MATH 283A Topics in Topology

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This document contains course notes from MATH 283A (taught at Stanford, spring 2020, by Ciprian Manolescu) transcribed by Shintaro Fushida-Hardy. Some additional thoughts and mistakes may have been added.

Contents

Chapter 1

Classification of 4-manifolds

1.1 Admin (lecture 1)

Topics in the course will include:

1. topological 4-manifolds: Freedman's classification (without proof)
2. presentations of smooth 4-manifolds: Kirby diagrams, trisections;
3. spin^c structures, the Dirac operator, the Seiberg-Witten equations;
4. applications of gauge theory: exotic smooth structures, Donaldson's diagonalizability theorem, the Thom and Milnor conjectures;
5. (time permitting) Khovanov homology and the combinatorial proofs of the Thom and Milnor conjectures.

There is no official textbook for the course, but the following resources could be useful:

- Robert Gompf and Andras Stipsicz, "4-Manifolds and Kirby Calculus"
- Alexandru Scorpan, "The Wild World of 4-manifolds"
- John Morgan, "The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds"
- John Moore, "Lectures on the Seiberg-Witten Invariants"
- Simon Donaldson and Peter Kronheimer, "The Geometry of Four-Manifolds"

Results from the following research articles will also be discussed:

- Peter Kronheimer and Tomasz Mrowka, "The Genus of Embedded Surfaces in the Projective Plane". Mathematical Research Letters. 1 (1994), 797–808,

- Mikhail Khovanov, , "A categorification of the Jones polynomial", Duke Mathematical Journal, 101 (2000), 359–426
- David Gay and Robion Kirby, "Trisecting 4-manifolds", Geom. Topol. 20 (2016), 3097-3132
- Peter Lambert-Cole, "Bridge trisections in $\mathbb{CP}^2$ and the Thom Conjecture", arXiv:1807.10131

Some highlights from the course are the following three results:

1. There exist smooth homeomorphic 4-manifolds which are not diffeomorphic.
2. The Thom conjecture, proved by Kronheimer and Mrowka in 1994: if $\Sigma \subset \mathbb{CP}^2$ is a smoothly embedded surface, and Σ represents an algebraic curve of degree d, then the genus of Σ is at least $(d-1)(d-2)/2$.
3. The Milnor conjecture, also proven by Kronheimer and Mrowka in the 90s: let $T_{p,q}$ denote the *torus knot* with p twists and q strands. Suppose $\Sigma \subset B^4$ is a smoothly and properly embedded surface, with $\partial\Sigma = \Sigma \cap \partial B^4 = T_{p,q}$. Then the genus of Σ is at least $(p-1)(q-1)/2$.

The original proofs of the above three results were analytic in nature. More precisely, the employed gauge theory - specifically the Yang-Mills and Seiberg-Witten equations.

Newer proofs are referred to as "combinatorial" in the literature, but this is a misnomer. The newer methods are algebraic and topological, without using analysis. A very important tool is Khovanov homology.

1.2 The futility of a full classification of 4-manifolds

The most fundamental and desired result is a classification of all 4-manifolds. Unfortunately, this is hopeless by combining the following two theorems:

Theorem 1.2.1 (Adyan-Rubin, 1955)**.** *There does not exist an algorithm which determines whether a given presentation of a group yields the trivial group.*

Theorem 1.2.2 (Markov, 60s)**.** *Given a finitely presented group G, there exists a smooth closed 4-manifold X with $\pi_1(X) = G$.*

Therefore smooth closed 4-manifolds are at least as complicated as finitely presented groups. It follows that we cannot classify smooth closed 4-manifolds up to homotopy, let alone up to diffeomorphism.

Proof of Markov's theorem. The proof proceeds in a few steps.

Step 1. Given any 4-manifolds X_1 and X_2, $\pi_1(X_1 \# X_2) = \pi_1(X_1) * \pi_1(X_2)$. This follows from Seifert-Van Kampen. Observe that

$$\pi_1(X_i) = \pi_1(X_i - B^4) *_{\pi_1(\mathbb{S}^3)} \pi_1(B^4) = \pi_1(X_i - B^4).$$

Therefore

$$\pi_1(X_1 \# X_2) = \pi_1(X_1 - B^4) *_{\pi_1(\mathbb{S}^3)} \pi_1(X_2 - B^4) = \pi_1(X_1) * \pi_1(X_2).$$

Step 2. Write $G = \langle g_1, \ldots, g_n | r_1, \ldots, r_m \rangle$. Suppose N is the connected sum of n copies of $\mathbb{S}^1 \times \mathbb{S}^3$. Then by step 1, $\pi_1(N) = *_{i=1}^n \mathbb{Z} = \langle g_1, \ldots, g_n \rangle$.

Step 3. Consider any relation r_j. These are represented by a loop $\gamma_i \subset N$. Since any two loops have dimension 1, and $1 + 1 < 4 = \dim N$, by the transversality theorem we can choose the r_j to be pairwise disjoint embedded submanifolds.

Step 4. Surgery on loops: fix a loop $\gamma \subset N$ representing a relation r. This has a tubular neighbourhood, homeomorphic to $\mathbb{S}^1 \times B^3$. Then

$$\partial(N - (\mathbb{S}^1 \times B^3)) = \mathbb{S}^1 \times \mathbb{S}^2 = \partial(B^2 \times \mathbb{S}^2).$$

Therefore the idea is to cut out $\mathbb{S}^1 \times B^3$ and glue in $B^2 \times \mathbb{S}^2$;

$$\widetilde{N} \coloneqq (N - (\mathbb{S}^1 \times B^3)) \sqcup_{\mathbb{S}^1 \times \mathbb{S}^2} (B^2 \times \mathbb{S}^2).$$

Once again we apply Seifert-Van Kampen. Writing $N = (N - (\mathbb{S}^1 \times B^3)) \sqcup_{\mathbb{S}^1 \times \mathbb{S}^2} (\mathbb{S}^1 \times B^3)$, we have

$$\pi_1(N) = \pi_1(N - (\mathbb{S}^1 \times B^3)) *_{\langle r \rangle} \langle r \rangle = \pi_1(N - (\mathbb{S}^1 \times B^3)).$$

Therefore we see that

$$\pi_1(\widetilde{N}) = \pi_1(N - (\mathbb{S}^1 \times B^3)) *_{\langle r \rangle} 1 = \pi_1(N - (\mathbb{S}^1 \times B^3))/\langle\langle r \rangle\rangle = \pi_1(N)/\langle\langle r \rangle\rangle.$$

Since all of the γ_i in step 3 were chosen to be disjoint, the above surgery can be carried out simultaneously on all of the γ_i, giving a closed smooth manifold M with fundamental group $\pi_1(N)/\langle\langle r_1, \ldots, r_m \rangle\rangle = G$. □

Question from class. Where does this proof fail in lower dimensions?

Answer. The surgery above required the use of a four manifold with trivial fundamental group, and boundary $\mathbb{S}^1 \times \mathbb{S}^2$. In three dimensions, one can show that there do not exist manifolds with trivial fundamental group and boundary $\mathbb{S}^1 \times \mathbb{S}^1$ (e.g. by comparing the first Betti number of the manifold to that of the boundary). □

1.3 The intersection form

We've observed that there is no hope of classifying all 4-manifolds, so instead we restrict to those with trivial fundamental group. *What are all of the closed simply connected smooth manifolds of dimension 4?* In this course we usually consider classifications up to diffeomorphism, but sometimes homeomorphism or homotopy are considered. Note that every simply connected manifold is orientable, so no generality is lost in assuming our 4-manifolds are oriented.

Question from class. Will we ever equip X with a metric?

Answer. For the traditional set-up with Seiberg-Witten equations and other PDE methods, a metric is necessary. However, in newer methods such as Khovanov homology, a metric is not required. □

We now study some invariants of an arbitrary oriented simply connected closed smooth manifold X.

First since X is connected, $H_0(X;\mathbb{Z}) = \mathbb{Z}$. By the universal coefficient theorem for cohomology, it follows that $H^0(X) = \mathbb{Z}$. Since X is oriented, Poincaré duality applies, from which we conclude that $H_4(X) = H^4(X) = \mathbb{Z}$.

Next since $\pi_1(X) = 0$, by Hurewicz's theorem we know that $H_1(X) = 0$. By the universal coefficient theorem we find that $H^1(X) = 0$. By Poincaré duality it follows that $H_3(X) = H^3(X) = 0$.

Finally we investigate $H_2(X)$. By Poincaré duality, it is isomorphic to its own dual. But by the universal coefficient theorem,

$$H^2(X) = \mathrm{Hom}(H_2(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(X), \mathbb{Z}) = \mathrm{Hom}(H_2(X), \mathbb{Z}),$$

so $H^2(X)$ is a free $\mathbb{Z}$-module. Thus $H_2(X) = H^2(X) = \mathbb{Z}^r$, where $r = b_2(X)$ is the second Betti number of X. By Hurewicz's theorem, we also know that $\pi_2(X) = H_2(X)$. In summary:

$$H_0 = H^0 = H_4 = H^4 = \mathbb{Z}, \quad \pi_1 = H_1 = H^1 = H_3 = H^3 = 0, \quad \pi_2 = H_2 = H^2 = \mathbb{Z}^r.$$

Recall that the cohomology is equipped with a *cup product*, $H^p(X) \times H^q(X) \to H^{p+q}(X)$. For an arbitrary oriented simply connected smooth 4-manifold, inspecting the cohomology groups above, the most interesting cup product should be that of H^2.

