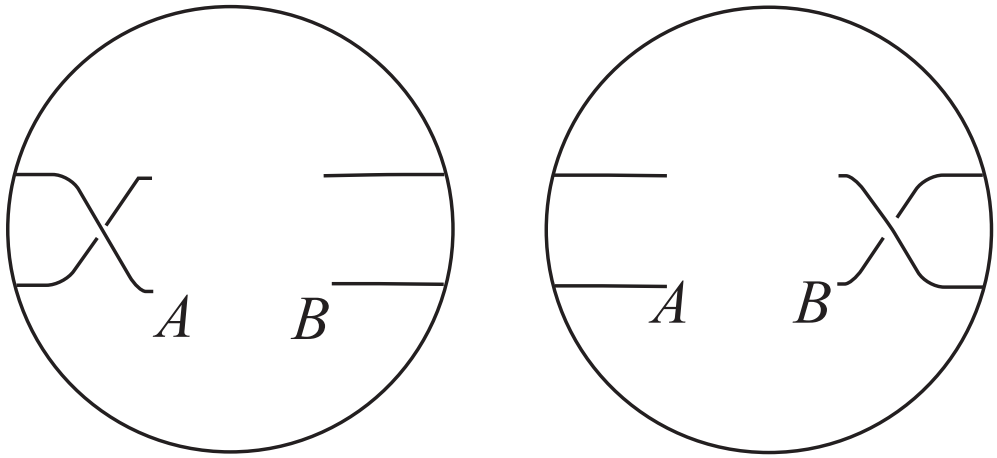


Figure 6: A branch AB forms overcrossing in the left picture and undercrossing in the right picture.

Note that links in such surfaces are well described by atoms. Indeed, fix (once for all) an orientation on $M\widetilde{\times}I$. Now, for a link diagram in M, we already have the frame of the atom: a framed 4-graph.

Now, the way for attaching black cells is the following (see Fig. 7). For a vertex v, we take two emanating non-opposite half-edges a and b. The corresponding virtual link contains two points projected in the vertex v, one of which is incident to the edge corresponding to a, and the other one is incident to the edge corresponding to b. In a neighbourhood of v, denote by c the small vector going from a point on the edge a to a point on b. If the basis (a, b, c) is positively oriented in our three-dimensional manifold, then the angle between half-edges a and b is decreed to be white, as well as the opposite angle. Otherwise they are both black.

Note that this choice does not depend on the ordering of the pair (a, b), nor on their directions.

In the case of general virtual links which are a particular case of twisted virtual links, the way of pasting black cells described above is agreed with the way described in Chapter 16, [10].

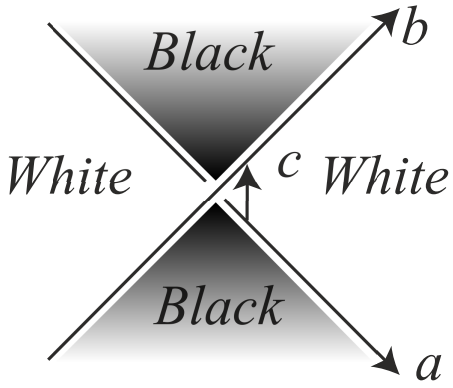


Figure 7: Constructing the atom from a diagram.

This leads to the following theorem.

Theorem 7.2

There is a well-defined map from the set of twisted virtual knots to the set of virtual knots modulo virtualisation.

Knots in such surfaces were considered by Asaeda, Przytycki and Sikora in [13], and Viro [40] (Bourgoin first considered stabilisations that led to twisted virtual knots). In [13] a Khovanov homology theory for such surfaces was constructed by using an additional topological information coming from surfaces. See also [12].
From Theorem 7.2 and the invariance of the Khovanov homology under virtualisation (Lemma 7.9), it follows immediately that the Khovanov homology constructed below can be generalised for twisted virtual knots.

Contents

Our aim is to define a homology theory for virtual knots (with arbitrary atoms) over an arbitrary ring in such a way that:

1. the homology we are defining is invariant under the (generalised) Reidemeister moves;
2. for the case of virtual knots with orientable atoms (also known as alternatible virtual knots) this homology theory coincides with the one constructed in the previous sections;
3. the tensor product of the complex with $\mathbb{Z}_2$ coincides with the theory constructed in Sec. 3;
4. the graded Euler characteristic of the complex which will be constructed coincides with the Jones polynomial.

The invariant of Rushworth [37] violates the last condition.

Remark 7.3

The coefficient ring might be an arbitrary abelian group with unit, for example, $\mathbb{Z}$.
For the sake of simplicity we shall sometimes abuse the notation and call modules over rings "linear spaces", not depending on whether the ring is a field or not.

If no $1 \to 1$-bifurcations occur, we may construct the Khovanov cube by using the standard differentials, the multiplication m (for $2 \to 1$-bifurcations) and the comultiplication Δ (for $1 \to 2$-bifurcations).

The situation with the $1 \to 1$-bifurcation (the essential phenomenon of the theory of virtual knots appearing because of the existence of non-orientable atoms) makes the problem more complicated. Indeed, if we wish to construct a grading-preserving theory without introducing any new grading, this partial differential should be identically equal to zero because of the grading reasons (there should be a map from V to V that lowers the grading by one). In the space V, the basis consists of two elements with gradings $+1$ and -1. If we set this partial differential to be equal to zero with all other differentials (m and Δ) defined in the standard way, we get a straightforward generalisation for the $\mathbb{Z}_2$ case.
Below we involve two additional structures: The basis change in the space V corresponding to a circle and generated by $\{1, \mathrm{X}\}$ (the homology group of the unknot) while passing from one crossing to another and the exterior product of "circles" instead of their usual tensor products.

Notational agreement. Given an unordered set of vector spaces, enumerate them arbitrarily: $V_1, \ldots, V_n$. We shall define a new space not depending on the ordering of the spaces, which will be denoted[1] by $V_1 \wedge V_2 \wedge \cdots \wedge V_n$ as follows. Consider all possible tensor products of these spaces and identify them according to the following rule. Let $x_i \in V_i$, $i = 1, \ldots, n$. We set $x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_n} = \mathrm{sign}(\sigma)\, x_1 \otimes \cdots \otimes x_n$. We shall denote such tensor product $x_1 \otimes \cdots \otimes x_n$ of elements $x_i \in V_i$ by $x_1 \wedge x_2 \wedge \cdots \wedge x_n$. We call this space the ordered tensor product.

Remark 7.4

To avoid confusion, note that, in writing $X \wedge X$, we always assume that the first X and the second X belong to different (but possibly isomorphic) spaces; thus $X \wedge X$ is not zero (unlike the wedge product of 1-forms).

[1] In the case of coincidence of the linear spaces $V = V_1 = \cdots = V_n$ we shall use also the notation $V^{\wedge n}$.

Let us consider a virtual diagram K.
To handle it and to make the whole cube anticommutative we have to add two ingredients, sensitive to orientability of the atom.

1. With each circle C in each state we associate a vector space of graded dimension[2] equal to $q + q^{-1}$. Namely, given an orientation o of the circle C; we associate with this circle the graded vector space generated by elements 1 and $X_{C,o}$ of gradings 1 and -1, respectively. The orientation change of the circle (passing to $-o$) leads to $X_{C,-o} = -X_{C,o}$.
2. Given a state s of a virtual link diagram K having l circles $C_1, \ldots, C_l$, with this state, we associate an ordered tensor product $V^{\wedge l}$; as a basis of this product we take the product $(p^1)_{C_{a_1}} \wedge (p^2)_{C_{a_2}} \wedge \cdots \wedge (p^l)_{C_{a_l}}$, where $(p^i)_{C_{a_i}}$ represents an element from $V_{C_{a_i}}$.

[2]From now on, we have passes from the notation 1 and X to the notation 1 and X (before 1 play the role of unity). This leads to the same homology theory up to a grading shift and a normalization. In the sequel we should not pay attention to these normalizations and shifts, this agrees with [11] in verbatim.

Thus, we have defined the chain space of the complex corresponding to the virtual diagram K. We denote it by [[K]]. All the basis elements of this space correspond to some states of K with an additional choice of the elements ± 1 or $\pm X$. Let s be a state of K with the set of circles $C_1, \ldots, C_l$, whence for these circles we have chosen elements $\gamma_1, \ldots, \gamma_l$, each of them being ± 1 or $\pm X$. Then these elements form a chain of the complex [[K]] having the height h, where h is the number of B-smoothings of s, and the grading which is equal to $h + \#1 - \#X$, where $\#1$ is the number of elements of type ± 1 among $\gamma_1, \ldots, \gamma_l$, and $\#X$ is the number of elements $\pm X$ among $\gamma_1, \ldots, \gamma_l$.
Our next goal is the description of the differential ∂ in this complex, which increases the height by one and does not change the grading.
Set n_+ = the number of crossings  , n_- = the number of crossings  .

Denote by $\mathcal{C}(K)$ the complex obtained from $[[K]]$ by the height shift and the grading shift: $\mathcal{C}(K) = [[K]]\{n_+ - 2n_-\}[-n_-]$; i.e. the height of each chain decreases by n_-, and the grading increases by $(n_+ - 2n_-)$; all differentials change respectively. Here we assume that $[[K]]$ is a complex, this fact will be proved below.
Whatever the differential ∂ is, from the construction of chains of the complex $\mathcal{C}(K)$ follows Theorem 7.5.

Theorem 7.5

For any virtual diagram K we have $\chi(\mathcal{C}(K)) = \hat{J}(K)$.

We shall think of all classical crossings as oriented upwards:  and  .

Consider a state s of a diagram of an oriented virtual link. Choose a classical crossing and consider all circles of the state s incident to this crossing. There are one or two such circles. Fix orientations on these circles according to the orientation of the edge emanating upwards to the right (and opposite to the orientation of the edge incoming to the crossing from the bottom left; see Fig. 8, upper part). As we shall see further, in the case of one circle, these two orientations defined locally can be uncoordinated, but this case can be treated easily.
Thus, the orientations of these circles of the state s locally agree with the orientation of the edge emanating upwards to the right (as well as with the edge incoming from the bottom-right) and disagree with the orientation on the left side. We orient the half-edges as shown in the lower-left part of Fig. 8. Thus, we have fixed a choice of the generator X for any circle incident to a given crossing. Note that for another crossing for the same circle the choice of X may differ from this one by a sign.

Differentials will be defined according to the orientations of circles at classical crossings and local orderings of components with the following rule.
The orientations described above are well defined unless the case when the edge corresponding to the crossing of the diagram bifurcates one circle to one circle. In such cases, we set the partial differential to be zero.

