On super-rigidity of Gromov's random monster group

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It is a *finitely generated infinitely presented random group* Γ_α, where α belongs to a probability space. It is constructed as follows:

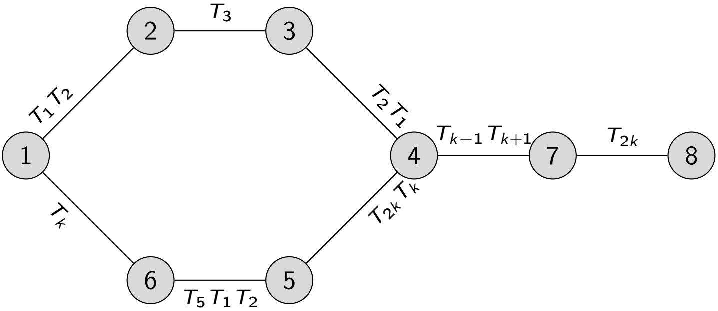
Let F_k be the free group of rank k with a symmetric generating set $T = \{T_1, \cdots, T_{2k}\}$ and Ω be a graph. A symmetric F_k-labelling α of Ω is a map from the edge set of Ω to F_k so that $\alpha(e^{-1}) = \alpha(e)^{-1}$ for every edge e.

Consider $\Omega = \sqcup_{n\in\mathbb{N}}\Omega_n$, where $\{\Omega_n\}_{n=1}^{\infty}$ is a sequence of finite connected graphs and denote by $\mathcal{A}(\Omega, T^j)$ the set of symmetric T^j-labellings of Ω, where T^j is the collection of j-length words consisting of letters from T.

If $\Omega_n = (V_n, E_n)$, $\mathcal{A}(\Omega, T^j)$ can be written as $\prod_{n=1}^{\infty} (E_n)^{T^j}$.

What is Gromov's random monster group?

We then define the group Γ_α as the quotient of F_k by the normal closure of the set of words corresponding to closed loops of Ω_n. We endow $\mathcal{A}(\Omega, T^j)$ with the product measure coming from the uniform measure on $\mathcal{A}(\Omega_n, T^j)$.



The properties of the sequence $\{\Omega_n\}_{n=1}^{\infty}$:

The sequence of finite connected graphs $\{\Omega_n\}_{n=1}^{\infty}$ should have the following properties:

(a) the maximum degree of the vertices of Ω_n is less than equal to d and $|\Omega_n| \to \infty$ as $n \to \infty$;

(b) $\{\Omega_n\}_{n=1}^{\infty}$ is dg-bounded, i.e., there exists $C > 0$ so that $diam(\Omega_n) \leq Cgirth(\Omega_n)$ for all $n \in \mathbb{N}$;

(c) $\{\Omega_n\}_{n=1}^{\infty}$ is of logarithmic girth, i.e., $girth(\Omega_n) \geq Clog|\Omega_n|$ for all n and for some $C > 0$;

(d) $\{\Omega_n\}_{n=1}^{\infty}$ is a sequence of expander graphs (or sometimes $\{\Omega_n\}_{n=1}^{\infty}$ is a p-expander with respect to the metric space l^p, where $p \in (1, \infty)$,i.e., $|V_n| \to \infty$ as $n \to \infty$ and there exists $C > 0$ such that

$$\frac{1}{|V_n|^2} \sum_{u,v \in V_n} d_Y\big(f(u), f(v)\big)^p \leq \frac{C}{|E_n|} \sum_{\overline{uv} \in E_n} d_Y(f(u), f(v))^p,$$

for every $f : V_n \to Y$).

Examples of such sequence $\{\Omega_n\}_{n=1}^{\infty}$:

(1) The first construction is due to Margulis. We consider the Cayley graphs $\{\Omega_p = Cay\big(SL_2(\mathbb{Z}/p\mathbb{Z}), \{A_p, B_p\}\big)\}$, where p runs over all odd primes and A_p and B_p are the following two matrices, respectively:

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{2} & \bar{1} \end{bmatrix}$$

(2) The second construction is the famous Ramanujan graphs, constructed by Lubotzky-Phillips-Sarnak .

What Gromov's random monster group is good for?