Definition 1.3.1. The *intersection form* of X is the symmetric unimodular bilinear form

$$Q : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$

induced from the cup product by Poincaré duality.

Suppose we consider the intersection form with real coefficients instead of integral coefficients. We find that the intersection form then contains less information. Why is this? With real coefficients, unimodular bilinear forms are classified by rank and signature. That is, any two unimodular matrices sharing the same rank and signature are similar over $\mathbb{R}$. If A is a unimodular matrix of rank r and signature p, then over R

$$A \sim \operatorname{diag}(1, \ldots, 1) \oplus \operatorname{diag}(-1, \ldots, -1),$$

where the first diagonal matrix has size $p \times p$, and the second $(r-p) \times (r-p)$.

To see that unimodular matrices are more difficult to classify over $\mathbb{Z}$, we introduce an invariant:

Definition 1.3.2. Let $A : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a bilinear form. A is *even* if $A(a,a) = 0 \mod 2$ for all $a \in \mathbb{Z}^r$. Evidently parity is a similarity invariant.

Example. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These are both rank 2 signature 1 matrices, hence similar over $\mathbb{R}$. However, A is not even while B is even, so they are not similar over $\mathbb{Z}$.

Why do we call Q the intersection form? This follows from the following theorem:

Theorem 1.3.3. *Let X be a smooth 4-manifold. Then any $\alpha \in H_2(X;\mathbb{Z})$ is represented by $[\Sigma]$ for some smoothly embedded surface $\Sigma \subset X$.*

Proof. There is an isomorphism between equivalence classes of complex line bundles over X and $H^2(X;\mathbb{Z})$ defined by sending each bundle E to its first chern class $c_1(E)$. Thus fix any complex line bundle over X representing α, and consider a generic section of the bundle. Then the zero-set of the section defines a surface (which can be assumed to be smoothly embedded by transversality) that represents α.

In general this proof holds in codimension 2. □

With this in mind, the intersection form Q can be thought of as taking two surfaces which are transverse and counting their signed intersections.

Remark. Recall that $\pi_2 = H_2$ in the case of simply connected 4-manifolds, so every class $\alpha \in H_2$ can be represented by the image of $f : \mathbb{S}^2 \to X$. But hey, doesn't this contradict the Thom conjecture? The key here is that the image of f is an *immersed* submanifold, while the Thom conjecture concerns embedded submanifolds. The Thom conjecture is an special case of the *minimum genus problem*:

$$\text{What is } \min\{\text{genus}(\Sigma) : \Sigma \text{ embedded surface}, [\Sigma] = \alpha \in H_2\}?$$

In the next lecture, we prove that the intersection form determines the homotopy type of a simply connected 4-manifold.

Theorem 1.3.4 (Whitehead)**.** *Let X_1, X_2 be closed simply connected topological 4-manifolds. Then X_1 is homotopy equivalent to X_2 if and only if their intersection forms are similar over $\mathbb{Z}$.*

To end the lecture we look at some examples of 4-manifolds and their intersection forms.

Example.

- If $X = \mathbb{S}^4$, then $Q = 0$ (the empty matrix.)
- If $X = \mathbb{CP}^2$ (with the complex orientation) then $Q = (1)$.
- If $X = \overline{\mathbb{CP}^2}$ (the projective plane with the reverse orientation), then $Q = (-1)$.
- If $X = \mathbb{S}^2 \times \mathbb{S}^2$, then Q is the anti-diagonal matrix adiag$(1, 1)$.
- The connected sum $X_1 \# X_2$ has intersection form $Q_{X_1} \oplus Q_{X_2}$.

Inspecting the above examples, we can extract some non-trivial facts.

1. There is no orientation reversing diffeomorphism $\mathbb{CP}^2 \to \mathbb{CP}^2$, since $Q_{\mathbb{CP}^2} \neq Q_{\overline{\mathbb{CP}^2}}$.
2. Let Q_1 denote the intersection form of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, and Q_2 the intersection form of $\mathbb{S}^2 \times \mathbb{S}^2$. Then $Q_1 = (1) \oplus (-1)$, so it has the same rank and signature as Q_2. It follows that they are similar over $\mathbb{R}$, so homology with real coefficients cannot distinguish $\mathbb{S}^2 \times \mathbb{S}^2$ from $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. However, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ aren't even homotopy equivalent, since their integral intersection forms have different parity.

1.4 Intersection form $\rightsquigarrow$ homotopy type? (lecture 2)

At the end of the previous lecture, we mentioned a theorem due to Whitehead:

Theorem 1.4.1 (Whitehead)**.** *Let X_1, X_2 be closed simply connected topological 4-manifolds. Then X_1 is homotopy equivalent to X_2 if and only if their intersection forms are similar over $\mathbb{Z}$.*

In other words, closed simply connected 4-manifolds are completely determined up to homotopy by their intersection forms. We now give a proof sketch.

Proof. Observe that $H_2(X) \cong H_2(X - B^4) \cong \pi_2(X - B^4)$, so generators of $H_2(X)$ can be represented by maps $f_i : \mathbb{S}^2 \to X - B^4$, with $i = 1, \ldots, r = b_2(X)$. Thus we can define a map

$$f : \bigvee_{i=1}^{r} \mathbb{S}^2 \to X - B^4$$

which induces isomorphisms on H_*. By relative Hurewicz, f induces isomorphisms on π_*, and by Whitehead's theorem (not this one - the usual one), f is a homotopy equivalence. It follows that X is homotopic to $\left(\bigvee_{i=1}^r \mathbb{S}^2\right) \sqcup_h e^4$, where h is a map $h : \mathbb{S}^3 \to \bigvee_{i=1}^r \mathbb{S}^2$. It remains to understand the map h.

Claim: $\pi_3\left(\bigvee_{i=1}^r \mathbb{S}^2\right) = \{\text{symmetric } r \times r \text{ matrices over } \mathbb{Z}\}$. (Thus each h corresponds to an intersection form Q.) The idea behind this correspondence is that each element $[h]$ of $\pi_3(\mathbb{S}^2)$ can be represented by the "linking number" $lk(L, L')$ of loops L and L' defined to be the preimages of points x, x' under the map h. This arises from the *Pontryagin-Thom construction*. More generally, for $h : \mathbb{S}^3 \to \bigvee_{i=1}^r \mathbb{S}^2$, we obtain a matrix $Q_{ij} = lk(L_i, L'_j)$ of linking numbers corresponding to $[h]$. □

Question from class. Why did we use $X - B^4$ instead of just X at the start of the proof?

Answer. We needed to kill H^4 by excision. □

1.5 Intersection form $\rightsquigarrow$ homeomorphism type?

It is natural to ask whether or not the intersection form restricts 4-manifolds any further. How much can we say about the homeomorphism type of a manifold if we know its intersection form? What can we say about its diffeomorphism type? This is answered by a celebrated theorem by Freedman, which earned him a Field's medal.

Theorem 1.5.1 (Freedman, 1982)**.** *(a) For every unimodular symmetric bilinear form Q, there exists a topological simply connected closed 4-manifold X with $Q_X = Q$.*

(b) If Q is even, X is unique up to homeomorphism.

(c) If Q is odd, there are exactly two homeomorphism types of such an X, and at most one of them admits a smooth structure.

In particular, the following corollary is immediate by combining (b) and (c):

Corollary 1.5.2. *If X is a priori a smooth manifold, then its homeomorphism type is completely determined by Q_X.*

The above theorem shows that it may be possible to detect smoothability by using invariants. This is indeed the case, one such invariant being the *Kirby–Siebenmann invariant* for simply connected n-manifolds; $KS_X \in H^4(X; \mathbb{Z}/2)$. Whenever KS_X is non-zero, X does not admit a smooth structure. It turns out that the intersection form affects the Kirby–Siebenmann invariant. Suppose X is a 4-manifold. Whenever Q_X is even, KS_X vanishes. Whenever Q_X is odd, KS_X is either 0 or 1.

Example. By Freedman's theorem, there exists a topological 4-manifold X with $Q_X = (1)$ which is not smoothable. (This manifold is now denoted $*\mathbb{CP}^2$, since it is the one other simply connected closed 4-manifold with the same intersection form as $\mathbb{CP}^2$.) One can show that $KS_X = 1$.

1.6 Homeomorphism type $\rightsquigarrow$ diffeomorphism type?

A question that has not been addressed above is the uniqueness of smooth structures that may exist on topological manifolds. This section is dedicated to studying diffeomorphism types of manifolds, given a homeomorphism type.

Definition 1.6.1. A *smooth structure* on a topological manifold X is a diffeomorphism equivalence class of smooth manifolds homeomorphic to X. If X is a priori equipped with a smooth structure, another smooth structure is said to be *exotic* if it doesn't contain X. That is,

$$X \overset{\text{homeo}}{\cong} X', \quad X \overset{\text{diffeo}}{\ncong} X'.$$

Example.

- In dimensions at most 3, every topological manifold admits a unique smooth structure (Moise, 50s).
- For $n \neq 4$, $\mathbb{R}^n$ admits a unique smooth structure. On the other hand, $\mathbb{R}^4$ has uncountably many. (Donaldson, Gompf, Taubes etc, 80s).
- If X^4 is closed, it has at most countably many smooth structures. This is because every smooth structure in 4 dimensions is uniquely determined by a piecewise linear structure, but a closed manifold admits at most countably many finite simplicial complexes (and hence countably many piecewise linear structures). Note that a closed 4-manifold admitting countably many smooth structures has been exhibited; namely $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$, for $k \geq 2$ (due to Akhmedov-Park).
- For $n \neq 4$, X^n admits finitely many smooth structures. A well known example is exotic spheres:
  - $n = 4$: It is unknown how many smooth structures $\mathbb{S}^4$ admits.
  - $n = 1, 2, 3, 5, 6$: $\mathbb{S}^n$ has a unique smooth structure.
  - $n = 7$: $\mathbb{S}^7$ admits 28 smooth structures (including orientation).