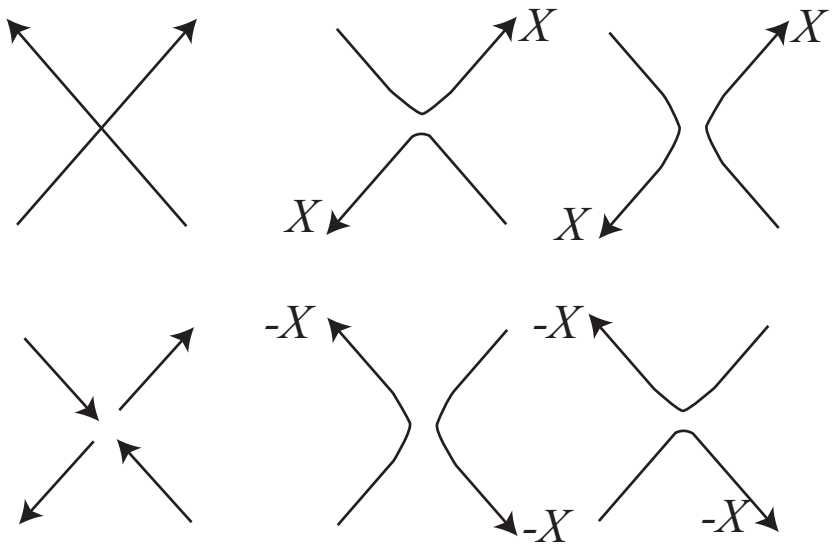


Figure 8: Definition of a basis at a crossing.

Assume we have a $1 \to 2$ or $2 \to 1$-bifurcation at a crossing.
If we deal with two circles incident to the crossing from the opposite sides, we order them in such a way that the upper (respectively, left) circle is locally first; the lower (respectively, right) one is thus, the second. In the sequel, when defining partial differentials we assume that all circles are ordered in such a way that the circles we deal with are in the very first position in our tensor product; this can always be obtained by means of a permutation, which might lead to a sign change. The map on the other circles is identical.
Let there be given an edge of the bifurcation cube where the number of circles is changed by one. This bifurcation corresponds to a certain crossing; we have two options $2 \to 1$ or $1 \to 2$. In those states when we have two circles incident to the crossing, the circles are ordered. Moreover, all three circles are oriented, thus, we have chosen a basis for the space corresponding to each of these circles.

Now we define the maps $\Delta\colon V \to V \wedge V$ and $m\colon V \wedge V \to V$ locally according to the prescribed choice of generators at the crossing and local ordering (see Fig. 9):

$$\Delta(1) = 1_1 \wedge X_2 + X_1 \wedge 1_2;\ \Delta(X) = X_1 \wedge X_2 \tag{3}$$

and

$$m(1_1 \wedge 1_2) = 1;\ m(X_1 \wedge 1_2) = m(1_1 \wedge X_2) = X;\ m(X_1 \wedge X_2) = 0. \tag{4}$$

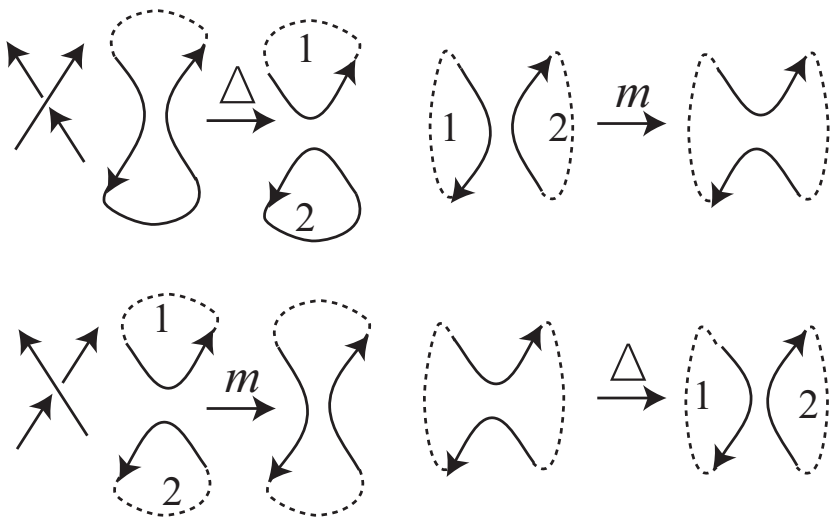


Figure 9: Defining operations m and Δ.

Note that the map m is surjective and the map Δ is injective.
If we have some circles $C_1, \ldots, C_l$ not incident to the crossing in question, and elements $\gamma_1, \ldots, \gamma_l$ on them, the formulae for the partial differentials ∂' are written as:

$$\begin{aligned}\partial'(1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \Delta(1) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= 1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l + X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \quad (5) \\ \partial'(X \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \Delta(X) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l = X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l\end{aligned}$$

(in the case of a $1 \to 2$-bifurcation) and

$$\begin{aligned}\partial'(1_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= m(1_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= 1 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \\ \partial'(X_1 \wedge 1_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= \partial'(1_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) \\ &= m(X_1 \wedge 1_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \quad (6) \\ &= m(1_1 \wedge X_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l \\ &= X \wedge \gamma_1 \wedge \cdots \wedge \gamma_l, \\ \partial'(X_1 \wedge X_2 \wedge \gamma_1 \wedge \cdots \wedge \gamma_l) &= m(X_1 \wedge X_2) \wedge \gamma_1 \wedge \cdots \wedge \gamma_l = 0\end{aligned}$$

(in the case of a $2 \to 1$-bifurcation).

After that we define the differential ∂ on the chain space corresponding to the state s as the sum of partial differentials acting on the state s.

Example 7.6

Thus, if we wish to comultiply the second factor X_2 in $X_1 \wedge X_2$, we get $X_1 \wedge X_2 = -X_2 \wedge X_1 \to -X_2 \wedge X_3 \wedge X_1 = -X_1 \wedge X_2 \wedge X_3$, where X_3 belongs to the newborn third component (under the condition that at the crossing of splitting the circle X_2 is locally first (i.e. upper and left), and the circle X_3 is locally second).

Given an oriented diagram K of a virtual link, we have constructed a set of bigraded groups with the differential ∂. Denote the set of groups by [[K]]. The differential increases the height and does not change the grading.

Our goal is to prove the main theorem.

Theorem 7.7

The set of groups [[K]] together with the differential ∂ is a well-defined bigraded complex, i.e. $\partial^2 = 0$. Herewith the differential preserves the grading and increases the height by one.

The complex $\mathcal{C}(\mathrm{K})$ is obtained from [[K]] by the height shift and grading shift. From the constructions it will follow that the homology of the complex $\mathcal{C}(\mathrm{K})$ coincides with the homology constructed for the case of virtual knots with orientable atoms.
Further, from the proof of Theorem 7.7 the claim of Theorem 7.5 follows by construction.
The complex with coefficients in $\mathbb{Z}_2$ coincides with the complex over $\mathbb{Z}_2$ described in Sec. 3.

Theorem 7.8

The homology of the bigraded complex $\mathcal{C}(\mathrm{K})$ is an invariant of the virtual link K under generalised Reidemeister moves.

We first prove Theorem 7.7. After that, we shall prove Theorem 7.8; its proof will be more technical and it will follow the standard scheme described above some additional sign checks for partial differentials, appearing while ordering and orienting the circles, will be needed. We shall also show that the homology of $\mathcal{C}(\mathrm{K})$ coincides with the homology constructed for the case of virtual knots with orientable atoms.

We first prove two lemmas that establish some properties of our complex $\mathcal{C}(\mathrm{K})$ and simplify further arguments.

Let K be a virtual diagram. Consider a classical crossing v of it. Let the diagram K′ be the diagram obtained from K by the virtualisation of v. Then there exists a one-to-one correspondence between the sets of classical crossings of the diagrams K and K′. It generates a one-to-one correspondence ϕ between the states (for the corresponding vertices we have either A-smoothings or B-smoothings). Note that such a bijection does not change the number of circles in the states; it follows from the fact that all states can be restored from the atom, and the atom does not change under virtualisations. Let us orient circles of corresponding states identically outside the crossing v. This identification defines the map g: $[[K]] \to [[K']]$ of the chain spaces according to the following rule. For any state s and the corresponding state ϕ(s), the diagrams K and K′ look identical outside a neighbourhood of v. Thus, we can establish the bijection between oriented circles of s and oriented circles of ϕ(s), that leads to the definition of g. We shall use the same notation g for maps of vector spaces (modules) corresponding to the circles in states s and ϕ(s).

Let C_s be the subspace of the space $[[K]]$ associated with a state s of the diagram K. Denote the corresponding space for K' by $C_{s'}$.

Lemma 7.9

Let K, K' be two diagrams obtained one from another by the virtualisation. Then there is a grading-preserving chain map $f\colon [[K]] \to [[K']]$ that maps C_s isomorphically to $C_{s'}$ and commutes with the local differentials.
In particular, if $[[K]]$ is a well-defined complex, then so is $[[K']]$; herewith their homology groups are isomorphic.

Proof

Suppose the diagram K′ is obtained from the diagram K by the virtualisation at a crossing v.

The map f is constructed according to the crossing type of v ( or ). By construction, partial differentials of the complex [[K′]] coincide with the images of partial differentials of [[K]] under g, except, maybe, those partial differentials corresponding to the crossing v. Furthermore, differentials corresponding to v split our cube to the "lower subcube" and the "upper subcube", as shown in Fig. 10.

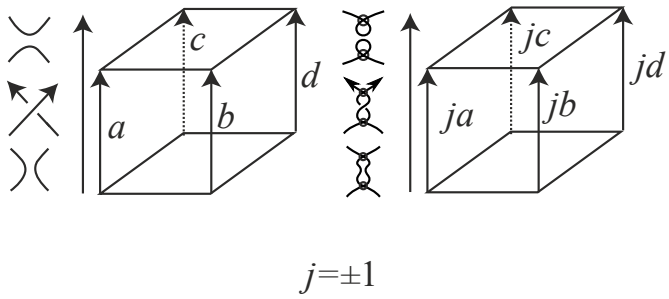


Figure 10: The behaviour of the cube under the virtualisation.

Now, the remaining partial differentials differ possibly by signs on edges corresponding to the crossing v. Our goal is to show that they either all agree or all differ by −1 sign, as shown in Fig. 10.

proof

Indeed, the bases at all crossings but v agree for K and K′. This leads to the identification of chains of the corresponding complexes. For this isomorphism for every circle C incident to v and the circle g(C) corresponding to it in the corresponding state of the diagram K′ we have $g(X_{C,o_K}) = -X_{g(C),o_{K'}}$, where o_K and $o_{K'}$ are the orientations of the circles C and C′ at the crossing v of the diagrams K and K′ chosen according to the rule depicted in Fig. 8. The latter identity holds because in any state s the circle C that tends from the upper-right to the crossing v of K, corresponds to the circle $\phi_*(C)$ in the state $\phi(s)$ that tends to v from the upper-left, this corresponds to the change X to −X in the local basis of spaces V corresponding to circles of the state incident to the given crossing; see Fig. 8. If we dealt with the usual tensor product case regardless of the circle ordering, the transformation $X \to -X$ would leave m invariant and change Δ to $-\Delta$.

Proof

Assume now that the crossing v is positive (). All maps of type m corresponding to v, represent bifurcations of two circles (a left one and a right one) into one circle. After the virtualisation, the circles interchange their roles; see Fig. 11.