(a) If one chooses $\{\Omega_n\}_{n=1}^{\infty}$ to be a sequence of expander graphs, the group Γ_α does not coarsely embed into a Hilbert space, because the sequence $\{\Omega_n\}_{n=1}^{\infty}$ "weakly" embeds in the Cayley graph of Γ_α;

(b) It provides a counter-example for the Baum-Connes conjecture with coefficients in commutative C*-algebra (due to N. Higson, V. Lafforgue, and G. Skandalis).

Known rigidity properties of Gromov's monster: Property (T)

Definition

A discrete countable group Λ has Property (T) if any affine isometric action of Λ on a Hilbert space has a fixed point.

Theorem

(Gromov'03, Silberman'03) Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of connected finite graphs, of vertex-degree between 3 and d for some fixed $d \geq 3$ satisfying girth$(\Omega_n) \to \infty$ as $n \to \infty$ and $\{\Omega_n\}_{n=1}^{\infty}$ is a sequence of expander graphs. Then Γ_α has Property (T) for almost every $\alpha \in \mathcal{A}(\Omega, T^j)$.

Known rigidity properties of Gromov's monster: Property F_{L^p}

Definition

A discrete countable group Λ has Property F_{L^p} if any affine isometric action of Λ on any L^p-space has a fixed point.

Theorem

(Naor-Silberman'11) Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of logarithmic girth p-expanders with respect to l^p-spaces. Then, Γ_α has Property F_{L^p} for almost every $\alpha \in \mathcal{A}(\Omega, T^j)$.

Known rigidity properties of Gromov's monster: hyperbolically rigid

Theorem

(Gruber-Sisto-Tessera'20) Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of connected finite graphs, of vertex-degree between 3 and d for some fixed $d \geq 3$. Assume, $|\Omega_n| \to \infty$ and that there exists $C > 0$ so that $diam(\Omega_n) \leq Cgirth(\Omega_n)$ for all $n \in \mathbb{N}$. Then, for every $j \geq 1$ and almost every $\alpha \in \mathcal{A}(\Omega, T^j)$, we have that Γ_α cannot act non-elementarily on any geodesic Gromov hyperbolic space.

Some digression: Gromov hyperbolic space

Definition

We say a geodesic metric space X is *Gromov hyperbolic*, or δ*-hyperbolic*, if there is a number $\delta \geq 0$ for which every geodesic triangle in X satisfies the δ-slim triangle condition, i.e. any side is contained in a δ-neighbourhood of the other two sides.

Some digression: elementary and non-elementary action

Let G be a group acting isometrically on a Gromov hyperbolic space X. By $\Lambda(G)$ we denote the set of limit points of G on ∂X, the Gromov boundary of X. That is, $\Lambda(G)$ is the set of accumulation points of any orbit of G on ∂X. The possible actions of groups on hyperbolic spaces break into the following 4 classes according to $|\Lambda(G)|$:

(1) $|\Lambda(G)| = 0$. Equivalently, G has bounded orbits. In this case the action of G is called *elliptic*.

(2) $|\Lambda(G)| = 1$. Equivalently, G has unbounded orbits and contains no 'loxodromic' elements. In this case the action of G is called *parabolic*.

(3) $|\Lambda(G)| = 2$. Equivalently, G contains a 'loxodromic' element and any two 'loxodromic' elements have the same limit points on ∂X. In this case the action of G is called *lineal*.

(4) $|\Lambda(G)| = \infty$.

The action of G is called *elementary* in cases (1)-(3) and *non-elementary* in case (4).

What is super-rigidity and hereditary super-rigidity?

Definition

For a countable discrete group G, if for any collection of homomorphisms $\phi_\alpha : \Gamma_\alpha \to G$, ϕ_α has finite image in G for a.e. $\alpha \in \mathcal{A}(\Omega, T^j)$, we say that Γ_α is *super-rigid* with respect to G for a.e. $\alpha \in \mathcal{A}(\Omega, T^j)$.

Definition

For a countable discrete group G, if for any collection of homomorphisms $\phi_\alpha : \Gamma'_\alpha \to G$, ϕ_α has finite image in G for all finite index subgroup Γ'_α of Γ_α and for a.e. $\alpha \in \mathcal{A}(\Omega, T^j)$, we say that Γ_α has *hereditary super-rigidity* with respect to G for a.e. $\alpha \in \mathcal{A}(\Omega, T^j)$.

Known super-rigidity properties of Gromov's monster group

Γ_α has super-rigidity with respect to following groups:

(a) linear groups (due to Naor-Silberman);

(b) groups with a-F_{L^p}-menability;

(c) K-amenable groups.