  In principle we can count the number of smooth structures on $\mathbb{S}^n$ for $n \geq 5$, in which case it reduces to understanding homotopy groups.

Remark. An interesting "non-example" is whether or not exotic smooth structures exist on the following manifolds:

$$\mathbb{S}^4, \mathbb{CP}^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, \mathbb{S}^2 \times \mathbb{S}^2.$$

The idea is that increasing topological complexity (Betti numbers) allows more space for constructions, making it easier to find exotic structures.

In fact, a remarkable open question is the following:

Open question. Does every closed smooth 4-manifold admit infinitely many smooth structures?

That is, we have not yet exhibited a single closed 4-manifold admitting only finitely many smooth structures. A more familiar open problem is the last remaining open version of the Poincaré conjecture:

Conjecture. Every homotopy 4-sphere necessarily diffeomorphic to $\mathbb{S}^4$. Here after we denote this conjecture by SPC4 (smooth Poincaré conjecture, dimension 4).

By Freedman's theorem, every homotopy 4-sphere is homeomorphic to a 4-sphere. Therefore the above question boils down to figuring out whether or not spheres admit exotic smooth structures. Two families of potential counter-examples to SPC4 will now be described.

Example. The first family of potential counter examples are constructed via surgery using *balanced presentations* of the trivial group. In other words, presentations $P = \langle g_1, \ldots, g_m \mid r_1, \ldots, r_m \rangle$ (so the number of generators and relations is equal). Then X_P is the simply connected manifold obtained from $\#^m(\mathbb{S}^1 \times \mathbb{S}^3)$ via surgery along loops, as in the construction in lecture 1 (where we proved that every finitely presented group arises as the fundamental group of a closed 4-manifold.)

We know that $\pi_1(X_P) = 0$, while $H_1 = H_3 = 0$ and $H_0 = H_4 = \mathbb{Z}$. Therefore to prove that X_P is a homotopy sphere, it remains to show that $H_2 = 0$. Suppose we know that the Euler characteristic of X_P is 2. But

$$2 = \chi(X_P) = 1 - 0 + b_2 - 0 + 1 = 2 + b_2,$$

so this proves that $H_2 = 0$. It turns out that (from the fact that P is balanced) we can deduce that $\chi(X_P) = 2$. (See homework.) It follows from Freedman's theorem that X_P is homeomorphic to the 4-sphere, but it's unknown what the diffeomorphism type of X_P is.

Some examples of balanced presentations of the trivial group are

$$P = \langle x, y \mid x^4 y^3 = y^2 x^2, x^6 y^4 = y^3 x^3 \rangle, \quad P' = \langle x, y \mid x^4 = y^5, xyx = yxy \rangle.$$

It is currently open whether or not X_P is diffeomorphic to the sphere, while $X_{P'}$ was famously shown to be diffeomorphic to the sphere 8 years ago.

Example. The second family of potential counter examples is constructed using *Gluck twists.* (Again, see the homework.) The idea is to consider knotted embeddings $\mathbb{S}^2 \cong \Sigma \to \mathbb{S}^4$. Let V be a neighbourhood of Σ, diffeomorphic to $\mathbb{S}^2 \times D^2$. Now consider the manifold

$$G_\Sigma = (\mathbb{S}^4 - V) \sqcup_\varphi (\mathbb{S}^2 \times D^2),$$

where $\varphi : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ is a *Gluck twist*: $\varphi(x, \theta) = (\mathrm{rot}_\theta(x), \theta)$. It is left as an exercise to prove that $H_2(G_\Sigma) = 0$, so that G_Σ is homeomorphic to $\mathbb{S}^4$.

1.7 Classification of symmetric $\mathbb{Z}$-bilinear forms

Earlier in the lecture we observed that the intersection form holds all of the information of 4-manifolds up to homotopy, and moreover "almost all" of the information of 4-manifolds up to homeomorphism. In particular, Freedman showed that every unimodular symmetric $\mathbb{Z}$-bilinear form arises as the intersection form of a simply connected closed topological 4-manifold. Therefore in the rest of this lecture we attempt to understand such bilinear form.

The following discussion can be found in *Serre, A Course in Arithmetic*. In the following discussion, two bilinear forms are considered equal if they are *similar*. Let $Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form. Then the *rank* of Q is $\mathrm{rk}(Q) = r$, and the *signature* of Q is $\sigma(Q) = N_+ - N_-$ where $N_\pm$ is the number of eigenvalues with sign $\pm$. Finally, parity is defined by whether or not $Q(a, a) = 0 \mod 2$ for all a, (with such a Q being called *even*).

Remark. Recall that, over $\mathbb{R}$, unimodular symmetric bilinear forms are classified by rank and signature.

Question from class. What about over $\mathbb{Q}$?

Answer. Unimodular symmetric bilinear forms over $\mathbb{Q}$ are classified by rank, sign, discriminant, and Hasse-Witte invariants (corresponding to p-norms.) □

Theorem 1.7.1. *Classification of unimodular symmetric bilinear forms over $\mathbb{Z}$. (Proof omitted.)*

1. *Q indefinite, odd: then $Q = m(1) \oplus n(-1)$, for $m, n > 0$.*
2. *Q indefinite, even: then $Q = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus nE_8$, where $m \geq 0$, and n is an integer. Here E_8 denotes*

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & & & & \\ 0 & 2 & 0 & -1 & & & \\ -1 & 0 & 2 & -1 & & & \\ & -1 & -1 & 2 & -1 & & \\ & & & -1 & 2 & \ddots & \\ & & & & \ddots & \ddots & -1 \\ & & & & & -1 & 2 \end{pmatrix}.$$

   E_8 can also be written as $2I - A$, where A is the adjacency matrix of the Dynkin diagram of the exceptional simple Lie group E_8.

3. *Q definite: complicated (whether or not Q is even or odd). For example, E_{16} can be involved.*

Thus when Q is indefinite, it is determined uniquely by parity, rank, and signature. However, when Q is definite, this no longer holds: for example, $9(1)$ is not similar to $E_8 \oplus (1)$, but they are both odd, with rank and signature 9.

Which Q appears as Q_X for a closed simply connected topological 4-manifold? By Freedman's theorem, all of them do. What if X is smooth?

Theorem 1.7.2 (Rokhlin, 1952)**.** *If X is smooth and simply connected, with Q_X even, then $16 \mid \sigma(Q_X)$.*

This is non-trivial. From algebraic arguments (using the classification above) we can only conclude that 8 divides the signature of Q_X.

Corollary 1.7.3. *There exists an "E_8-manifold", i.e. a simply connected closed topological 4-manifold X with $Q_X = E_8$, and this is not smoothable.*

What about the 4-manifold corresponding to $E_8 \oplus E_8$? This time 16 divides the signature, but in fact it is still not smoothable! This is a corollary of the following groundbreaking result due to Donaldson, which was part of a wave of physical methods flowing into maths.

Theorem 1.7.4 (Donaldson diagonalizability theorem, 1982)**.** *Let X^4 be a smooth closed simply connected manifold. Then if Q_X is definite, it is diagonalizable (over $\mathbb{Z}$). That is, $Q_X = \pm r(1)$.*

The original proof used the Yang-Mills equations. Newer proofs used Seiberg-Witten theory, and even more recently Heegard-Floer homology. How about indefinite forms, of which we have a better classification?

1. For Q indefinite and odd, $Q_X = m(1) \oplus n(-1)$ is realised by $X = (\#^m \mathbb{CP}^2)\#(\#^n \overline{\mathbb{CP}^2})$.
2. For Q indefinite and even, we see later that for $|m| \leq (2/3)n$, $Q_X = n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus mE_8$ is realised by X being a connected sum of K3 surfaces and copies of $\mathbb{S}^2 \times \mathbb{S}^2$. A special case is $Q_X = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2E_8$, which is realised by the *Fermat quartic*,
$$X = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3.$$

The above restriction that $|m| \leq (2/3)n$ is quite curious. However, there is a conjecture that this is not a restriction at all!

Conjecture (11/8-conjecture (Matsumoto))**.** If X is a simply connected closed smooth 4-manifold, then we necessarily have that $|m| \leq (2/3)n$.

Note that $b_2 = 2n + 8|m|$, and $\sigma = -8m$, so the condition that $|m| \leq (2/3)n$ is equivalent to the condition that $b_2 \geq (11/8)|\sigma|$. This explains the naming.

1.8 Summary of homeomorphism types of X^4 (lecture 3)

Recall the *11/8-conjecture* from the previous lecture:

Conjecture (11/8-conjecture (Matsumoto))**.** If X is a simply connected closed smooth 4-manifold, then we necessarily have $b_2 \geq (11/8)|\sigma|$.

Using Seiberg-Witten theory, a slightly weaker version of the conjecture has been known for several years:

Theorem 1.8.1 (10/8-theorem (Furuta))**.** *If X is a simply connected closed smooth 4-manifold, then we necessarily have $b_2 \geq (10/8)|\sigma|$.*

The 10/8-theorem is equivalent to the statement that $|m| \leq n$, where m and n are as in the classification of $\mathbb{Z}$-bilinear forms from the previous lecture. Most recently, a slight improvement to the 10/8-theorem was achieved:

Theorem 1.8.2 (Hopkins, Lin, Shi, Xu)**.** *If X is a simply connected closed smooth 4-manifold, with $m = 2p \geq 4$, then*

$$n \geq \begin{cases} 2p+2 & p \equiv 1,2,5,6 \\ 2p+3 & p \equiv 3,4,7 \\ 2p+4 & p \equiv 0 \end{cases} \qquad \text{mod } 8.$$

Here we can assume m is even by Rokhlin's theorem. In fact, it was shown that this is the best bound that can be achieved using Seiberg-Witten theory.