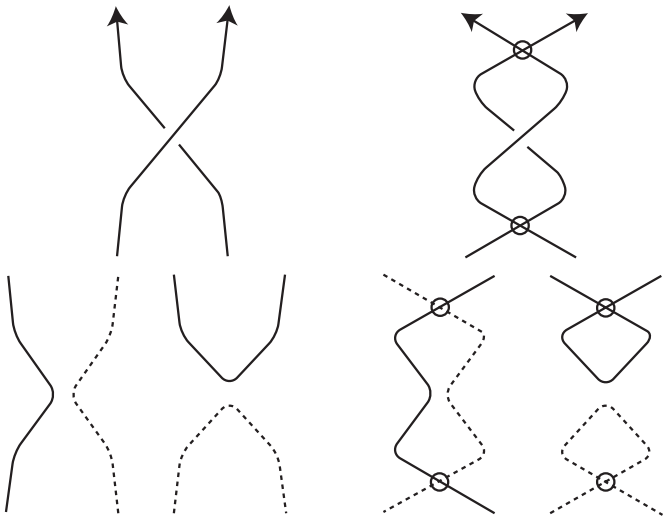


Figure 11: Virtualisation.

Proof

Globally we get a sign change for all m-type partial differentials. For partial differentials of type Δ we have one circle that bifurcates into two ones, the upper one, and the lower one; the "up-down" position remains unchanged under virtualisation, that preserves all Δ-type partial differentials. The first component is shown locally by solid line, whence the second component is shown by a dashed line. Summing up (and recalling the sign change of the partial differential Δ because of passing $X \to -X$), we see that the virtualisation of a positive crossing changes the signs of all partial differentials corresponding to this crossing.

Proof

Now divide the chain space [[K]] and [[K′]] into two parts each, according to the smoothing of v; we call one part of the cube “upper”, the remaining part being lower. Now set f: [[K]] → [[K′]] as g for all elements from the lower subcube and as −g for the upper subcube. Evidently, this map commutes with partial differentials. Indeed, the commutativity of the map f with partial differentials inside one of the subcubes follows from the fact that the map g is anticommutative; therefore, the map f commutes.

Thus if the initial cube were anticommutative, then the constructed map would be an isomorphism in homology.

Similar arguments show that the virtualisation of a negative crossing does not change the cube at all. The minus sign that appears on edges corresponding to Δ is canceled by the minus sign caused by the permutation of circles (the right one and the left one). This completes the proof of the lemma.

This lemma means that the homology of a virtual diagram with two classical crossings (if well defined) can be restored from an atom endowed with an orientation of the link components.
Thus, to prove that the cube [[K]] anticommutes, we can make some preliminary virtualisations for classical crossings of K and consider the analogous question for the obtained diagram K′.
To check the anticommutativity of the cube [[K]] we have to consider all 2-faces of it. Each 2-face is represented by fixing a way of smoothing some $(n-2)$ classical crossings of K; see Fig. 12. The remaining two crossings can be smoothed arbitrarily; the four possibilities correspond to the vertices of the 2-face.
In Fig. 12 the bifurcation cube is shown in the left part and the 2-face and the corresponding atom are shown in the right part. The atom can be restored from a knot diagram, as described above in Chapter 16 of [10].

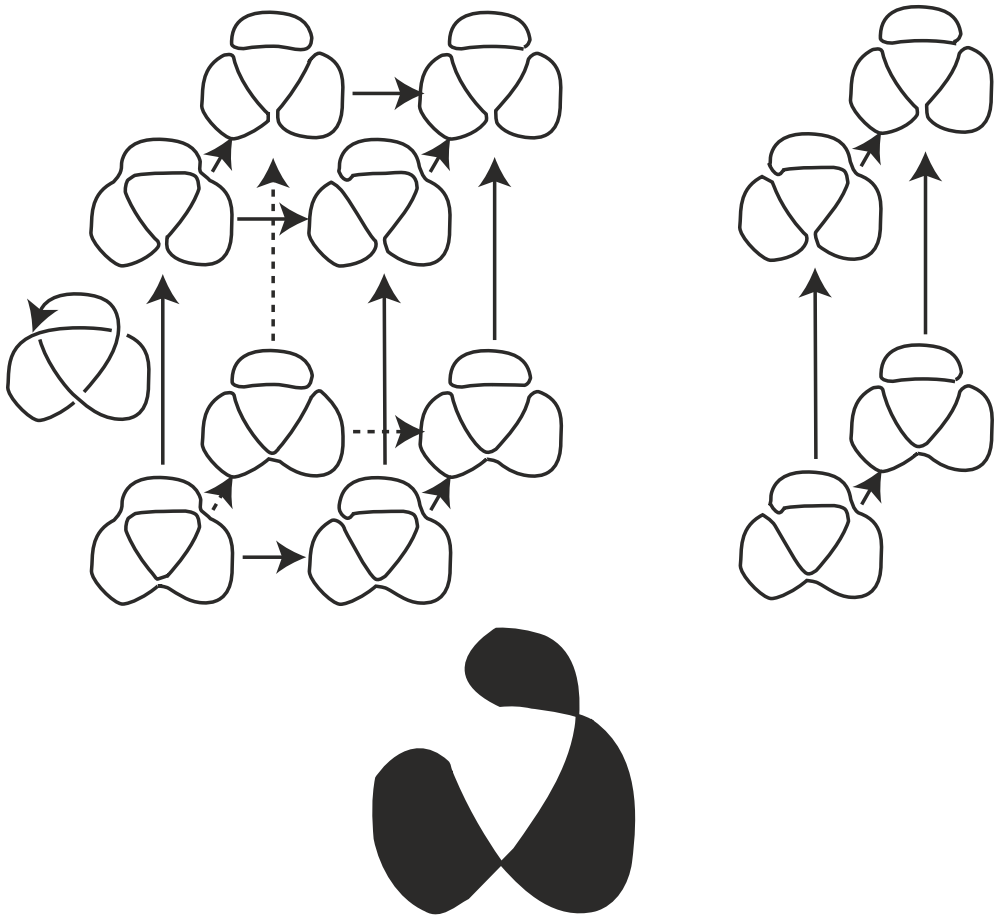


Figure 12: A 2-face generates an atom.

Now, for these four states, there are some "common" circles which do not touch any of the two vertices in question (in the case depicted in Fig. 12 there are no such circles). After removing these circles, we get an atom with two vertices.
What we actually have to check is that any face corresponding to any possible atom with two vertices anticommutes.
For the two vertices of such an atom, we have some local orientations of the link at each of these vertices; they are needed to fix the local ordering of components (see Fig. 8) when defining the differentials.
Note that globally these orientations might not agree on the circles; namely, an edge of the atom with two vertices consists of several edges of the diagram which might have opposite orientations; see Fig. 13.

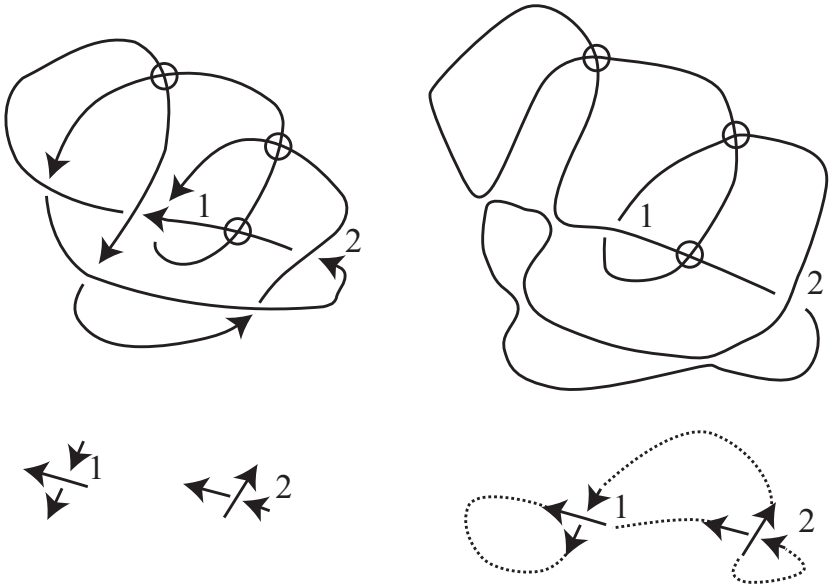


Figure 13: Orientation for atom crossings.

It turns out, however, that these local orientations can be chosen arbitrarily without losing the anticommutativity property and without changing the homology.

Namely, fix an atom with two vertices. All possible occurrences of this atom in the cube correspond to local orientations of edges at these vertices. Fix an orientation for one crossing v_1 and choose two distinct orientations for the second crossing v_2 that differ from each other by the clockwise $\frac{\pi}{2}$-turn of the arrows; see Fig. 14. Thus, we get two pictures and two two-dimensional discrete cubes, Q_1 and Q_2. These cubes coincide as sets of linear spaces. Let V_s and $V_{s'}$ be linear spaces of Q_1 and Q_2 corresponding to some fixed state s and the state s' corresponding to it.

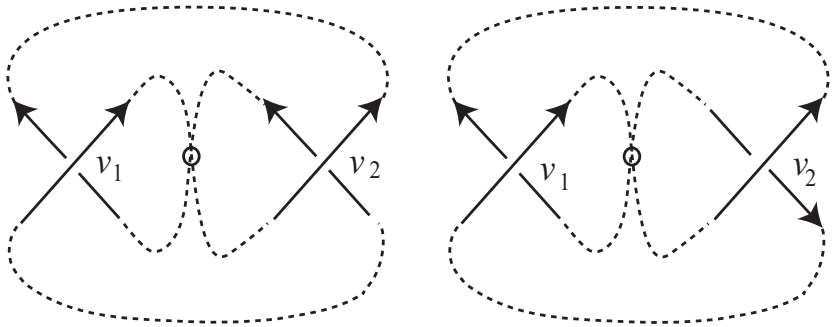


Figure 14: Q_1 and Q_2.

Lemma 7.10

If Q_1 is anticommutative, then so is Q_2. Moreover, there exists a grading preserving chain map $f\colon Q_1 \to Q_2$ that takes V_s isomorphically to $V_{s'}$ and commutes with partial differentials.

Proof.

The proof of Lemma 7.10 is very much similar to that of Lemma 7.9. A sketch of the proof goes as follows. After rotating all arrows at v_2 in the counterclockwise direction, we get the local sign change of X for all circles incident to this crossing. Analogously to Lemma 7.9, we consider two complexes and identify their chain spaces by means of the map g (analogous to the map g from Lemma 7.9) in such a way that the differentials corresponding to any other crossing coincide. After that we correct g, as in Lemma 7.9, to get a map f that commutes with all partial differentials, which would yield the statement of the lemma. If we dealt with the usual unordered tensor product, this would lead to the sign change of all partial differentials of type Δ corresponding to v_2.

Furthermore, in the case of a positive crossing, all differentials of type m corresponding to this crossing, change their sign, too.

In the case of negative crossings, partial differentials of type $2 \to 1$ do not change, and $1 \to 2$-bifurcations change the sign again. Thus, we have the same situation as in Lemma 7.9, which completes the proof of Lemma 7.10. □

Let us continue the proof of Theorem 7.7.
Lemma 7.10 means that in order to check the anticommutativity of all possible faces, it is sufficient to enumerate all atoms with two vertices and check the anticommutativity for each of them. We first fix a representation of such an atom in $\mathbb{R}^2$ (i.e. an immersion of its frame preserving the A-structure); such immersions differ by a possible virtualisation which does not change the complex (up to isomorphism) by Lemma 7.9; then we choose a local orientation, which does not matter either by Lemma 7.10.
Note that among atoms with two vertices there are disconnected atoms; i.e. those for which each edge connects some vertex with itself. For such atoms in the case of ordinary tensor product we get by evident reasons commutative 2-faces. In the case of ordered tensor products the corresponding faces will obviously anticommute.

