Branching Brownian motion in a periodic environment and pulsating travelling waves

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Outline

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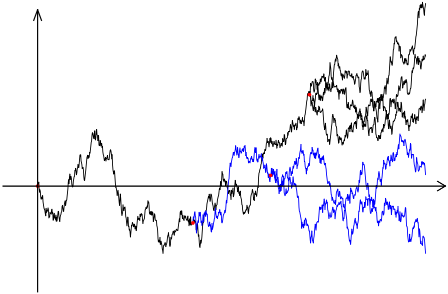
1. Branching Brownian motion and travelling waves
2. BBM in a periodic environment
   - Model
   - Main results
3. Sketch of the proof

Branching Brownian motion (BBM)

- One ancestor particle at the origin at time $t = 0$.
- Motion: standard Brownian motion $B = \{B(t) : t \geq 0\}$.
- Branching rate: $\beta > 0$.
- The number of offspring: $1 + L$, $L \sim \{p_k, k \geq 0\}$.
- Each offspring performs the same behaviors (independently).



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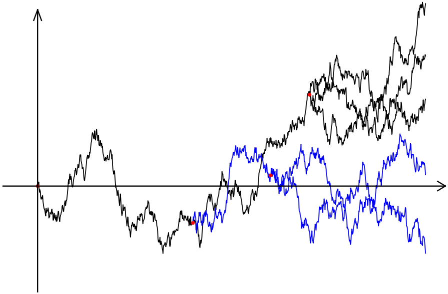

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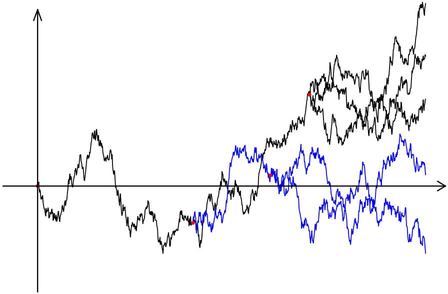

BBM and F-KPP equation

- Fisher (1937), Kolmogorov, Petrovskii and Piskounov (1937): F-KPP equation.
- $u(t,x) = \mathbb{E}_x(\prod_{v \in N_t} h(X_v(t)))$ satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta(f(u) - u)$$

  and $u(0,x) = h(x)$, where $f(s) = \mathbb{E}(s^{1+L})$. N_t is the set of particles at time t, and $X_v(t)$ is the position of the particle v at time t.
- McKean (1975): $h(x) = \mathbf{1}_{\{x>0\}}$. $u(t,x) = \mathbb{P}(M_t \leq x)$ satisfies F-KPP equation.

  M_t: maximal position of BBM at time t.
- Ikeda, Nagasawa and Watanabe (1968a, 1968b, 1969): BBM and F-KPP equation.

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F-KPP equation and travelling waves

- Bramson (1978, 1983): $m_t = \mathrm{med}\, M_t$ (when binary branching and $\beta = 1$) satisfies

$$m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + \text{constant} + o(1).$$

$$\mathbb{P}(M_t \leq \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t + x) \xrightarrow{t\to\infty} \Phi(x).$$

- Lalley and Sellke (1987): a probabilistic representation in terms of the derivative martingale.

- Φ_c (travelling wave solution with speed c) satisfies

$$\frac{1}{2}\Phi_c'' + c\Phi_c' + \beta(f(\Phi_c) - \Phi_c) = 0.$$

and the boundary condition $\lim_{x\to-\infty} \Phi_c(x) = 0, \ \lim_{x\to+\infty} \Phi_c(x) = 1.$

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- Kyprianou (2004): alternative probabilistic proofs and martingale convergence.

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$$c^* := \min_{\lambda>0} c_\lambda = \sqrt{2\beta m}, \qquad \lambda^* := \arg\min_{\lambda>0} c_\lambda = \sqrt{2\beta m}.$$

Theorem (Kyprianou 2004)

The limit $W(\lambda) := \lim_{t\uparrow\infty} W_t(\lambda)$ *exists* $\mathbb{P}$*-a.s.*

(1) If $|\lambda| \ge \lambda^*$*, then* $W(\lambda) = 0$ $\mathbb{P}$*-a.s.*

(2) If $|\lambda| \in [0, \lambda^*)$*, then* $W(\lambda) = 0$ $\mathbb{P}$*-a.s. when* $\mathbb{E}(L\log^+ L) = \infty$ *or* $W(\lambda)$ *is an* $L^1(\mathbb{P})$*-limit when* $\mathbb{E}(L\log^+ L) < \infty$.