Summarising results so far, we have established the following:

Theorem 1.8.3. *Let X^4 be a simply connected closed smooth 4-manifold. Then the homeomorphism type of X is determined uniquely by*

$$\sigma(Q_X), \quad \text{parity}(Q_X), \quad \chi(X).$$

This follows from Freedman's theorem, Donaldson's diagonalisability theorem, and the classification of symmetric unimodular $\mathbb{Z}$-bilinear forms.

Equivalently, X is determined up to homeomorphism by b_2^+, b_2^-, and the parity of Q, where $b_2 = b_2^+ + b_2^-$ is the second Betti number of X, and b_2^+ is the number of positive eigenvalues of Q_X, while b_2^- is the number of negative eigenvalues. Then $\sigma = b_2^+ - b_2^-$ and $\chi = 2 + b_2$.

1.9 Crash course on characteristic classes

Using characteristic classes, it is possible to calculate $\sigma(Q_X)$ and $\chi(X)$ in some cases. First we define the four characteristic classes.

Definition 1.9.1 (Chern class)**.** Let $E \to X$ be a complex vector bundle with rank r. (X can be any paracompact topological space, but is typically a manifold.) Then for each $k \in \mathbb{N}$, $c_k(E) \in H^{2k}(X;\mathbb{Z})$ is uniquely determined by the following four properties:

1. *Rank.* $c_0(E) = 1$, $c_k(E) = 0$ for $k > r$.
2. *Functoriality.* If $f : Y \to X$, then $f^* c_k(E) = c_k(f^* E)$.
3. *Product.* If $E, F \to X$, then $c(E \oplus F) = c(E) \smile c(F)$. Here c is the *total Chern class*, $c(E) = c_0(E) + c_1(E) + \cdots \in H^*(X;\mathbb{Z})$. Thus for each k,
$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \smile c_{k-i}(F).$$
4. *Normalisation.* If $X = \mathbb{CP}^n$ and $E = TX$, then $c(E) = (1+\omega)^{n+1}$, where $\omega \in H^2(\mathbb{CP}^n) = \mathbb{Z}$ is the Poincaré dual of $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$.

Geometrically, the chern class c_k corresponds to the Poincaré dual of the locus where $r + 1 - k$ generic sections of E are linearly dependent. The Chern class enjoys a few more notable properties:

Lemma 1.9.2. Let $E \to X$ be a complex vector bundle as above. Then

1. $c_1(E) = c_1(\Lambda^r E)$. The line bundle $\Lambda^r E \to X$ is also denoted $\det E \to X$.
2. If $L_1, L_2 \to X$ are line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.
3. For each k, $c_k(E^*) = (-1)^k c_k(E)$, where E^* is the dual bundle.

Next we define the Stiefel-Whitney classes. These are the real analogue of Chern classes. Every complex structure induces an orientation so integral homology was used above, but for Stiefel-Whitney classes we use mod 2 homology.

Definition 1.9.3 (Stiefel-Whitney class)**.** Let $E \to X$ be a real vector bundle with rank r. Then for each $k \in \mathbb{N}$, $w_k(E) \in H^k(X;\mathbb{Z}/2\mathbb{Z})$ is uniquely determined by the following four properties:

1. *Rank.* $w_0(E) = 1$, $w_k(E) = 0$ for $k > r$.
2. *Functoriality.* If $f : Y \to X$, then $f^* w_k(E) = w_k(f^* E)$.

3. *Product.* If $E, F \to X$, then $w(E \oplus F) = w(E) \smile w(F)$. Here w is the *total Stiefel-Whitney class*, defined analogously to above.

4. *Normalisation.* If $X = \mathbb{RP}^n$ and $E = TX$, then $w(E) = (1 + \omega)^{n+1}$, where $\omega \in H^2(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$ is the Poincaré dual of $\mathbb{RP}^{n-1} \subset \mathbb{RP}^n$.

In addition, we have the following properties.

Lemma 1.9.4. Let $E \to X$ be a real vector bundle as above. Then

1. Suppose E is endowed with a complex structure. Then $w_{2k+1}(E) = 0$ for each k, and $w_{2k}(E) = c_k(E) \mod 2$.

2. $w_1(E) = 0$ if and only if E is orientable.

3. If $w_1(E) = 0$, then $w_2(E) = 0$ if and only if E is *spinnable*. We now explain what this means: oriented real vector bundle of rank r are in bijective correspondence with principal $\mathrm{SO}(r)$-bundles, with a correspondence given by clutching maps. But $\mathrm{SO}(r)$ has a double cover, namely $\mathrm{Spin}(r) \to \mathrm{SO}(r)$. For $r \geq 3$, since $\pi_1(\mathrm{SO}(r)) = \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Spin}(r)$ is in fact the universal cover of $\mathrm{SO}(r)$. A *spin structure* on E is a lift of E to a $\mathrm{Spin}(r)$-bundle.

The third characteristic class is again defined for real vector bundles, but via a complexification.

Definition 1.9.5 (Pontryagin class)**.** Let $E \to X$ be a real vector bundle with rank r. Then for each k,

$$p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(X; \mathbb{Z}).$$

Lemma 1.9.6. The Pontryagin class inherits rank, functoriality, product, and normalisation properties from the Chern class.

The complex vector bundle $E \otimes_{\mathbb{R}} \mathbb{C} \to X$ is called the *complexification* of E. Since this is self-dual, by a property of the Chern class above, $2c_k(E \otimes \mathbb{C}) = 0$ for each odd k. Thus we only consider even Chern classes in the definition of the Pontryagin class. The final characteristic class is in fact the most familiar, as it relates directly to Euler characteristics.

Definition 1.9.7 (Euler class)**.** Let $E \to X$ be an oriented real vector bundle of rank r. Then $e(E) \in H^r(X; \mathbb{Z})$ is uniquely determined by the following properties:

1. *Orientation.* If $\overline{E}$ is E equipped with the opposite orientation, then $e(\overline{E}) = -e(E)$.

2. *Functoriality.* If $f : Y \to X$ is orientation preserving, then $f^* e(E) = e(f^* E)$.

3. *Product.* If $E, F \to X$ are oriented, then $e(E \oplus F) = e(E) \smile e(F)$.

4. *Normalisation.* If E possesses a nowhere-vanishing section, then $e(E) = 0$.

Geometrically, the Euler class is the Poincaré dual of the zero set of a generic section of E. In addition, we have the following properties.

Lemma 1.9.8. Let $E \to X$ be a real oriented vector bundle as above. Then

1. $w_r(E) = e(E) \mod 2$.
2. If E is endowed with a complex structure, $e(E) = c_{r/2}(E)$. In particular, $p_{r/2}(E) = c_r(E \otimes_{\mathbb{R}} \mathbb{C}) = e(E) \smile e(E)$.
3. If X is oriented, then choosing $E = TX$ gives $e(TX)[X] = \chi(X)$, where $\chi(X)$ is the Euler characteristic.

1.10 Classifying homeomorphism types of X^4 with characteristic classes

Suppose X^4 is a simply connected closed smooth 4-manifold. In this section we show that the homeomorphism type of X is determined completely by the characteristic classes of X. (Specifically the Stiefel-Whitney, Pontryagin, and Euler class.) We also try to determine as much as we can about the characteristic classes, given the premise for X.

First we study the Euler class. Since X is orientable as shown in lecture 1, we assume X is oriented. Then $e(X)$ is determined entirely by the Euler characteristic $\chi(X)$ and vice versa.

Next we study the Pontryagin class. Since $p_i(TX) \in H^{4i}(X;\mathbb{Z})$ and $p_0(TX) = 1$, the only non-trivial Pontryagin class is $p_1(TX)$. By the *Hirzebruch signature theorem*, we know that $L_1(X)[X] = \sigma(X)$ where $L_1(X)$ is the first L-class of X. But $L_1(X) = \frac{1}{3}p_1(TX)$, so it follows that $p_1(TX)[X] = 3\sigma(X)$. Thus the first Pontryagin class is completely determined by the signature $\sigma(X)$ and vice versa.

Finally we investigate the Stiefel-Whitney classes. We know that $w_i(TX) \in H^i(X;\mathbb{Z}/2\mathbb{Z})$, and $w_0(TX) = 0$ since X is oriented. How about $w_2 \in H^i(X;\mathbb{Z}/2\mathbb{Z})$?

Lemma 1.10.1. $w_2(TX)$ is a *characteristic element* of X, i.e. $\langle w_2, \alpha\rangle = \langle \alpha, \alpha\rangle \mod 2$ for all $\alpha \in H^2(X;\mathbb{Z})$.