Definition

A discrete group G has *Property a-F_{L^p}-menability* if there a proper affine isometric action of G on an L^p space for some $p \in (1, \infty)$.

Example

Groups with Haagerup Property, Hyperbolic groups etc.

The proof follows directly from the following three facts:

(a) the image of a group with Property F_{L^p} has Property F_{L^p};

(b) the subgroup of an a-F_{L^p}-menable group is a-F_{L^p}-menable;

(c) a discrete group with Property F_{L^p} and Property a-F_{L^p}-menability is finite.

K-amenable groups

Definition

(A rough definition) Let us consider a discrete countable group G and the epimorphism $\lambda_G : C^*G \to C_r^*G$ induced by the left-regular representation of G, where C^*G is the maximal C^*-algebra of G and C_r^*G is the reduced C^*-algebra of G . A characterization of amenability is that G is amenable if and only if λ_G is an isomorphism. Roughly speaking we say that G is *K-amenable* if λ_G induces isomorphisms in K-theory, i.e.,

$$(\lambda_G)_* : K_i(C^*G) \to K_i(C_r^*)$$

is an isomorphism for $i = 0, 1$.

The proof follows directly from the following three facts:

(a) the image of a group with Property (T) has Property (T);

(b) the subgroup of an K-menable group is K-menable;

(c) a discrete group with Property (T) and K-menability is finite.

(J. Cuntz. K-theoretic amenability for discrete groups. J. reine angew. Math., 344:180-195, 1983.)

Main Theorem

Theorem

(D.'23)
Let $\{\Gamma_\alpha\}_{\alpha\in\mathcal{A}(\Omega,T^j)}$ *be the Gromov's random monster group as defined before. Then, for any collection of homomorphisms* ϕ_α *from* Γ_α *to a discrete countable group* G, ϕ_α *has finite image for almost all* $\alpha\in\mathcal{A}(\Omega,T^j)$ *if* G *is any of the following types of groups:*

(a) *mapping class group* $MCG(S_{g,b})$*;*

(b) *braid group* B_n*;*

(c) *outer automorphism group of a free group* F_N *,* $Out(F_n)$*;*

(d) *automorphism group of a free group* F_N*,* $Aut(F_N)$*;*

(e) *hierarchically hyperbolic group.*

Proof of (a)

Step 1: We will prove by induction on the complexity $\tau(S)$ of a surface S. We define complexity of a surface $S = S_{g,b}$, denoted by $\tau(S)$, by the quantity $(3g - 3 + b)$. $\tau(S) \leq 0$ if and only if S is one of the following surfaces: annulus, sphere, pair of pants, disc or torus. If S is a annulus, sphere, pair of pants or disc, $MCG(S)$ is trivial; if S is a torus, $MCG(S)$ is $SL_2(\mathbb{Z})$. In first case, the theorem is trivially true. Since $SL_2(\mathbb{Z})$ has Haagerup property, we obtain that the theorem is true for second case.

Step 2: We assume that $S_{g,b}$ is any surface with $\tau(S_{g,b}) > 0$ and the theorem is true for all surfaces T with $\tau(T) < \tau(S_{g,b})$. Let $\phi_\alpha : \Gamma_\alpha \to MCG(S_{g,b})$ be a group homomorphism and $H_\alpha = \phi_\alpha(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}(\Omega, T^j)$. Let H_α be infinite. We will prove the theorem by contradiction.

Step 3: Consider the 'curve complex' $C(S_{g,b})$ of $S_{g,b}$. Its 1-skeleton is given by the following data: Vertices - There is one vertex of $C(S_{g,b})$ for each isotopy class of essential simple closed curves in $S_{g,b}$. Edges- There is an edge between any two vertices of $C(S_{g,b})$ corresponding to isotopy classes a and b with geometric intersection number of a and b being zero.

Proof of (c)

Step 4: By Masur-Minsky (in 1999) The curve complex $C(S_{g,b})$ is a δ-hyperbolic metric space, where δ depends on $S_{g,b}$.

Step 5 By Ivanov (in 1992), every subgroup $H \leq MCG(S_{g,b})$ either

- contains two 'pseudo-Anosov diffeomorphisms' of $S_{g,b}$ that generate a rank two free subgroup of H, or
- is virtually cyclic and virtually generated by a 'pseudo-Anosov diffeomorphism', or
- H is reducible.