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Travelling waves

Subcriticality. When $|c| < c^* = \sqrt{2\beta m}$, travelling waves do not exist.

Criticality. When $|c| = c^*$ and $\mathbb{E}(L(\log^+ L)^2) < \infty$, then there is a unique travelling wave with speed c^* given by

$$\Phi_{c^*}(x) = \mathbb{E}(\exp\{-e^{-\lambda^* x}\partial W(\lambda^*)\}).$$

Asymptotic:

$$1 - \Phi_{c^*}(x) \sim \text{const} \cdot x e^{-\lambda^* x}, \quad \text{as } x \to \infty.$$

Supercriticality. When $|c| > c^*$ and $\mathbb{E}(L \log^+ L) < \infty$ then there is a unique travelling wave with speed $c = c_\lambda$ ($|\lambda| \in [0, \lambda^*)$) given by

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Outline

BBM in a periodic environment (BBMPE)

Model:

- Branching rate: g, where $g \in C^1(\mathbb{R})$ is positive and 1-periodic.

$$\mathbb{P}_x\left(d_v - b_v > t \mid b_v, \{X_v(s) : s \geq b_v\}\right) = \exp\left\{-\int_{b_v}^{b_v+t} g(X_v(s))\mathrm{d}s\right\},$$

  where b_v is birth time of v, d_v is death time.

- Offspring: $1 + L$, $L \sim \{p_k, k \geq 0\}$.

Lubetzky, Thornett and Zeitouni (2022): studied the the maximum of BBMPE with binary branching.

$$\lim_{t\to\infty} \left|\mathbb{P}(M_t \leq m_t + y) - \mathbb{E}[\exp\{-\Theta b(m_t + y)e^{-\lambda^* y}\}]\right| = 0,$$

where Θ is a random variable and $b \in C(\mathbb{R})$ is positive and 1-periodic.

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F-KPP equation and the pulsating travelling waves

The F-KPP equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + g(x)(f(u) - u).$$

Pulsating travelling waves to the F-KPP equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + g \cdot (f(u) - u), \\ u(t + \frac{1}{\nu}, x) = u(t, x - 1). \end{cases}$$

as well as the boundary condition

$$\lim_{x \to -\infty} u(t,x) = 0, \quad \lim_{x \to +\infty} u(t,x) = 1, \text{ when } \nu > 0,$$

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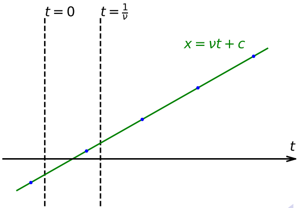

Pulsating travelling waves

- Hamel (2008): The monotonicity and exponential decay of pulsating travelling fronts for the following corresponding equation (A, q, $f(\cdot, u)$ periodic)

$$u_t - \nabla \cdot (A(z)\nabla u) + q(z) \cdot \nabla u = f(z, u).$$

- Hamel and Roques (2011): uniqueness of pulsating travelling fronts.
- Hamel, Nolen, Roquejoffre and Ryzhik (2016): generalize Bramson's results to the periodic case.

Eigenvalue and eigenfunction

As in Hamel et al. (2016), let $\gamma(\lambda)$ and $\psi(\cdot,\lambda)$ be the principal eigenvalue and positive eigenfunction of the periodic problem

$$\begin{cases} \frac{1}{2}\psi_{xx} - \lambda\psi_x + (\frac{1}{2}\lambda^2 + mg(x))\psi = \gamma(\lambda)\psi, \\ \psi(x+1,\lambda) = \psi(x,\lambda). \end{cases}$$

Let

$$\nu^* := \min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}, \quad \lambda^* := \arg\min_{\lambda>0} \frac{\gamma(\lambda)}{\lambda}.$$

Main results

$$W_t(\lambda) := e^{-\gamma(\lambda)t} \sum_{v \in N_t} e^{-\lambda X_v(t)} \psi(X_v(t), \lambda),$$

$$\partial W_t(\lambda) := e^{-\gamma(\lambda)t} \sum_{v \in N_t} e^{-\lambda X_v(t)} (\psi(X_v(t), \lambda)(\gamma'(\lambda)t + X_v(t)) - \psi_\lambda(X_v(t), \lambda)).$$

Theorem 1 (Ren, Song and Y. 2022+)

For every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{W_t(\lambda) : t \geq 0\}$ is a martingale. The limit $W(\lambda, x) := \lim_{t \uparrow \infty} W_t(\lambda)$ exists $\mathbb{P}_x$-a.s.