Proof. Let $\alpha \in H^2(X;\mathbb{Z})$. From lecture 1, α is represented by an embedded oriented surface, i.e. α is the Poincaré dual $[\Sigma]$, where $\Sigma \to X$ is a smooth oriented embedding. But $TX|_\Sigma = T\Sigma \oplus N\Sigma$, and on each of these we have $w(T\Sigma) = 1 + w_2(T\Sigma)$ and $w(N\Sigma) = 1 + w_2(N\Sigma)$ (since all higher w_k vanish). Thus by the product axiom

$$w(TX)|_\Sigma = (1 + w_2(T\Sigma))(1 + w_2(N\Sigma)).$$

It follows that $w_2(TX) = w_2(T\Sigma) + w_2(N\Sigma)$. In particular, pairing with α gives

$$\langle w_2(TX), \alpha\rangle = w_2(T\Sigma)[\Sigma] + w_2(N\Sigma)[\Sigma] = e(T\Sigma)[\Sigma] + e(N\Sigma)[\Sigma] \mod 2.$$

The last equality applies; we know that $T\Sigma$ and $N\Sigma$ are oriented since Σ and X are oriented. But $(T\Sigma)[\Sigma]$ is the Euler characteristic of Σ, which vanishes mod 2. On the other hand, $e(N\Sigma)[\Sigma] = \langle [\Sigma], [s(\Sigma)] \rangle$ where s is a section of $N\Sigma$ transverse to the zero section. But then $s(\Sigma)$ is itself a representative of α, so in summary

$$\langle w_2(TX), \alpha \rangle = e(N\Sigma)[\Sigma] = \langle \alpha, \alpha \rangle \mod 2.$$

□

Corollary 1.10.2. *With X as above, Q_X is even if and only if TX is spinnable.*

Proof. Q_X is even if and only if $\langle \alpha, \alpha \rangle = 0 \mod 2$ for all $\alpha \in H^2(X; \mathbb{Z})$. But by the above lemma, $w_2(TX)$ is a characteristic element, so equivalently Q_X is even if and only if $\langle w_2(TX), \alpha \rangle = 0 \mod 2$ for all α. Since Q_X is non-degenerate, this holds if and only if $w_2(TX)$ vanishes, i.e. exactly when TX is spinnable. □

In summary, the data of $e(TX)$ is equivalent to that of the Euler characteristic of X, the data of $p_1(TX)$ is equivalent to that of the signature of Q_X, and the data of $w_2(TX)$ is equivalent to that of the parity of Q_X. Therefore we have the following:

Corollary 1.10.3. *Let X be a closed simply connected smooth 4-manifold. Then the classes $e(X), p_1(TX)$, and $w_2(TX)$ determine X up to homeomorphism.*

1.11 Algebraic surfaces as smooth 4-manifolds

To finish this lecture we explore some examples of smooth 4-manifolds for which we can compute characteristic classes. Namely, these are *algebraic surfaces*. Specifically, we consider

$$Z_d = \{[z_0 : \cdots : z_3] \in \mathbb{CP}^3 : P(z_i) = 0\},$$

where P is a homogeneous degree d polynomial, and the system of equations $\{\partial P/\partial z_i = P = 0, i\}$ has no non-zero solutions. Then Z_d is a smooth manifold. In fact, the diffeomorphism type of Z_d depends only on d and not on P. For example, we can always choose $P = z_0^d + \cdots + z_3^d$. (This is a homework problem.) Concretely, for each d, we have the following:

1. $Z_1 = \mathbb{CP}^2 \subset \mathbb{CP}^3$. This is automatic because P is a linear equation.
2. $Z_2 = \mathbb{CP}^1 \times \mathbb{CP}^1 \cong \mathbb{S}^2 \times \mathbb{S}^2$. We can choose our polynomial to be $xy = uv$. Then a diffeomorphism $Z_2 \to \mathbb{CP}^1 \times \mathbb{CP}^1$ is given by $[x : y : u : v] \mapsto ([x : u], [y : v])$.
3. $Z_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}^2}$. This is a homework problem.
4. Z_4 is a *K3 surface*. These are all diffeomorphic, but algebraically distinct.

5. Z_d for $d \geq 5$ are all "surfaces of general type".

We now compute some characteristic classes associated to the Z_d above. First by applying the Veronese embedding and Lefschetz hyperplane theorem, we conclude that Z_d is simply connected. Write $X = Z_d$, and $X = \mathbb{CP}^3 \cap V \subset \mathbb{CP}^m$ where V is some hyperplane, and $\mathbb{CP}^m$ is the codomain of the Veronese embedding.

Let $H \to \mathbb{CP}^3$ be the *hyperplane line bundle*, i.e. the dual bundle of the tautological bundle. We study characteristic classes of this bundle to better understand X. We begin with Chern classes. First define

$$h = c_1(H) = PD(\mathbb{CP}^2) \in H^2(\mathbb{CP}^3) = \mathbb{Z}.$$

Here $PD(\mathbb{CP}^2)$ denotes the Poincaré dual of $[\mathbb{CP}^2]$. Now consider $X \subset \mathbb{CP}^3$. Its normal bundle is given by $H^{\otimes d}|_X$, so

$$c_1(NX) = c_1(H^{\otimes d})|_X = d\eta,$$

where $\eta = h|_X \in H^2(X;\mathbb{Z})$. It follows that

$$c(T\mathbb{CP}^3|_X) = c(TX)c(NX) = (1 + c_1(TX) + c_2(TX))(1 + d\eta).$$

On the other hand,

$$c(T\mathbb{CP}^3|_X) = (1+\eta)^4 = 1 + 4\eta + 6\eta^2$$

by the normalisation axiom. Solving the system of equations gives

$$c_1(TX) = (4-d)\eta, \quad c_2(TX) = (d^2 - 4d + 6)\eta^2.$$

Next we can use the Chern classes to determine the Euler characteristic. Specifically, we have

$$\chi(X) = e(TX)[X] = c_2(TX)[X] = (d^2 - 4d + 6)(\eta^2[X]).$$

But $\eta^2[X] = d$, because $h[X] = d$ in $\mathbb{CP}^3$. This gives

$$\chi(X) = d^3 - 4d^2 + 6d,$$

which also determines all Betti numbers of X (since we already knew all Betti numbers other than b_2).

Next we determine the signature of Q_X. Recall that $\sigma(Q_X) = \frac{1}{3}p_1(TX)[X]$. But the Pontryagin class is defined using the Chern classes which we have already understood! Specifically,

$$p_1(TX) = -c_2(TX \otimes \mathbb{C}) = -c_2(TX \oplus T^*X) = \sum_{i=0}^{2} c_i(TX) \smile c_{2-i}(T^*X).$$

Since $c_i(TX) = (-1)^i c_i(T^*X)$, this gives $p_1(TX) = c_1^2(TX) - 2c_2(TX)$. (Note that this calculation holds for all complex algebraic surfaces!) In particular, we now find that the signature is given by

$$\sigma(X) = \frac{d(4-d^2)}{3}.$$

Finally, we determine the Stiefel-Whitney classes. Since $w_2 = c_1 \mod 2$, we find that Q is even if and only if d is even. In summary, we have the following results:

Proposition 1.11.1. Let Z_d be as above. Let Q denote its intersection form. Then the parity of Q is the parity of d, and

$$\chi(Z_d) = d^3 - 4d^2 + 6d, \quad \sigma(Q) = \frac{d(4-d^2)}{3}.$$

One can verify that Z_d agrees with the 11/8-conjecture.

Example. We now fix $d = 4$. Then $X = Z_4$ is a *K3 surface*. Since $c_1(TX) = (4-d)\eta$, $c_1(TX)$ vanishes. Thus X is a *Calabi-Yau manifold.* We further find that $b_2 = 4^3 - 4^3 + 24 - 2 = 22$, and $\sigma(Q_X) = -16$. Finally, d is even, so Q_X is even. Since Q_X is even and indefinite, by the classification of symmetric unimodular bilinear forms,

$$Q_X = n\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus m(-E_8).$$

Solving for n and m using σ and b_2, we find that $n = 3$ and $m = 2$.

In the above two pages, we determined the homeomorphism type of the complex algebraic surfaces Z_d. Of course, similar calculations can be carried out on alternative algebraic surfaces:

Example. Let $H \to \mathbb{CP}^2$ be the hyperplane line bundle, and s a generic section of $H^{\otimes 2p}$. Denote its zero set by $B_p \subset \mathbb{CP}^2$. We can define a new bundle by

$$R_p = \{\xi : \xi^2 = s\} \to \mathbb{CP}^2.$$

This is a two to one cover away from B_p, so R_p is a double cover of $\mathbb{CP}^2$ branched over B_p. Using similar methods to above, we have

$$\pi_1(R_p) = 1, \quad b_2^+ = p^2 - 3p + 3, \quad b_2^- = 3p^2 - 3p + 1, \quad R_p \text{ spin } \Leftrightarrow p \text{ odd}.$$

Fixing p, this gives

1. $R_1 = \mathbb{S}^2 \times \mathbb{S}^2$,
2. $R_2 = \mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2}$,

3. R_3 is a K3 surface,

4. R_p for $p \geq 4$ is a surface of general type.

We finish the lecture with a caveat into the classification of algebraic surfaces.

Theorem 1.11.2 (Enriques-Kodaira classification of (smooth projective) algebraic surfaces)**.** *Let K denote the canonical bundle of X, and p_n the dimension of $H^0(K^{\otimes n})$ for each $n \geq 1$. Then define the* Kodaira dimension *κ by*

$$\kappa = \begin{cases} \text{smallest } k \text{ such that } \frac{p_n}{n^k} \text{ is bounded } (k = 0, 1, 2 \text{ in dimension } 4) \\ -\infty \text{ if all } p_n \text{ vanish.} \end{cases}$$

Smooth projective algebraic surfaces are classified as follows:

1. *If $\kappa = -\infty$, then X is a rational or ruled surface. For example, $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$.*

2. *If $\kappa = 0$, then X is a K3 surface, diffeomorphic to T^4, hyperelliptic, or an Enriques surface. (Note that all K3 (and T^4) surfaces are diffeomorphic, but they are algebraically distinct.)*

3. *If $\kappa = 1$, then X is elliptic.*

4. *If $\kappa = 2$, then X is a surface of general type. These are essentially unclassifiable.*

Chapter 2

Representations of 4-manifolds

2.1 Morse functions and handle decompositions (lecture 4)

Definition 2.1.1. Let X be a smooth manifold and $f : X \to \mathbb{R}$ a smooth function. f is a *Morse function* if its critical points are all non-degenerate. That is, locally at a critical point $p \in \mathrm{Crit}(f)$ we can model X to have coordinates

$$f(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2 + c.$$

Then k is called the *index* of p.