Some (connected) atoms with two vertices are inessential in the following sense. We have set the $1 \to 1$ differential to be zero. By parity reasons, in the 2-face of any atom there might be 0, 2 or 4 such edges. The case when we have no such edges is orientable. When we have four edges representing differentials of type $1 \to 1$, then the proof follows from the identity $0 = 0$. The same takes place in the case when in the diagram, the anticommutativity of which we prove, we have two compositions of maps and one of the maps at each composition is zero.

There are some inessential atoms, where two vertices are not connected to each other. For any of them, anticommutativity is obvious. There are six essential connected atoms with two vertices, as shown in Fig. 15. All these atoms except the first one are orientable.

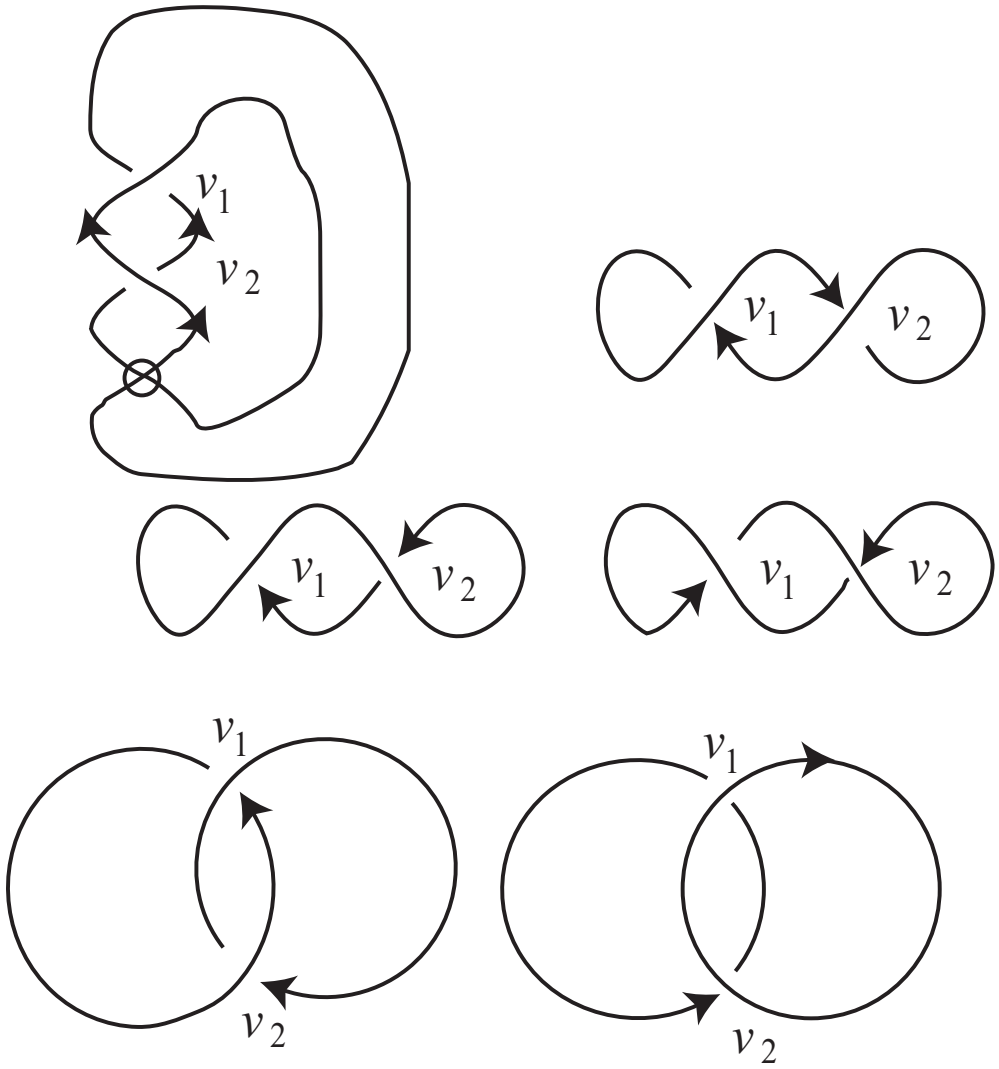


Figure 15: Essential atoms with two vertices.

For the first one, an accurate calculation corresponding to Fig. 16

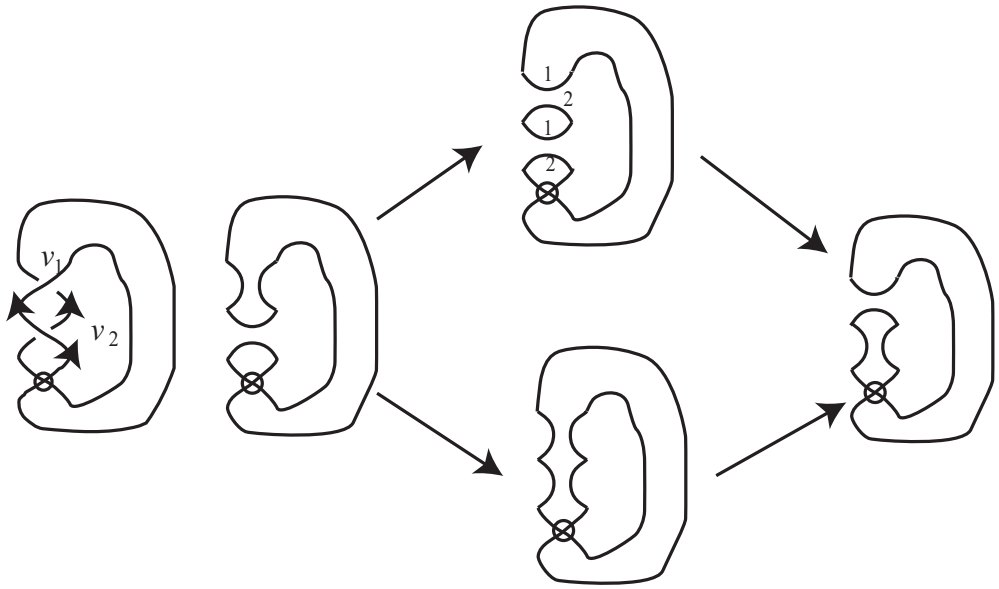


Figure 16: The non-orientable atom.

Indeed, the lower composition is zero. Substituting X into the upper composition, we get $\pm X \wedge X$ at the first step and zero at the second step. If we start with 1, we get $1_{1,o_{v_1}} \wedge X_{2,o_{v_1}} + X_{1,o_{v_1}} \wedge 1_{2,o_{v_1}}$ at the first step; here the first index is the local number of the circle (the first circle is big and the second one is small), and the second index is the name of the vertex. When passing to the second vertex v_2, the first and second circles change their roles: The circle number 1 becomes the lower one and number 2 becomes the upper one. Also, for the big circle, X changes to $-X$. Thus we get $-X \wedge 1 + 1 \wedge X$ which is mapped by m to zero.

Let us now check orientable atoms. For any of them, we fix an orientation as shown in Fig. 15. Such an orientation gives a coordinated orientation of circles at two crossings which are under consideration in the sense of Fig. 8. After that, we can fix the bases $\{1, X\}$ for all circles at vertices according to the rule shown in Fig. 8.

Now, the anticommutativity is checked as follows. If we dealt with the usual (unordered) tensor product case, everything would commute. Now, the enumeration of circles might cause minus signs on some edges. We have to check that for any of these five atoms the total sign would be minus.

For instance, in Fig. 17 we have an oriented atom with two vertices. The analogous check of the unordered tensor product case means the usual associativity $\mathrm{m} \circ (\mathrm{m} \otimes \mathrm{Id}) = \mathrm{m} \circ (\mathrm{Id} \otimes \mathrm{m})$, where the circles are enumerated from the left to the right. In the left part of the figure, one pair of numbers of the circles 1 and 2 is drawn upside down to underline which circle is assumed to be locally the first (left); the other one is the second (right).

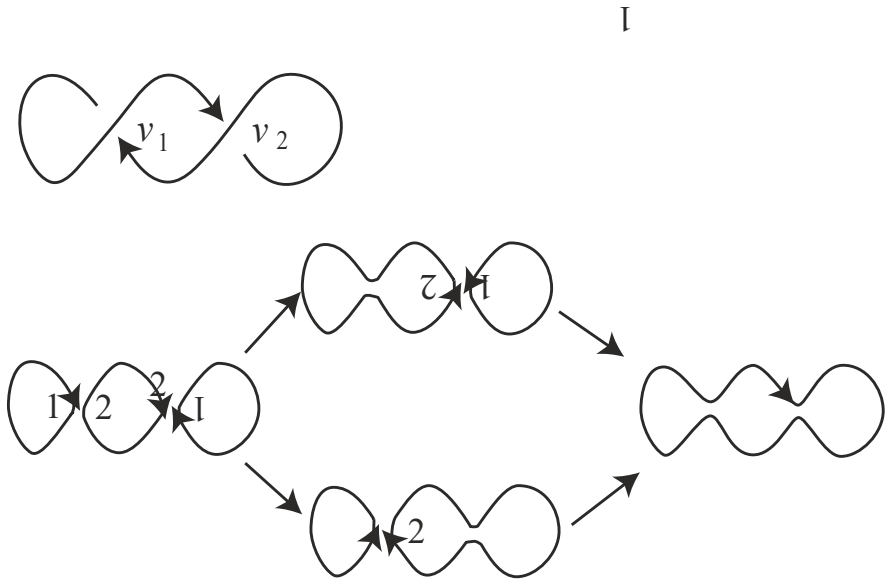


Figure 17: An orientable two-vertex atom.

Here we have to take into account the global ordering of the components. Note that for three components, we always have to apply $m \wedge Id$ first, taking those components to be multiplied with the first and second positions.
Thus, $m \circ (m \wedge Id)$ applied to $A_1 \wedge A_2 \wedge A_3$ gives us $m(m(A_1, A_2), A_3) = -(A_1 \cdot A_2 \cdot A_3)$; here $\cdot$ means the usual multiplication in Khovanov's sense:
$X \cdot X = 0$; $X \cdot 1 = 1 \cdot X = X$; $1 \cdot 1 = 1$. Here the minus sign appears at the second crossing; we have two branches oriented downwards; thus, the rightmost circle occurs to be locally the left one.
On the other hand, if we consider the second crossing first, we get $A_1 \wedge A_2 \wedge A_3 = (A_2 \wedge A_3) \wedge A_1 = -(A_3 \wedge A_2) \wedge A_1 \rightarrow -(A_2 \cdot A_3) \wedge A_1 = A_1 \wedge (A_2 \cdot A_3)$. Applying m to that, we get $A_1 \cdot A_2 \cdot A_3$.
All other atoms are checked analogously. Note that our setup gives directly an anticommutative cube, unlike the Khovanov original setup, where we got an anticommutative cube from a commutative one by adding some minus signs on edges. Thus, Theorem 7.7 is proven. Therefore, Theorem 7.5 is also proven.