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Theorem 2 (Ren, Song and Y. 2022+)

For every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}$, $\{\partial W_t(\lambda) : t \geq 0\}$ is a martingale. For all $|\lambda| \geq \lambda^*$, $\partial W(\lambda, x) := \lim_{t\uparrow\infty} \partial W_t(\lambda)$ exists $\mathbb{P}_x$-a.s.

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Main results

Theorem 3 (Ren, Song and Y. 2022+)

(i) Suppose $u(t,x)$ is a pulsating travelling wave with speed $\nu > \nu^*$ and $\lambda \in (0, \lambda^*)$ satisfies $\nu = \frac{\gamma(\lambda)}{\lambda}$. If $\mathbb{E}(L \log^+ L) < +\infty$, then there exists $\beta > 0$ such that

$$1 - u\left(\frac{y-x}{\nu}, y\right) \sim \beta e^{-\lambda x} \psi(y, \lambda) \text{ as } x \to +\infty \text{ uniformly in } y \in [0,1].$$

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Main results

Theorem 4 (Ren, Song and Y. 2022+)

(i) **Supercriticality case.** If $|\nu| > \nu^*$ and $\mathbb{E}(L \log^+ L) < \infty$, then there is a unique pulsating travelling wave with speed ν given by

$$u(t,x) = \mathbb{E}_x\left(\exp\left\{-e^{\gamma(\lambda)t}W(\lambda,x)\right\}\right),$$

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Outline

Spine decomposition

- $\Xi_t(\lambda) := e^{-\gamma(\lambda)t - \lambda B_t + m\int_0^t g(B_s)\mathrm{d}x}\psi(B_t, \lambda)$, a Π_x-martingale.
- $W_t(\lambda) = e^{-\gamma(\lambda)t}\sum_{v\in N_t} e^{-\lambda X_v(t)}\psi(X_v(t), \lambda)$.
- Define

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\lambda}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{W_t(\lambda)}{W_0(\lambda)}.$$

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Under $\widetilde{\mathbb{P}}_x^{\lambda}$:

- A distinguished and randomized spine.
- Particles on the spine:

  Motion $X_\xi(t)$: diffusion process. Infinitesimal generator,

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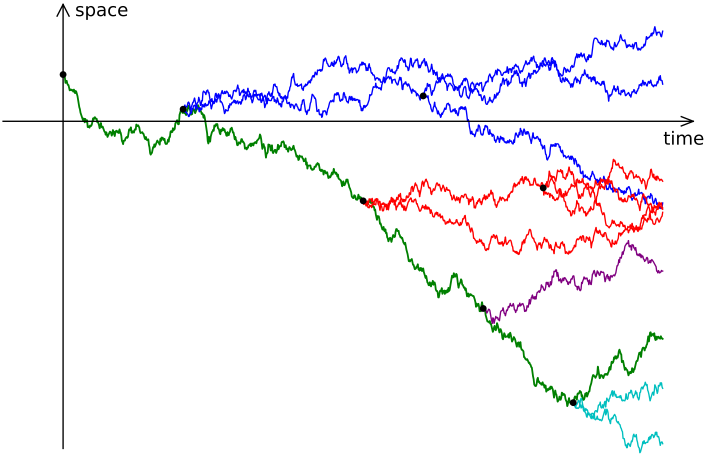

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We adapt the ideas from Harris (1999).