The critical points on a Morse function are necessarily discrete. Therefore if X is compact, a Morse function has finitely many critical points. By perturbing them, the critical values can all be assumed to be distinct.

Example. The height function on a torus as shown in figure 2.1 is a Morse function.

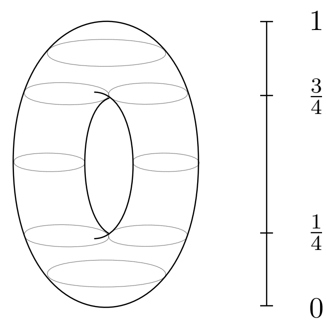


Figure 2.1: A torus with its height function next to it.

Morse functions contain topological information about a manifold in the following way:

Proposition 2.1.2. Suppose p is a critical point of index k. Then passing from the sublevel set $X_{\leq p-\varepsilon}$ to $X_{\leq p+\varepsilon}$, the diffeomorphism type of the sublevel set changes by the attachment of a k-handle.

The remainder of this section is dedicated to unpacking what this means.

Definition 2.1.3. Let Y be an n-manifold with boundary. Let $\varphi : \mathbb{S}^{k-1} \to \partial Y$ be an embedding, with trivial normal bundle. Fix a framing $N\mathbb{S}^{k-1} \cong \mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$. Then a *$k$-handle* is $D^k \times D^{n-k}$, glued to Y along a tubular neighbourhood of $\varphi(\mathbb{S}^{k-1})$. That is, Y' is obtained by gluing a k-handle to Y if $Y' = Y \sqcup_{\mathbb{S}^{k-1} \times D^{n-k}} (D^k \times D^{n-k})$, where $\mathbb{S}^{k-1} \times D^{n-k}$ is a neighbourhood of $\varphi(\mathbb{S}^{k-1})$ in ∂Y. (See figure 2.2.)

Example. In the case of the height function on a torus, as we pass the sub-level set at height 1/4, the topology changes by the addition of a 1-handle.

Relevant terminology is introduced in figure 2.2.

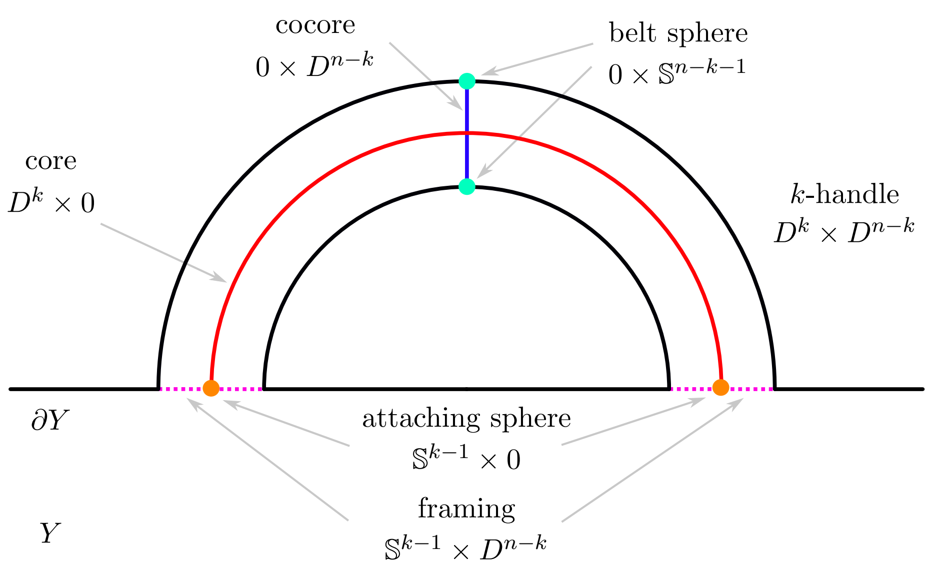


Figure 2.2: Handle attachment terminology

Remark. It is interesting to observe that the boundary of Y' as above is obtained from a straight forward surgery along the attaching sphere: simply remove the neighbourhood $\mathbb{S}^{k-1} \times D^{n-k}$ of the attaching sphere, and glue in $D^k \times \mathbb{S}^{n-k-1}$.

Proposition 2.1.4. A result from Morse theory is that *every X^n admits a handle decomposition.* Without loss of generality, suppose f is a Morse function on X such that the critical points are arranged with increasing index. Then

$$X = X_0 \mid X_1 \mid \cdots \mid X_n,$$

where each X_i is a union of i-handles, and the vertical line represents that X_i and X_{i+1} glue together along boundary components.

Example. The torus T^2 admits a Morse function with indices $0, 1, 1, 2$. Thus T^2 admits a handle decomposition consisting of a 0-handle, two 1-handles, and a 2-handle.

Remark. The homology type of a manifold can be read off its handle decomposition! The cores of k-handles are k-cells, so $C_k(X)$ is generated by k-handles $(h^k_\alpha)_{\alpha\in A}$, with the boundary map given by

$$\partial h^k_\alpha = \sum_\beta \langle h^k_\alpha, h^{k-1}_\beta \rangle h^{k-1}_\beta.$$

Here the angle-brackets denote the *incidence number*, also called the *algebraic intersection number.* It is the signed count of intersections between attaching spheres of h^k_α and belt spheres of h^{k-1}_β.

2.2 Handle moves

Theorem 2.2.1 (Cerf)**.** *Every two monotone handle decompositions of X are related by a finite sequence of handle slides and creation/cancellation of handle pairs.*

By a monotone handle decomposition, we mean the manifold is decomposed into ordered levels as in the previous proposition. We now describe the moves.

Definition 2.2.2. We first describe a *handle slide* between handles h^k_α and h^k_β, with attaching spheres $\mathbb{S}^{k-1}_\alpha$ and $\mathbb{S}^{k-1}_\beta$. Since the normal bundles of attaching spheres are trivial (and in particular we have chosen a framing), there is a push-off $\mathbb{S}^{k-1\prime}_\beta$ of the attaching sphere $\mathbb{S}^{k-1}_\beta$. We then update the attaching sphere of h^k_α to be a connected sum $\mathbb{S}^{k-1\prime}_\alpha := \mathbb{S}^{k-1}_\alpha \# \mathbb{S}^{k-1\prime}_\beta$. See figure 2.3.

Definition 2.2.3. Next we describe creation/cancellation of handle-pairs. Suppose h^k and h^{k-1} are handles on Y such that the attaching sphere of h^k and the belt sphere of h^{k-1} intersect at exactly one point. (That is, they have a *geometric intersection number* of 1.) Then Y is diffeomorphic to $Y \cup h^k \cup h^{k-1}$. For an example, see figure 2.4.

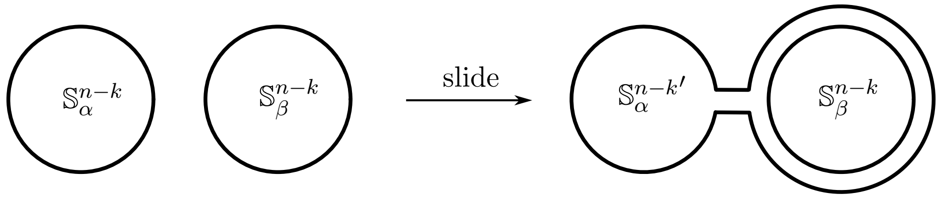


Figure 2.3: A *handle slide*; an operation on attaching spheres.

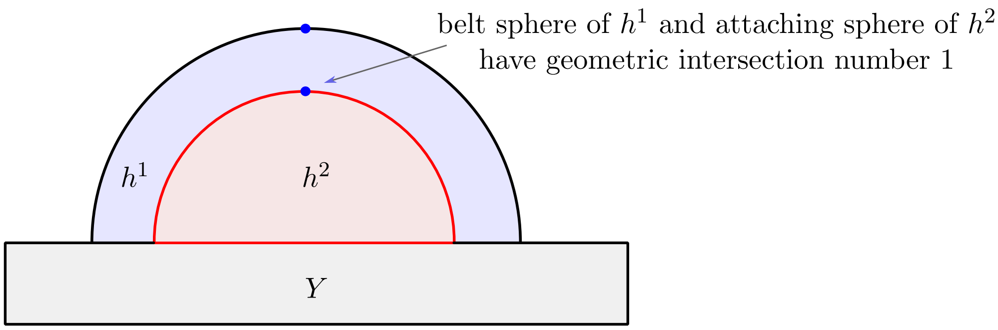


Figure 2.4: A *handle pair creation*; attaching a 1-handle and 2-handle without changing the diffeomorphism type.