Step 6 An element ϕ of the mapping class group $MCG(S_{g,b})$ acts loxodromically on $C(S_{g,b})$ if and only if ϕ is 'pseudo-Anosov'.

Proof of (c)

Step 7: Using Property (T) and hyperbolic rigidity of Γ_α, we obtain that H_α is reducible, which implies that H_α fixes a finite, non-empty, collection C of disjoint, non-peripheral, simple closed curves on $S_{g,b}$. Since H_α leaves C invariant, it induces a permutation on the connected components $\{S_1, \cdots, S_n\}$ $(n > 0)$ of $S_{g,b} \setminus C$. Then, there exists a finite index subgroup H_α^0 of H_α which leaves each subsurface S_i invariant (here each S_i is a compact surface with boundary), and there is a homomorphism

$$\psi_\alpha : H_\alpha^0 \to MCG(S_1) \times \cdots MCG(S_n).$$

Step 8: Let $p_i : MCG(S_1) \times \cdots MCG(S_n) \to MCG(S_i)$ be the natural projection for all $i = 1, \cdots, n$. Then $(p_j \circ \psi_\alpha)(H_\alpha^0)$ is infinite for at least one $1 \leq j \leq n$. But, by induction, since $\tau(S_j) < \tau(S_{g,b})$, $(p_j \circ \psi_\alpha)(H_\alpha^0)$ must be finite, a contradiction.

Some digression:stability result

Theorem

(D.'23) Let $1 \to N \xrightarrow{i} G \xrightarrow{q} G/N \to 1$ *be a short exact sequence. If* Γ_α *has hereditary super-rigidity with respect to* N *and super-rigidity with respect to* G/N*, then it has super-rigidity with respect to* G*.*

Proof of (b)

Proof of (d) follows from the stability theorem and the following short exact sequence

$$1 \to \mathbb{Z} \to B_n \to MCG_x(S_{n+1}) \to 1$$

where S_{n+1} is the sphere with $(n+1)$ punctures, $MCG(S_{n+1})$ is the mapping class group of S_{n+1} and $MCG_x(S_{n+1})$ denotes the subgroup of $MCG(S_{n+1})$ which fixes a fixed puncture x.

Proof of (c)

Step 1: Let $\phi_\alpha : \Gamma_\alpha \to Out(F_N)$ be a group homomorphism and $H_\alpha = \phi_\alpha(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}(\Omega, T^j)$.

Step 2: Consider the free factor complex FF_N of $Out(F_N)$. The free factor complex FF_N is an abstract simplicial complex associated to a free group F_N . The set of vertices $V(FF_N)$ of FF_N is defined as the set of all F_N-conjugacy classes $[A]$ of proper free factors A of F_N. Two distinct vertices $[A]$ and $[B]$ of FF_N are joined by an edge whenever there exist proper free factors A, B of F_N representing $[A]$ and $[B]$ respectively, such that either $A \le B$ or $B \le A$.

Step 3: By Bestvina-Feign (in 2014), the free factor complex FF_N is hyperbolic. $Out(F_N)$ acts isometrically on FF_N.

Proof of (c)

Step 5: By Handel-Mosher (in 2009), every subgroup of $Out(F_N)$ (finitely generated or not) either

- contains two 'fully irreducible' automorphisms that generate a rank two free subgroup, or
- is virtually cyclic and virtually generated by a 'fully irreducible' automorphism, or
- virtually fixes the conjugacy class of a proper free factor of F_N .

Step 6: An automorphism $\psi \in Out(F_N)$ acts loxodromically on FF_N if and only if ψ is fully irreducible.

Step 7: By using Property (T) and hyperbolic rigidity of Γ_α, we obtain that H_α virtually fixes the conjugacy class of a proper free factor of F_N. Let $F_N = L * L'$, where L is a proper free factor of F_N and there exist a finite index subgroup H_α^0 of H_α fixing the conjugacy class of L.

Step 8: This situation is similar to the situation of mapping class group. We apply a similar kind of argument to conclude this theorem.

Proof of (d)

Proof of (f) follows from the stability theorem and the following short exact sequence

$$1 \to F_N \to Aut(F_N) \to Out(F_N) \to 1$$

Bibliography

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Thank you!