Supercritical case ($\lambda < \lambda^*$):

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- $v(-t, Y_t)$ converges to some limit ξ_x Π_x^λ-a.s.
- Coupling method: $\exists$ a constant $\beta \geq 0$ such that $\xi_x = \beta$ for $\forall x \in \mathbb{R}$.
  - $\{(Y_t^1, Y_t^2), \widetilde{\Pi}_{(x,y)}^\lambda\}$ with $\{Y_t^1\}$ and $\{Y_t^2\}$ being independent.
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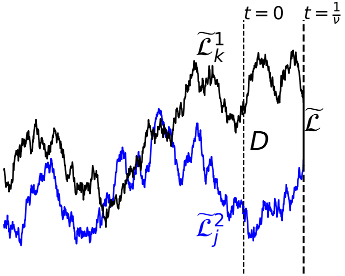

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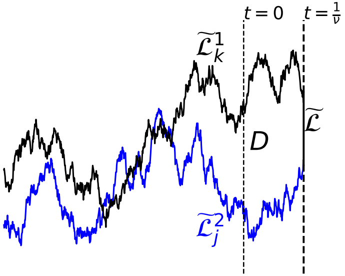

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For the general branching mechanism:

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- For any $r, c \in (0,1)$, $\sum_{n=0}^\infty A(cr^n) < +\infty$ iff $\mathbb{E}(L \log^+ L) < +\infty$ (Athreya and Ney, 1972, *Branching processes*).

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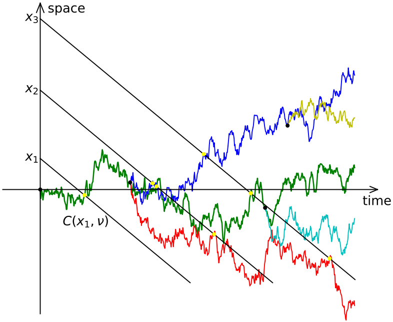
Uniqueness



Stopping lines

- $C(x,\nu)$: the random collection of particles stopped at the line $y+\nu t=x$. $(\nu \geq \nu^*)$
- Let $Y_x = \sum_{v\in C(x,\nu)} \delta_{\{X_v(\sigma_v)\}}$.

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$$M_x(\nu) := \prod_{v\in C(x,\nu)} u(-\sigma_v, X_v(\sigma_v)) = e^{\langle Y_x, \log u(\frac{\cdot - x}{\nu}, \cdot)\rangle}.$$
- Then $\{M_x(\nu) : x \geq z\}$ is a $\mathbb{P}_z$-martingale.
- Chauvin (1991): product martingales for BBM.

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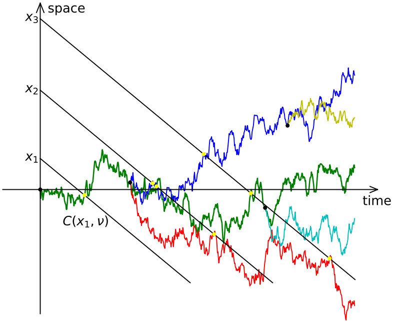

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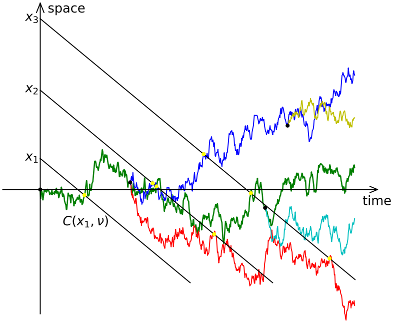

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Define

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$W_{C(x,\nu)}(\lambda)$ converges to $W(\lambda, z)$ when $|\lambda| \in [0, \lambda^*)$ and $\mathbb{E}(L \log^+ L) < \infty$.

By the dominated convergence theorem,

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Kyprianou (2004): branching rate is constant

- $\psi \equiv 1$, $W_{C(x,\nu)}(\lambda) = e^{-\lambda x}\langle Y_x, 1\rangle$, $M_x(\nu) = e^{\langle Y_x, 1\rangle \log \Phi(x)}$.
- $\langle Y_x, 1\rangle$ continuous time branching process.
- $1 - \Phi(x) \sim \mathrm{const} \cdot e^{-\lambda x}$ as $x \to \infty$.

Further questions

- Kyprianou, Liu, Murillo-Salas and Ren (2012): super-Brownian motion with branching mechanism

$$\phi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) n(\mathrm{d}r).$$

- Probabilistic proofs for the existence, asymptotics and uniqueness of travelling wave solutions

$$\frac{1}{2}\Phi_c'' + c\Phi_c' - \phi(\Phi_c) = 0.$$

- Consider super-Brownian motion with a periodic branching mechanism

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Thank you!