Proof of Theorem 7.8

Let us prove Theorem 7.8.

Remark 7.11

Throughout the rest of the proof of Theorem 7.8, we shall not care about height and degree shifts. The proof of their coincidence for diagrams differed by Reidemeister moves repeats verbatim that in the classical case.

First, note that the complex $\mathcal{C}(\mathrm{K})$ itself does not change at all if we perform the detour move. Therefore, the homology does not change.
In the case of classical Reidemeister moves, the proof goes along the line of the proof for classical links.
Let us be more specific. The case of the first Reidemeister move is evident (see the main theorem in Lecture 12).
As in the case of the first Reidemeister move, the invariance under the second Reidemeister move repeats the proof given in the classical case (see the main theorem in Lecture 12).

proof

In addition to the classical case, we should pay attention to orientations of circles when we prove the invariance under the second Reidemeister move. But for the second Reidemeister move, we can choose orientations of all circles incident to a given crossing locally agreed (such that under passing along one circle from one crossing to the other one the variable X does not change the sign); see Fig. 18.

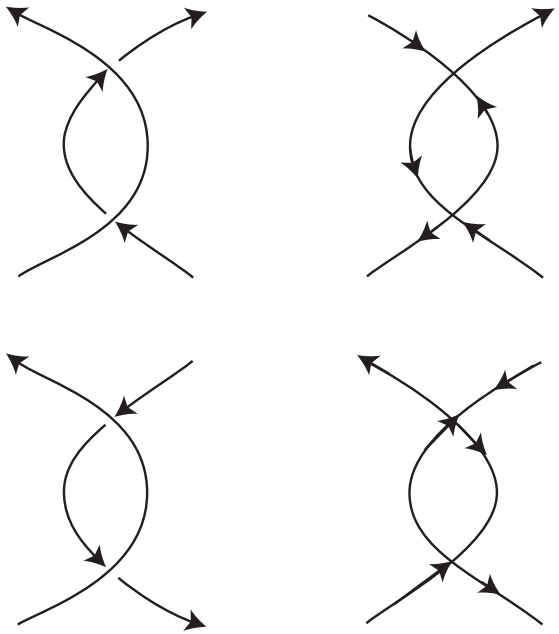


Figure 18: Orientations of upper-right agrees for Ω_2.

Let us now consider the third Reidemeister move shown in Fig. 19 (page 114).
It is well known (see, e.g. [36]) that any variant of the third Reidemeister move can be obtained as a composition of Ω_1, Ω_2 and one prefixed version of the third Reidemeister moves, in which a choice for over/undercrossing and orientations of edges is chosen.
Consider only one case, shown in Fig. 21, with crossing smoothings as in Fig. 19.
At any crossing in Fig. 21 there is a local rule for orientations for all edges incident to it, according to the rule shown in Fig. 8. If two crossings are adjacent, the orientation might or might not be coordinated. We see that the orientation (defined according to Fig. 8) in the third crossing (left picture) does not agree with the orientations in the first and second crossings analogously; for the right picture, the second crossing disagrees with the first one and with the third one.
Note that the rule in Fig. 8 does not depend on types of crossings, but does depend on the orientations of branches.

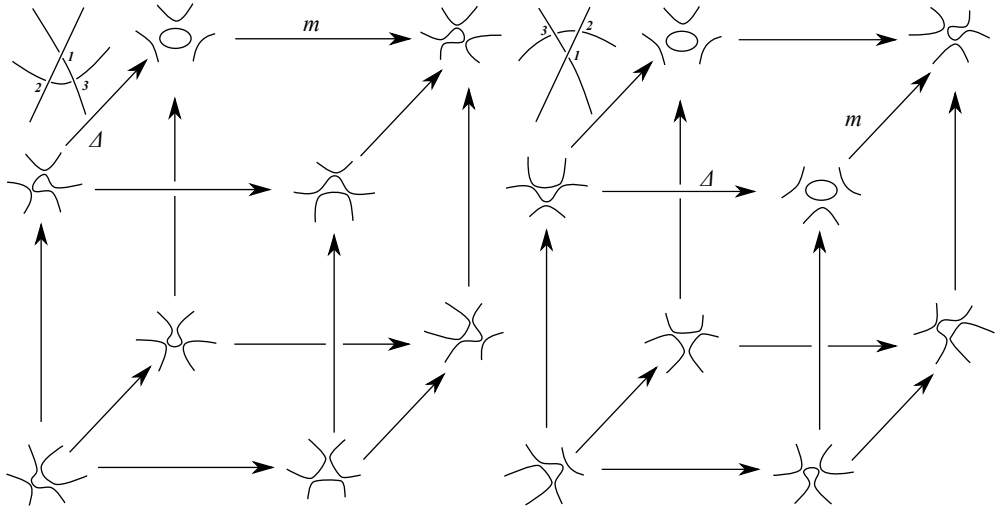


Figure 19: Behaviour of Khovanov's complex under Ω_3

Proof

Apply virtualisations to crossings 1, 2 of the first diagram and to the second crossing of the second diagram; after that, all local orientations (in the sense of variable X) will be coordinated; see Fig. 20.

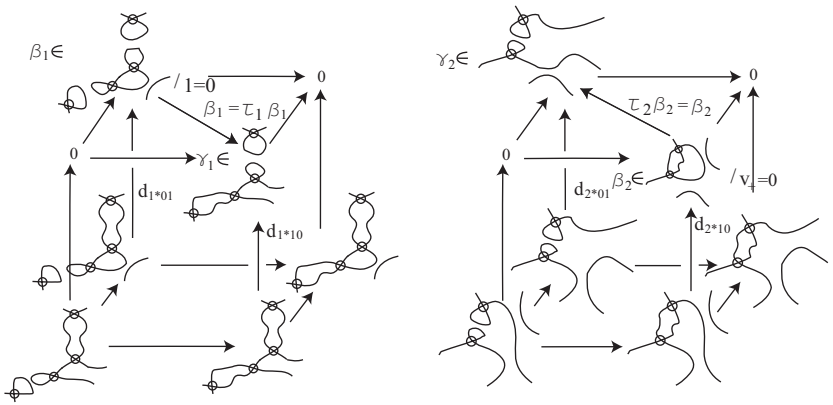


Figure 20: The diagrams after the virtualisation.

The positive smoothings at crossing 1 are the same (up to virtualisations) for both diagrams. The negative smoothing of them gives rise to two pictures obtained one from another by a sequence of (virtualisations and) two classical Reidemeister moves.

proof

Thus, the complexes of the two diagrams in question can be rearranged to have coinciding bottom levels, and top levels having the same homology (in both cases we applied Ω_2).

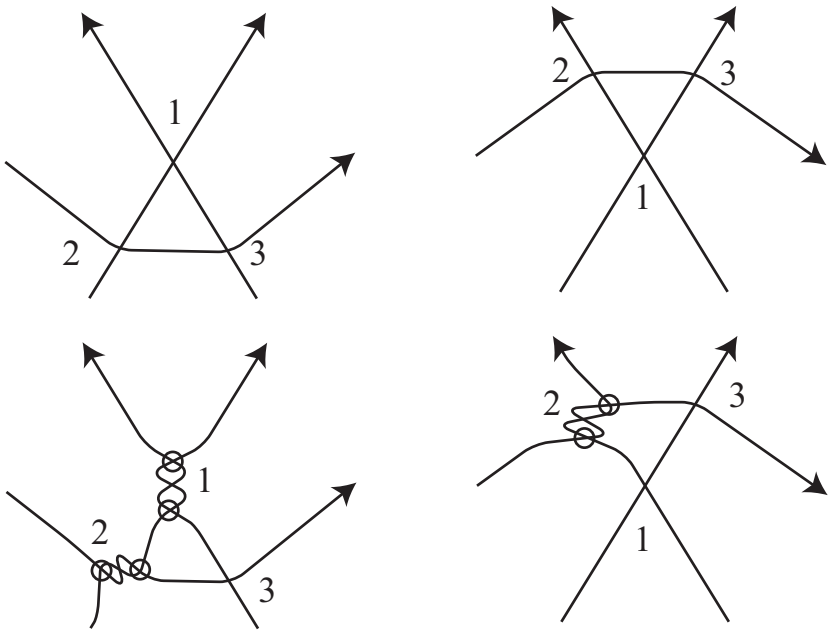


Figure 21: Virtualizing crossings under Ω_3 to make all bases agree.

The main thing to check is that the differentials going upwards agree for these complexes; i.e. the "upwards" maps in both cases either coincide or differ by a sign. These complexes are shown in Fig. 22.
In our situation, the only difference from the classical case which may occur is that they differ by a minus sign (because of ordered tensor products taken instead of the usual tensor products).
In the classical case the final complexes (after factorising) have the form shown in Fig. 22. In Fig. 22 the virtualisation applied by us in Fig. 21 is not designated. The picture shows only what circles are transformed, but does not show what circle is the first at a crossing, and what circle is the second (for this it is necessary to take into consideration the virtualisation in Fig. 21.

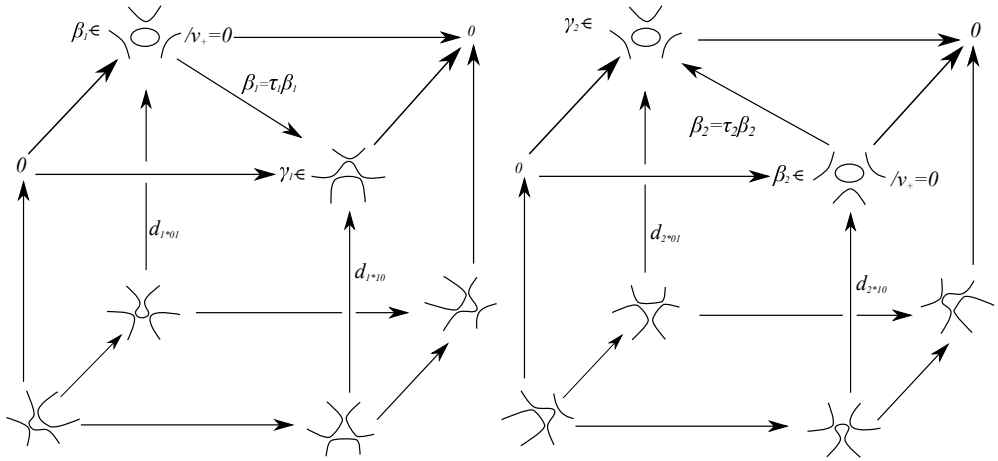


Figure 22: Invariance under Ω_3

Here $1 = 0$ ($v_+ = 0$) in the left upper corner of Fig. 22 means that the space corresponding to the given state is factorised by the subspace where the small circle is marked by 1. Here τ_1 and τ_2 are not differentials; they are chain maps taking an element to the element which is minus homologous to the initial one.

To establish the isomorphism in homology, it is sufficient to show that $\tau_1 \circ d_{1*01} = d_{2*01}$ and $d_{1*10} = \tau_2 \circ d_{2*10}$. In this case we shall show that all the maps "upwards" in both complexes differ by a sign (since in both cases τ_i is minus the identity in homology). After that the homotopy equivalence of the two complexes corresponding to the third Reidemeister move is proved as in Lemma 7.9: By means of a natural map that identifies lower subcubes and minus that map that corresponds to the complex which the upper subcube is reduced to. The ordered tensor product case differs from the usual one, possibly, by signs on edges.