Proof. A sketch of the proof of Cerf's theorem is as follows. Any two monotone handle decompositions are induced from Morse functions f_0 and f_1. Relate the Morse functions by a family f_t. In general f_t is not Morse at each t, with two types of singularities occuring. The first is that critical points can cancel out (visualise a cubic graph being straightened so that the local minimum and maximum cancel out). The second is that the gradient field of f_t could have trajectories between two critical points of the same index. These two singularities correspond exactly to creation/cancellation, and handle sliding, respectively. □

2.3 H-cobordism theorem

Definition 2.3.1. A *cobordism* is a compact manifold with boundary W whose boundary decomposes as $\partial W = V_0 \sqcup V_1$, where V_0 and V_1 are themselves embedded smooth manifolds. A cobordism W between V_0 and V_1 is said to be an *h-cobordism* if ι_0, ι_1 are homotopy equivalences. The h stands for homotopy.

Theorem 2.3.2 (h-cobordism theorem)**.** *Let n be at least 5, and W a compact $n+1$-dimensional simply connected smooth h-cobordism between simply connected smooth n-*

manifolds V_0 and V_1. Then W is diffeomorphic to $V_0 \times [0,1]$.

Proof. We now give a proof sketch of the h-cobordism theorem. We choose a Morse function $f : W \to [0,1]$ such that $f^{-1}(0) = V_0$, and $f^{-1}(1) = V_1$. We assume without loss of generality that critical points are arranged in increasing order of index, so that $W = W_0 \mid \cdots \mid W_{n+1}$, where each vertical line represents a sum of cobordisms. Each W_i consists of i-handles. The proof outline is simple:

1. Eliminate 0-handles and $n+1$-handles, so that $W = W_1 \mid \cdots \mid W_n$.
2. Eliminate 1-handles and n-handles by trading them for 3-handles and $n-2$ handles, so that $W = W_2 \mid \cdots \mid W_{n-1}$.
3. Show that k-handles and $k+1$-handles (for $2 \leq k \leq n-1$) have incidence number 1.
4. Upgrade this result; show that belt spheres of k-handles and attaching spheres of $k+1$-handles (for $2 \leq k \leq n-1$) can be perturbed to have geometric intersection number 1. Apply *handle cancellation* to conclude that W is a trivial cobordism.

1. Note that the attaching sphere of a 0-handle is empty. Since our handle decomposition is monotone, any 0-handle is necessarily connected with other components via 1-handles. But the attaching sphere of a 1-handle consists of two points $a \sqcup b$, so to connected a 0-handle to another component, it is necessarily the case that a connects to the belt sphere of the 0-handle, and b connects to another component. Then by handle cancellation, the 0-handle and 1-handle cancel. This applies to $n+1$-handles, since these are dual to 0-handles by replacing f with the Morse function $-f$.

2. A similar procedure is used to replace 1-handles with 3-handles. Again by replacing f with $-f$, we trade n-handles with $n-2$-handles.

3. Since W is an *h-cobordism* rather than just a cobordism, we can conclude that $H_\bullet(W;V_0)$ is trivial. Recall that $C_\bullet(W;V_0)$ is generated by handles and is freely generated over $\mathbb{Z}$. Since the homologies vanish, up to isomorphism, the boundary maps decompose into a direct sum of identity maps $\mathbb{Z} \to \mathbb{Z}$. Thus the incidence numbers are $\langle h^k_\alpha, h^{k-1}_\beta \rangle = 1$.

4. We now know that the algebraic intersection numbers of belt spheres of h^{k-1} and attaching spheres of h_k are 1. These have dimensions $n-k+1$ and $k-1$. More generally, suppose P^{k-1} and Q^{n-k+1} are submanifolds of W, such that $P \cap Q$ is contained in a level set $Z^n = f^{-1}(x)$. Suppose their algebraic intersection number is 1. We use the *Whitney trick* to cancel intersection pairs so that their geometric intersection number is 1.

Suppose $a, b \in P \cap Q$ are distinct, with opposite sign. We can find a path from a to b in Q, and a path from a to b in P. Suppose these two paths bound and embedded disk. Then by the Whitney trick we can isotope P along the disk to cancel the intersections a and b. Therefore the goal is to find and embedded disk.

First we require that the loop is homotopically trivial so that it bounds at least an immersed disk, so we want $\pi_1(Z) = 0$. This comes from simple connectedness assumptions in the h-cobordism theorem premises. To ensure that the disk can be embedded, we use transversality results. We a generic perturbation of the disk to have trivial intersection with the disk, which happens when $2 + 2 < n$. Thus we also require $n \geq 5$ (as given as a premise in the h-cobordism theorem). Therefore we can make P and Q have geometric intersection number 1 as required. By applying handle cancellation, this completes the proof. ☐

Corollary 2.3.3 (Smooth Poincaré conjecture, $n \geq 6$)**.** *For $n \geq 6$, a smooth n-manifold homotopic to the n-sphere is homeomorphic to the n-sphere.*

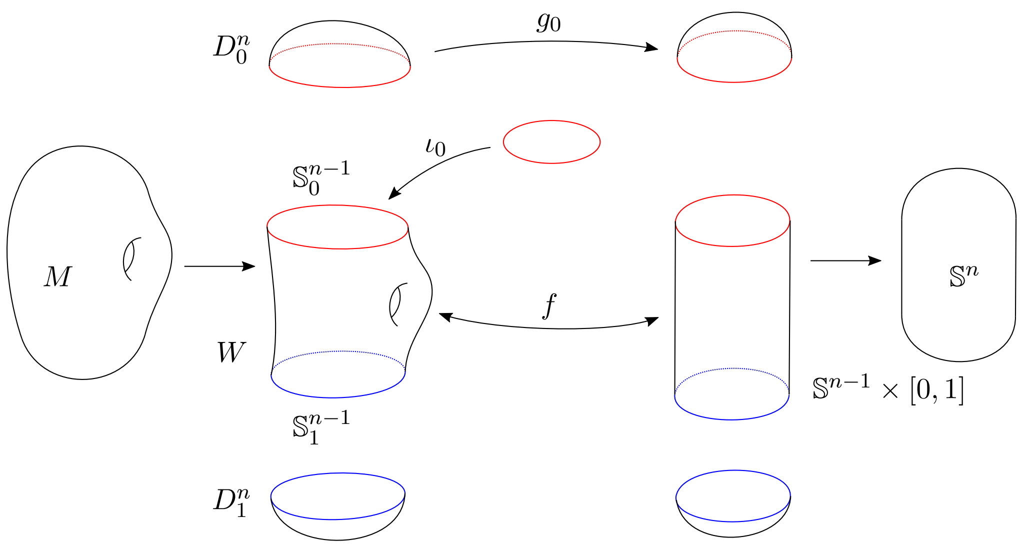


Figure 2.5: Proof of the Poincaré conjecture (in dimensions at least 6).

Proof. The proof follows figure 2.5.

Suppose M is a smooth n-manifold ($n \geq 6$) with the homotopy type of an n-sphere. Any two distinct points are contained in disjoint disks D_0^n and D_1^n. By cutting along the boundary of the disks, we obtain a decomposition of M as shown in figure 2.5. Precisely, we write $M = D_0^n \cup W \cup D_1^n$, where $W = M \setminus \mathrm{int}(D_0^n \sqcup D_1^n)$.

Observe that W is a cobordism between spheres $\mathbb{S}_0^{n-1}$ and $\mathbb{S}_1^{n-1}$. Using the homology excision theorem and Whitehead's theorem, we can show that $\iota_0 : \mathbb{S}_0^{n-1} \hookrightarrow W$ is a homotopy equivalence. (The same result holds for $\mathbb{S}_1^{n-1}$). Therefore by the h-cobordism theorem, W is diffeomorphic (and in particular homeomorphic) to $\mathbb{S}^{n-1} \times [0,1]$, with the homeomorphism denoted by f in the figure.

f restricts to homeomorphisms on the boundary, e.g. $g_0 : \mathbb{S}_0^{n-1} \to \mathbb{S}^{n-1}$ as shown in the figure. But any homeomorphism of a sphere induces a homeomorphism of disks $D_0^n \to D^n$

by the *Alexander trick*. (One can simply take the radial extension of the homeomorphism.) Therefore we have homeomorphisms $g_0, g_1 : D_0^n, D_1^n \to D^n$ which agree with f on overlaps. The map $M \to D^n \cup (\mathbb{S}^{n-1} \times [0,1]) \cup D^n \cong \mathbb{S}^n$ defined piecewise by g_0, f, and g is therefore a homeomorphism. $\square$

Remark. The *topological* Poincaré conjecture is true in all dimensions. However, the h-cobordism theorem is false in dimension 4. The issue is that we cannot find embedded disks (only immersed) and the Whitney trick cannot be applied.

Proposition 2.3.4 (Freedman)**.** The *topological* h-cobordism theorem is true in dimension 4.

Freedman's approach for proving the topological h-cobordism theorem is to remove transverse double-points in immersed Whitney disks by adding "infinite towers of handles" called *Casson handles*. The topological h-cobordism theorem implies the 5-dimensional topological Poincaré conjecture. However, it also implies the 4-dimensional topological Poincaré conjecture when combined with the following result:

Theorem 2.3.5 (Wall)**.** *Let M, N be smooth closed simply connected 4-manifolds. Suppose they have equivalent intersection forms. Then they are h-cobordant.*

The proof strategy is to use the fact that the intersection forms are the same to construct a cobordism, and then use surgery to upgrade to an h-cobordism.