Let us check that the signs agree in our setup. We shall show that $\tau_1 \circ d_{1*01} = d_{2*01}$ (the remaining case $d_{1*10} = \tau_2 \circ d_{2*10}$ is completely analogous).
Let us view Fig. 22 and take into account the virtualisation of the right and left diagrams at crossings. The required identity will look like $p = q \circ \Delta^{-1} \circ \Delta$; see Fig. 23.

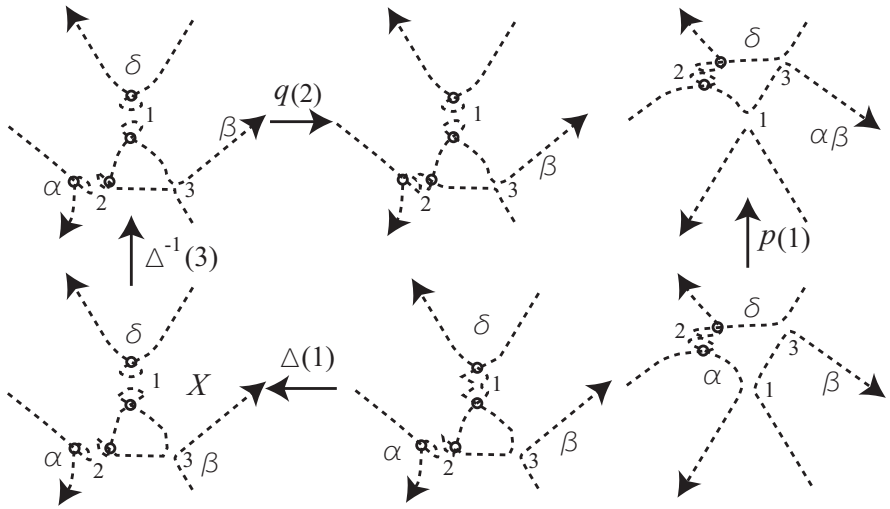


Figure 23: Checking the invariance under Ω_3.

Here d_{1*01} is a $1 \to 2$-bifurcation (we denoted it by Δ); $\tau_1 = \nu \circ \Delta^{-1}$, where ν is a partial differential and Δ^{-1} is assumed as an operation inverse to Δ (note that the space in the upper-left corner in which the element β_1 stays is factorised by $1 = 0$; i.e. the space associated with the small circle C, is one-dimensional with generator X). Then, the comultiplication map for which C is a resulting circle becomes an isomorphism.

View Fig. 23. For each of the maps in the brackets the number of a crossing is indicated which this map is applied to.

The maps p and q are just the usual local differentials, either both multiplications, or both comultiplications, or both zeros.

If $p = q = 0$, there is nothing to prove.

Consider the remaining cases. We have three fragments of circles α, β, δ. In the very initial state (which the map p in the right picture and Δ in the left picture are applied to) they may belong to one, two or three different circles. We shall first consider the case when all fragments containing α, β, δ belong to three different circles.
For simplicity we denote the elements of the algebra V (of type 1 or $\pm$X) related to these circles, by the same letters as fragments α, β, δ.
In our case, both operations p and q are multiplications.
Starting with $\alpha \wedge \beta \wedge \delta$, we get on the right picture the map d_{2*01}:

$$p\colon \alpha \wedge \beta \wedge \delta \to (\alpha \cdot \beta) \wedge \delta,$$

where $(\alpha \cdot \beta)$ means an ordinary product in the Frobenius algebra.

On the left picture we have:

$$\alpha \wedge \beta \wedge \delta = \delta \wedge \alpha \wedge \beta \stackrel{\Delta}{\to} \delta \wedge \mathrm{X} \wedge \alpha \wedge \beta.$$

Here we applied the comultiplication to δ to get two circles at the crossing number 1; the two resulting circles are denoted by δ (the upper one) and X (the lower one).
Now, $\delta \wedge \mathrm{X} \wedge \alpha \wedge \beta = -\beta \wedge \mathrm{X} \wedge \alpha \wedge \delta$. We then perform Δ^{-1} at crossing 3. This map joins the two circles marked by β and X.
At this crossing the generator X is related to the left circle, and β is related to the right circle. Thus, we have

$$-\beta \wedge \mathrm{X} \wedge \alpha \wedge \delta = \mathrm{X} \wedge \beta \wedge \alpha \wedge \delta \stackrel{\Delta^{-1}}{\to} \beta \wedge \alpha \wedge \delta.$$

Now, the operation q is the comultiplication at crossing 2, where the circle marked by β is the first (upper), and the one marked by α is the second one (lower). Thus, we get: $(\alpha \cdot \beta) \wedge \delta$.

Now assume that α and β form one circle (in the initial state), and δ forms a separate circle. Denote the mark (an element from V) corresponding to the first circle by A, and the mark corresponding to the second circle by δ.
The map p looks like:

$$A \wedge \delta \stackrel{\Delta}{\to} \sum_i A_{i,1} \wedge A_{i,2} \wedge \delta,$$

where $\sum_i A_{i,1} \otimes A_{i,2}$ is the result of application of the comultiplication to A in the ordinary sense (in the case of unordered tensor product); see Fig. 24.

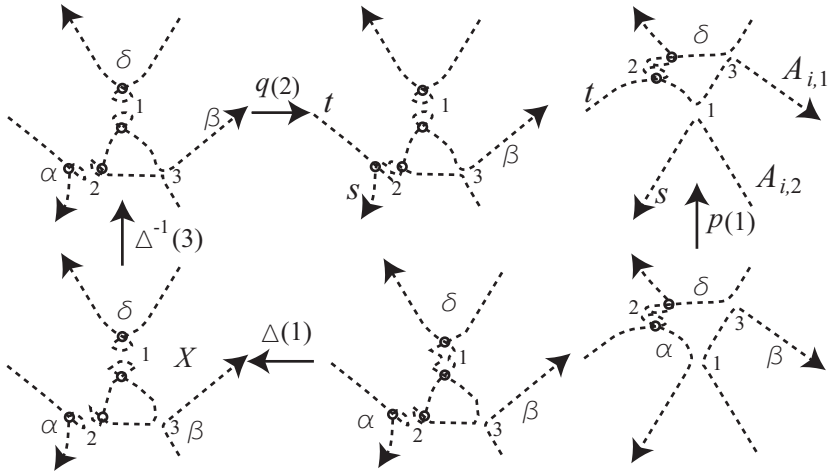


Figure 24: Checking the invariance under Ω_3.

In the further proof for simplicity of writing we shall not use the sum sign $\sum_i$.
In the left picture we have

$$A \wedge \delta = -\delta \wedge A \to -\delta \wedge X \wedge A$$

(at the first crossing the marking δ corresponds to the upper circle and X corresponds to the lower circle).
Then for the map Δ^{-1} at crossing 3 we have

$$-\delta \wedge X \wedge A = -X \wedge A \wedge \delta \to -A \wedge \delta$$

(here X was on the left side, and A was on the right side).
Finally, the map q at crossing 2 gives us

$$-A \wedge \delta \to -A_{i,1} \wedge A_{i,2} \wedge \delta.$$

Here $A_{i,1}$ corresponds to the locally upper component s at crossing 2, and $A_{i,2}$ is locally lower component t. But, in the right picture they have opposite ordering. More precisely, we have

$$-A_{i,1,s} \wedge A_{i,2,t} \wedge \delta.$$

In the first case (the map p) we had

$$\mathrm{A_{i,1,t}} \wedge \mathrm{A_{i,2,s}} \wedge \delta = -\mathrm{A_{i,2,s}} \wedge \mathrm{A_{i,1,t}} \wedge \delta.$$

These two results coincide because of cocommutativity of Δ in the ordinary case.
One can consider the remaining cases analogously.
Suppose that α and δ belong to one circle (the corresponding element being denoted by α), and β belongs to another circle. Then we have the following maps.
In the simplest case (the map p) we have

$$\alpha \wedge \beta \to (\alpha \cdot \beta).$$

On the left picture we have

$$\alpha \wedge \beta \to \alpha \wedge \mathrm{X} \wedge \beta = \mathrm{X} \wedge \beta \wedge \alpha \to \beta \wedge \alpha \to (\beta \cdot \alpha).$$

Consider the case of multiplication when β and δ form one circle (the corresponding element being denoted by β). We get:

$$\alpha \wedge \beta \to (\alpha \cdot \beta)$$

on the right picture (the map p) and

$$\alpha \wedge \beta = -\beta \wedge \alpha \to -\beta \wedge \mathrm{X} \wedge \alpha = \mathrm{X} \wedge \beta \wedge \alpha \to \beta \wedge \alpha \to (\beta \cdot \alpha)$$

on the left picture.
Finally, consider the case when at the beginning we have exactly one diagram, we get two comultiplications:

$$\mathrm{A} \to \mathrm{A_{i,1,t}} \wedge \mathrm{A_{i,2,s}}$$

in the simplest case (the map p) and

$$\mathrm{A} \to \mathrm{A} \wedge \mathrm{X} = -\mathrm{X} \wedge \mathrm{A} \to -\mathrm{A} \to -\mathrm{A_{i,1,s}} \wedge \mathrm{A_{i,2,t}} = \mathrm{A_{i,2,t}} \wedge \mathrm{A_{i,1,s}}.$$

Thus, we have proved the equality $\tau_1 \circ \mathrm{d}_{1*01} = \mathrm{d}_{2*01}$. The proof of the equality $\mathrm{d}_{1*10} = \tau_2 \circ \mathrm{d}_{2*10}$ is completely analogous.

Theorem 7.12

Let K be a virtual diagram for which the corresponding atom is orientable. Then the homology Kh(K) coincides with the Khovanov homology constructed in Lemma 4.2.

During the proof of this theorem, we denote our complex and our homology by $\mathcal{C}(K)$ and Kh(K), respectively, and the ones constructed in Section Khovanov homology of double knots by $\mathcal{C}'(K)$ and $Kh'(K)$ respectively.

Proof of Theorem 7.12

First we note that the shifts for $\mathcal{C}$ and $\mathcal{C}'$ are performed in the same manner. Thus, we can forget about additional normalizations of type $[-n_-]\{n_+ - 2n_-\}$.
First, we assume the diagram of K is chosen in such a way that all X's for all crossings and circles agree (that is, for a given state circle, while passing from one classical crossing P to another one Q, we get $X_{C,o_P} = X_{C,o_Q}$, not $X_{C,o_P} = -X_{C,o_Q}$). This is possible since the atom corresponding to K is orientable. Indeed, since the atom corresponding to K is orientable, we can globally define the orientation of all edges to be compatible with the orientation of the circles in each state. At each crossing of K this orientation may agree or disagree with the local orientation of edges determined by Fig. 8 (the orientation originates from the source–sink structure). Let us apply the virtualisation to all crossings of K where these orientations disagree. By Lemma 7.9, the homology of the complex $\mathcal{C}(K)$ remains the same, and the orientations of circles given locally at crossings according to the rule in Fig. 8 become compatible.

After that, we should just care about signs of local differential and enumeration of circles for any crossing.
We construct a homology-preserving map between two cubes. Fix an enumeration of the classical crossings of K. Let us associate the spanning tree for the cubes $\mathcal{C}(\mathrm{K})$ and $\mathcal{C}'(\mathrm{K})$ as follows. This tree consists of all edges of the form $(\alpha_1, \ldots, \alpha_l, *, 0, \ldots, 0), \alpha_j \in \{0, 1\}$; i.e. an edge in the direction x_{l+1} belongs to this tree if all the coordinates of $x_{l+2}, \ldots, x_n$ vanish; see Fig. 25.

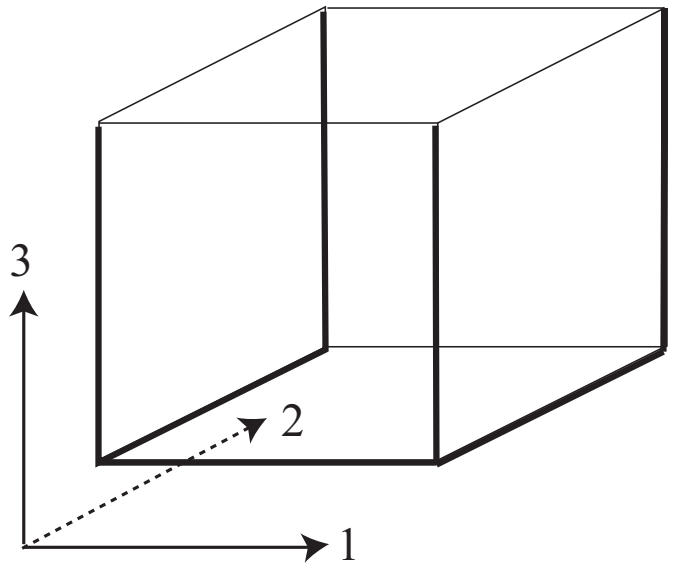


Figure 25: Choosing a spanning tree.

With each state s of the complex $\mathcal{C}(\mathrm{K})$ we associate the ordered tensor power $\mathrm{V}^{\wedge \mathrm{l}}$, and with the corresponding state for the complex $\mathcal{C}'(\mathrm{K})$ we associate $\mathrm{V}^{\otimes \mathrm{l}}$, where l is the number of circles in the state s. Enumerate the circles in the A-state in some way. Then the ordering determines a map between the space corresponding to the A-state s in $\mathcal{C}(\mathrm{K})$ and the space corresponding to some state g(s) of the complex $\mathcal{C}'(\mathrm{K})$. After that we can successively renumber the circles at all vertices of the tree in order that the identification of the chains in the corresponding states of the complexes $\mathcal{C}(\mathrm{K})$ and $\mathcal{C}'(\mathrm{K})$ commute with the partial differentials acting along the edges of the spanning tree. Thus we have constructed a map between the whole chain space of $\mathcal{C}(\mathrm{K})$ and the chain space of $\mathcal{C}'(\mathrm{K})$.
This map g commutes with all the partial differentials for the following reasons. Let ∂', ∂'' be the partial differentials corresponding to the same edge of the complexes $\mathcal{C}$ and $\mathcal{C}'$. Then we have $\mathrm{g} \circ \partial' = \pm \partial'' \circ \mathrm{g}$.

If the compatibility holds for three of four edges of some two-dimensional face, then it also holds for the fourth edge, since both complexes are anticommutative and no one of the partial differentials is the identical zero.

To complete the proof, we note that all the edges of the cube can be exhausted if we start from the spanning tree and successively add the missing edges of the two-dimensional faces (add the fourth edge provided that we have three).

As it was done in the definitions from the last lecture, we call by the height h(Kh(K)) of the Khovanov homology of a virtual link K the difference between the leading and lowest non-zero quantum gradings of non-zero Khovanov homology groups of the virtual link K. From Theorem 7.12 it follows that the definition given in Sec. 3 (using Khovanov homology for orientable atoms) is agreed with the definition for the ordinary case based on the construction of the present section.

Spanning tree decomposition for Khovanov homology considered in Chapter 7 of [10] (Theorem 7.9) remains valid for virtual links.

Theorem 8.1

The non-normalised Khovanov complex of a diagram K of a virtual link is isomorphic to some complex whose chain group looks like

$$\bigoplus_{s\in\mathcal{V}_1} \mathcal{A}[\beta(s) + w(K_s)]\{\beta(s) + 2w(K_s)\}, \tag{7}$$

where $\mathcal{A}$ is the homology group of the unknot and $\mathcal{V}_1$ is the set of states with one circle.

Note that in the proof of Theorem 8.1 we have not used the fact that a link is classical. Therefore, everything can be generalised word by word for virtual links in the case of the field on which the initial Khovanov complex is well defined.

We assert that this proof fits for all models of the Khovanov complex of virtual knots in those cases when this complex is well defined.

Contents

The Khovanov theory of virtual knots described earlier in this lecture is not unique to what one can get by looking at the Kauffman model and the (anti)commutative state cube. The present section is devoted to a generalisation of the Khovanov theory which uses Frobenius extensions for classical and virtual links.
Below, we show that Khovanov's universal construction $(\mathcal{A}, \mathcal{R})$ works in the case of orientable atoms straightforwardly, and write down the algebraic equations the partial differentials have to satisfy for the case of arbitrary virtual links.

Contents

With each virtual link diagram having an orientable atom, the universal $(\mathcal{R}, \mathcal{A})$-construction associates some bifurcation cube, the bigraded chain space with partial differentials, whose homology leads to an invariant of virtual links (after a normalization).
Here, with the state cube and the bifurcation cube we associate bigraded complexes with tensor powers of the ring $\mathcal{A}$ over the ring $\mathcal{R}$ staying in vertices of the cube; the tensor power corresponds to the number of circles in the given state; partial differentials in these cubes are defined by using m and Δ, and differentials are sums of partial differentials with signs.
From Khovanov's theory [27] it follows that there exists a local proof of the invariance for the universal $(\mathcal{R}, \mathcal{A})$-construction; i.e. there is a number of algebraic steps (equivalences, analogous to the cancellation principle and short exact sequences) which leads to the following.

Let us fix a classical Reidemeister move Ω_i. Then for any classical diagrams K and K′ which differ locally by a Reidemeister move Ω_i, there exists, see ahead, a consequence of algebraic transformations taking $Kh_U(K)$ to $Kh_U(K')$ and not depending explicitly on the behaviour of partial differentials of the Khovanov complexes for K and K′ except for those whose explicit form (μ or Δ) follows from the structure of our Reidemeister move Ω_i.

This argument leads to the fact that the universal $(\mathcal{R}, \mathcal{A})$-construction can be generalised for virtual diagrams with orientable atoms. Namely, given a diagram K with an orientable atom, we can construct the corresponding bifurcation cube with differentials, corresponding to the multiplication and comultiplication operations (with signs) and calculate its homology.

Furthermore, if two diagrams K, K′ have orientable atoms and are obtained from each other by some classical Reidemeister move Ω_i, then according to the principle described above, there is an isomorphism between the graded homologies $Kh_U(K) \cong Kh_U(L')$. Since the universal $(\mathcal{R}, \mathcal{A})$-constructionis tautologically invariant under the detour move (the bifurcationcube does not change), the following analogue of Lemma 4.3 holds.

Lemma 9.1

Let K, K′ be two diagrams with orientable atoms such that K′ differs from K by an application of a detour move or one of the three classical Reidemeister moves. Then $\mathrm{Kh_U(K)} \cong \mathrm{Kh_U(K')}$.

This argument together with Lemmas 4.5, 5.5 yields that the universal $(\mathcal{R}, \mathcal{A})$-construction works for

- the construction of the Khovanov homology theory $\mathrm{Kh_U}$ for framed virtual links by taking the 2l parallel copies;
- the construction of the Khovanov homology theory $\mathrm{Kh_U}$ for virtual knots by taking two-sheeted orienting coverings over the corresponding atoms.
- the construction of the Khovanov homology theory $\mathrm{Kh_U}$ for virtual knots obtained by taking parity projections, see [2].

More precisely, the following theorem holds.

Theorem 9.2

(1) Let l be a natural number. Then $Kh_U(D_{2l}(K))$ is an invariant of the framed virtual link K.
(2) The map $K \mapsto Kh_U(\widetilde{K})$ gives a well-defined invariant for virtual links.

Contents

As we have shown above, for virtual knots with orientable atoms the Khovanov homology with $\mathbb{Z}_2$-coefficients can be defined straightforwardly if we set all partial differentials of type $1 \to 1$ to be zero.

Let us now consider the universal $(\mathcal{R}, \mathcal{A})$-construction , and let us generalise it for the case of virtual knots.

Note that if with each knot we associate a well-defined complex,then the homology of this complex will be automatically invariant under classical Reidemeister moves (according to the locality of the invariance proof) and the detour move (there is nothing to prove in this case).

Thus, we have reduced the problem of finding an extension for the ring $\mathcal{A}$ in order to construct the Khovanov homology theory for arbitrary virtual link diagrams, to the following problem. Find an operator (a homomorphism of $\mathcal{R}$-modules) $\mathfrak{I}\colon \mathcal{A} \to \mathcal{A}$ corresponding to maps of type $1 \to 1$ in such a way that for every virtual diagram the bifurcation cube with partial differentials obtained from m, Δ, $\mathfrak{I}$, is anticommutative.

Thus, we require the commutativity of the cube in order to turn it into an anticommutative cube (just as it was done in the usual case). This problem is purely algebraic. In order to solve it, one has to consider all possible 2-faces of the bifurcation cube for a diagram K; there are finitely many such types (with each face, one associates some atom with two vertices). For each face, one has to check some algebraic conditions for the maps $\mathfrak{I}$, Δ and m.

For the space $\mathcal{A}$, let us take the basis $\{1, \mathrm{X}\}$, and for the space $\mathcal{A} \otimes \mathcal{A}$ we take the basis $\{1 \otimes 1, 1 \otimes \mathrm{X}, \mathrm{X} \otimes 1, \mathrm{X} \otimes \mathrm{X}\}$.
Then in these bases the maps Δ and m are represented by the following matrices:

$$\Delta = \begin{pmatrix} -\mathrm{h} & \mathrm{t} \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{m} = \begin{pmatrix} 1 & 0 & 0 & \mathrm{t} \\ 0 & 1 & 1 & \mathrm{h} \end{pmatrix}.$$

After that we shall use the sign of matrix multiplication instead of the composition of the operators. So, for example, we write $\mu \cdot \Delta$ instead of $\mu \circ \Delta$. One of the particular cases given here, is considered in detail in [39].

We look for a matrix

$$\mathfrak{I} = \begin{pmatrix} \mathrm{p} & \mathrm{q} \\ \mathrm{r} & \mathrm{s} \end{pmatrix},$$

which corresponds to bifurcations of type $1 \to 1$ and gives, at the same time, the (anti)commutativity of the bifurcation cube.
Let a coefficient ring $\mathcal{R}$ containing elements h and t with gradings 2 and 4, respectively, be given. Denote the obtained bifurcation cube by $[[\mathrm{K}]]_{\mathcal{R}}$. Let us define the differential as the sum of the partial differentials corresponding to edges (of type m, Δ, $\mathfrak{I}$) with signs arranged as it was done in previous lectures.

Lemma 9.3 and its proof

Lemma 9.3

The bifurcation cube $[[\mathrm{K}]]_{\mathcal{R}}$ is anticommutative if and only if the following properties hold:

$$\mathrm{m} \cdot \Delta = (\mathfrak{I})^2,$$

$$\Delta \cdot \mathfrak{I} = (\mathfrak{I} \otimes 1) \cdot \Delta = (1 \otimes \mathfrak{I}) \cdot \Delta, \tag{8}$$

$$\mathfrak{I} \cdot \mathrm{m} = \mathrm{m} \cdot (\mathfrak{I} \otimes 1) = \mathrm{m} \cdot (1 \otimes \mathfrak{I}). \tag{9}$$

To check the (anti)commutativity of the state cube it is necessary for us to consider all possible sorts of faces of the cube. Later on, we disregard additional signs on edges and prove the commutativity.
In the "simple" case where we have the field $\mathbb{Z}_2$ and null-differentials corresponding to bifurcations of type $1 \to 1$, everything was reduced to the "classical" cases, and as well as to the case depicted in Fig. 1.
For the $(\mathcal{R}, \mathcal{A})$-theory we have to check more cases, since maps of type $1 \to 1$ are not assumed to be zero, and the maps m (multiplication) and Δ (comultiplication) are more complicated than in the case of the homology Kh.

Lemma 9.3 and its proof

Each two-dimensional face of the cube represents a collection consisting of four states, see Section Khovanov homology of double knots. When pasing from one state to another, some circles are reconstructed and the others persist. Denote these four states by s_{00}, s_{01}, s_{10} and s_{11} depending on the values of two changing coordinates. Delete "common components" of the states s_{ij}; i.e. those components of the state s_{00} which do not connect to the crossings at which the substitution of the smoothing occurs. Then the given two-dimensional face of the cube will represent some virtual knot and, therefore, the atom corresponding to it. This atom will have exactly two vertices. If the atom is height, then the corresponding diagram is realized by a bifurcation of embedded circles into the plane, thus, the (anti)commutativity of the corresponding face belongs to the number of classical cases checked in [28, 39].

For atoms with disconnected frames the check is obvious. Further, each orientable atom with the connected frame having two vertices is height. Thus, the required verification is reduced to sorting out unoriented atoms with two vertices (all of them by definition are not height). Sorting out these atoms, eventually we shall come to relations which are satisfied identically, e.g. $\mathfrak{I} \circ \mu = \mathfrak{I} \circ \mu$; see Fig. 26. Three atoms giving non-trivial relations pointed out in the claim of the lemma are given in Figs. 27, 28

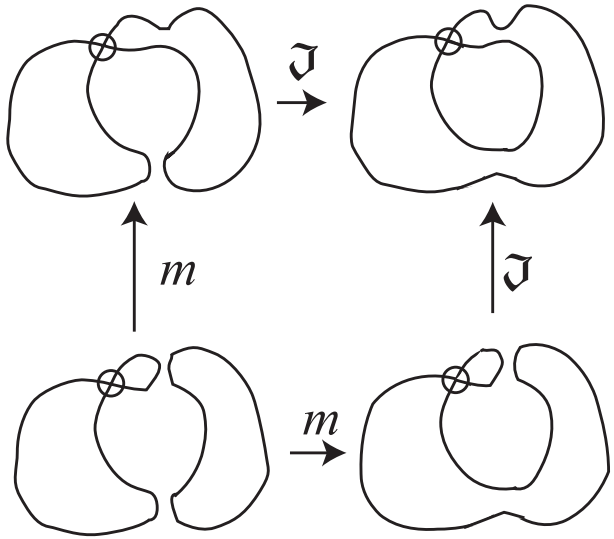


Figure 26: Bifurcation corresponding to tautological relation.

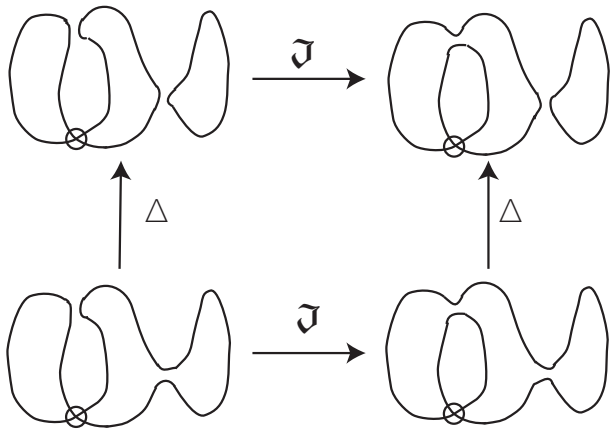


Figure 27: Relations $\Delta \cdot \mathfrak{J} = (\mathfrak{J} \otimes 1) \cdot \Delta = (1 \otimes \mathfrak{J}) \cdot \Delta$.

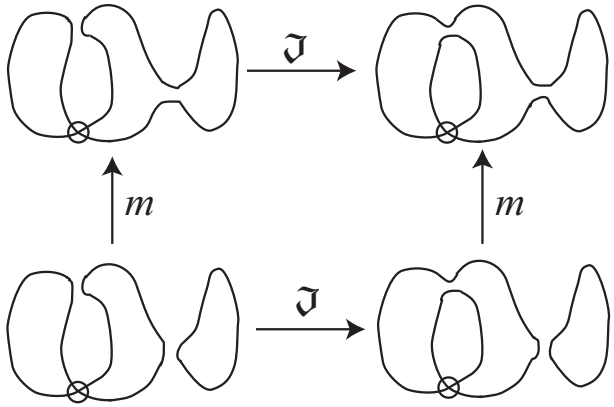


Figure 28: Relations $\mathfrak{J} \cdot \mathrm{m} = \mathrm{m} \cdot (\mathfrak{J} \otimes 1) = \mathrm{m} \cdot (1 \otimes \mathfrak{J})$.

We met the first equation already in the case of the general Khovanov homology $\mathcal{C}$ (there the composition $\mathrm{m} \cdot \Delta$ looks simpler). In the case of the universal $(\mathcal{R}, \mathcal{A})$-theory we have:

$$\mathrm{m} \cdot \Delta = \begin{pmatrix} -\mathrm{h} & 2\mathrm{t} \\ 2 & \mathrm{h} \end{pmatrix}.$$

If we want to construct a $\mathbb{Z}$-graded theory, then it is necessary for us that the matrix $\mathfrak{I}$ increases the grading of elements of the ring $\mathcal{R}$ by one. This means that all elements p, q, r, $\mathrm{s} \in \mathcal{R}$ should be homogeneous. In this case $\deg \mathrm{p} = 1$, $\deg \mathrm{q} = 2$, $\deg \mathrm{r} = 0$, $\deg \mathrm{s} = 1$; herewith it is possible that any of the elements p, q, r, s are equal to zero (in this case the grading is not defined). Then from the equality $(\mathfrak{I})^2 = \mathrm{m} \cdot \Delta$ it follows $\deg (2\mathrm{t}) = \deg \mathrm{t} = 3$, which leads us to a contradiction, if $2 \neq 0$.

Thus (as well as in the case of the general Khovanov homology), under this approach the $\mathbb{Z} \oplus \mathbb{Z}$-bigraded homology theory is possible only in the case of a field of characteristic two.
Let us consider the case of a field of characteristic two. It turns out that in this case we have a simple non-trivial solution. Namely, in the case $2 = 0$ the matrix $m \cdot \Delta$ is turned into the diagonal matrix

$$m \cdot \Delta = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}.$$

Let us add to the ring $\mathcal{R}$ a new element $u = \sqrt{h}$, $\deg u = 1$.
Now set $\mathcal{R}' = \mathbb{Z}_2[u, t]$, herewith the algebra $\mathcal{A}$ takes the form $\mathcal{A}' = \mathcal{R}'[x]/(X^2 - u^2 X - t)$, where $\deg X = 2$, $\deg t = 4$, $\deg u = 1$.

Set

$$\mathfrak{I} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}. \tag{10}$$

In this case the matrix $\mathfrak{I}$ is scalar, and Eqs. (8) and (9) are satisfied automatically.
Thus, we conclude with the following theorem.

Theorem 9.4

Over the field $\mathbb{Z}_2$ the pair of algebras $(\mathcal{R}', \mathcal{A}')$ together with multiplication m, comultiplication Δ defined by $\Delta(1) = 1 \otimes X + X \otimes 1 - u^2 \cdot 1 \otimes 1$, $\Delta(X) = X \otimes X + t1 \otimes 1$ and the scalar map $\mathfrak{I}$ looking like (10), gives an invariant homology theory for virtual links.

In the general case; i.e. in the case of the Khovanov homology for virtual links, we have the following.

Theorem 9.5

The restriction of Khovanov's universal theory for the case $h = 0$ (no restrictions on t) can be extended to virtual links by the method suggested in the section of Khovanov homology for virtual links.

The main idea of the proof of Theorem 9.5 is the following. Δ and m behave nicely under the involution $I\colon 1 \mapsto 1$, $X \mapsto -X$ that takes place while inverting the circle: The multiplication m does not change, and Δ changes the sign. Note that this takes place only for $h = 0$ (for arbitrary t). The case when $h \neq 0$ can be handled by using a more sophisticated twisting.
This generalises straightforwardly for the case when $h = 0$ (where all differentials of type $1 \mapsto 1$ are assumed to be zero). As a particular case, this leads to an analogue of Lee's theory, see [22, 23, 37].

Contents

In the classification and tabulation of (virtual) knots the important step is to describe diagrams having a minimal number of (classical) crossings. One of the main achievements in the development of knot theory is Kauffman–Murasugi–Thistlethwaite theorem and the classification of alternating links by Menasco and Thistlethwaite [35] following from this theorem.
In this section we shall mention theorems establishing the minimality of virtual and classical diagrams, see also [25, 5]. The proofs are analogous to the classical case.

Definition 10.1

By thickness (width) T(K) of the link diagram K we mean the maximum of all $T_R(K)$ over all rings R without additional grading.

The thickness T(K) of a virtual diagram K measures the number of diagonals between the two extreme diagonals in the Khovanov homology of K.

Lemma 10.2

For any diagram K (with a connected atom) of a virtual link we have: $T(K) \leqslant g(K) + 2$, where g(K) is the genus of the atom corresponding to K.

Definition 10.3

Let us call a virtual diagram K 2-complete, if $T(K) = g(K) + 2$.

Now we have the following

Theorem 10.4

Let $T(K) = g + 2$, span $\langle K \rangle = s$. Then the number of classical crossings of a connected diagram of the virtual link generated by K cannot be smaller than $s/4 + g$.
In particular, if a diagram with n crossings and the atom with genus g is 1-complete and 2-complete, then it is minimal.

Theorem 10.4 holds in any category in which the Khovanov complex is well defined and invariant. So, if we are interested in the invariance of a classical diagram in the category of classical diagrams, we can consider the thickness in the classical category.

Contents

Problems

1. Investigate the functoriality of Khovanov homlogy with $\mathbb{Z}$-coefficients under cobordisms
2. Conjecture: can two non-isotopic classical links can be obtained from each other by a finite sequence of generalised Reidemeister moves and virtualisations?

Contents

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