Corollary 2.3.6. *Topological Poincaré conjecture in dimension 4.*

Proof. Suppose M is a 4-dimensional homotopy sphere. By Wall's theorem, there is an h-cobordism W between M and a 4-sphere. By Freedman's topological h-cobordism theorem, $\mathbb{S}^4 \times [0,1] = W = M \times [0,1]$. Therefore M is homeomorphic to $\mathbb{S}^4$. $\square$

Earlier it was remarked that the smooth h-cobordism theorem fails in dimension 4. However, the following result is an alternative which does hold, also due to Wall:

Theorem 2.3.7 (Wall)**.** *Let M, N be smooth closed simply connected 4-manifolds. Suppose they have equivalent intersection forms. Then M and N are* stably diffeomorphic*. In other words, there exists $k \geq 0$ such that*

$$M \# k(\mathbb{S}^2 \times \mathbb{S}^2) \cong N \# k(\mathbb{S}^2 \times \mathbb{S}^2).$$

2.4 Handle decompositions of 3 and 4 manifolds (lecture 5)

Example. We first consider the case of 3-manifolds. Suppose

$$X^3 = X_0 \mid X_1 \mid X_2 \mid X_3$$

where each X_i is a union of 3-dimensional i-handles. (Without loss of generality we have arranged the handles monotonically, and without loss of generality X_0 and X_3 are both single 3-balls.) We can denote

$$H_g = X_0 \mid X_1, \quad H_g' = X_2 \mid X_3,$$

and $\Sigma_g \coloneqq \partial H_g$. Then

$$X = H_g \sqcup_{\Sigma_g} H_g'$$

is called the *Heegaard splitting* of X. What do H_g and H_g' look like? H_g is a *boundary connected sum* of 1-handles; $\natural^g(\mathbb{S}^1 \times D^2)$. A boundary connected sum $A\natural B$ is obtained by identifying a small disks in ∂A to one in ∂B. Thus $\partial(A\natural B) = \partial A \# \partial B$.

By reversing the Morse function ($f \mapsto -f$) we see that H_g' can also be realised as $\natural^k(\mathbb{S}^1 \times D^2)$ for some k. In fact, since H_g' and H_g have the same boundary, and k is the genus of the boundary of H_g', we must have that $k = g$. Therefore H_g' is topologically the same as H_g!

Example. Next we consider 4-manifolds. This time we write

$$X^4 = X_0 \mid X_1 \mid X_2 \mid X_3 \mid X_4,$$

and again we assume $X_0 \cong X_4 \cong B^4$. What does $X_0 \mid X_1$ look like? As in the 3-manifold case, we have

$$X_0 \mid X_1 \cong \natural^k(\mathbb{S}^1 \times D^3).$$

Similarly we know that $X_3 \mid X_4$ is of the same form.

What about 2-handles? The attaching sphere of a 2-handle is a copy of $\mathbb{S}^1$. These attaching spheres can be *knotted*. Precisely, the boundary of $X_0 \mid X_1$ is $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$ (which is a three manifold and hence knots and links are non-trivial), and the attaching spheres of all the 2-handles of X are given by a *link* $L \subset \#^k(\mathbb{S}^1 \times \mathbb{S}^2)$.

For each component S of L, we require a *framing* which describes the way a neighbourhood $S \times D^2$ embeds into $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$. (E.g. annulus vs mobius strip.) This is characterised by the self-linking number $lk(S, S) \in \mathbb{Z}$. (Note that the correspondence between self linking number and framing depends on $H_1(\mathbb{S}^3) = 0$.)

In summary, all 2-handles are determined by the data of a link L, where each component is decorated with an integer.

2.5 Kirby diagrams

By the above discussion, we can represent all 2-handles by a link with each component decorated by an integer representing the self-linking number. On the other hand, every 1-handle is determined by its attaching sphere $\mathbb{S}^0$. In a *Kirby diagram* we represent the figure on the left (figure 2.6) by the diagram on the right: The two green spheres represent

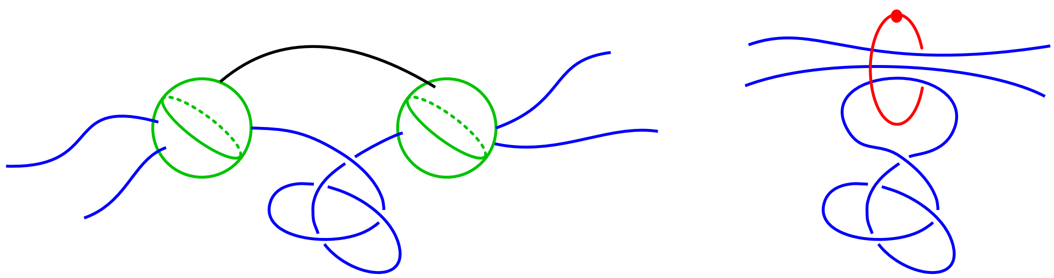


Figure 2.6: 1-handles in a Kirby diagram.

the framings of the attaching sphere of a 1-handle. The blue curves are attaching spheres of 2-handles. The black curve is a path that one can use to idenity the two green spheres.

This gives a systematic way of representing a 1-handle by a link component, in the same way that 2-handles are described by link components. To distinguish them, 1-handles are always denoted by a dot. In summary, $X_0 \mid X_1 \mid X_2$ is specified by a link, where some components are decorated with dots and the rest by integers.

Note that restricting to only dotted components must give an unlink. Now we aim to understand the higher handles. We know that $\partial(X_0 \mid X_1 \mid X_2)$ is of the form $\#^\ell(\mathbb{S}^1 \times \mathbb{S}^2)$ for some ℓ, and the union of 3 and 4 handles is of the form $\natural^\ell(\mathbb{S}^1 \times D^3)$. But then the attaching of higher handles is automatic by the following theorem!

Theorem 2.5.1 (Laudenbach, Poenaru)**.** *Every self-diffeomorphism of $\#^\ell(\mathbb{S}^1 \times \mathbb{S}^2)$ extends to a self-diffeomorphism of $\natural^\ell(\mathbb{S}^1 \times D^3)$.*

This means that any two ways of gluing 3 handles to $X_0 \mid X_1 \mid X_2$ extends to a diffeomorphism of the entire 4-manifold, so up to diffeomorphism there is a unique way of gluing the higher handles. Therefore no information is lost (when representing a 4-manifold) by simply specifying the 1 and 2 handles, and writing "union 3 handles".

Theorem 2.5.2. *A* Kirby diagram *is a link diagram where each component is decorated with integers or a dot, and these correspond to 2 handles and 1 handles respectively. The dotted components must form an unlink. Every Kirby diagram corresponds to a 4-manifold (possibly with boundary), and specifies the 4-manifold up to diffeomorphism.*

Above we mentioned that we also have Kirby diagrams for manifolds with boundary. What does this look like? If X^4 has boundary Y^3, we can give a handle decomposition $X_0 \mid X_1 \mid X_2 \mid X_3$ (with no 4 handle). We require $\partial(X_0 \mid X_1 \mid X_2) = Y\#(\#^\ell(\mathbb{S}^1 \times \mathbb{S}^2))$.

Definition 2.5.3. Let D_1 and D_2 be Kirby diagrams. We write $D_1 \sim D_2$ if the corresponding 4-manifolds are diffeomorphic. We write $D_1 \sim_\partial D_2$ if the boundaries of the corresponding 4-manifolds are diffeomorphic.

Example. Consider $X^4 = \mathbb{S}^4$. This has a handle decomposition consisting of one 0 handle and one 4 handle. Therefore it corresponds to the empty diagram.

Another handle decomposition is given by a 0 handle, 1 handle, 2 handle, and 4 handle. The corresponding diagram is then a Hopf link, with one component decorated with an integer, and one with a dot.

Further we can dualise the decomposition, so that the sphere breaks into a 0 handle, 2 handle, 3 handle, and 4 handle. Then the corresponding Kirby diagram necessarily consists of a single unknot (union 3-handles). This is decorated with the integer 0.

Example. What is the 4 manifold corresponding to the diagram with an unknot labelled with a non-zero integer n? This is a D^2-bundle over $\mathbb{S}^2$, with Euler number n. It's boundary is the *Lens space* $L(n, 1)$. Note that $\partial(L(0, 1)) = \mathbb{S}^1 \times \mathbb{S}^2$.

Example. What about the diagram with a single dotted unknot? The corresponding handle is $\mathbb{S}^1 \times D^3$, so it is boundary diffeomorphic to $D^2 \times \mathbb{S}^2$, which is the 2-handle represented by the unknot with integer 0.

Example. What about the 4-manifold represented by a single unknot, with label 1? The boundary is $L(1, 1) = \mathbb{S}^3$. The corresponding manifold is in fact $\mathbb{CP}^2$. (One can find a Morse function on $\mathbb{CP}^2$ with three critical points of index 0,2, and 4.) Similarly the unknot labelled with -1 corresponds to the manifold $\overline{\mathbb{CP}^2}$.

Example. How about the four manifold represented by a Hopf link, both components labelled with 0? This is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$. This is because the height function on $\mathbb{S}^2$ has two critical points of index 0 and 2 respectively, so the height function on $\mathbb{S}^2 \times \mathbb{S}^2$ has four critical points, of index 0,2,2, and 4.

Example. If D_1 and D_2 are Kirby diagrams for M_1 and M_2, their disjoint union is a diagram for $M_1 \# M_2$ (if the M_i are closed). If the M_i has boundary, the Kirby diagram corresponds to the boundary connected sum.

Example. As a last example let's look at something crazy. What does a K3 surface look like? By Harer, Kas, and Kirby, a diagram for K3 is given in figure 2.7 (sourced from Mandelbaum: *Four dimensional topology: an introduction*).

We also observe that the homology of a 4-manifold can be read off the Kirby diagram: we know that $C_k(X)$ is generated by k-handles, and the boundary map ∂_k computes incidence numbers between k handles and $k - 1$ handles.

For example, if a diagram consists only of 2-handles, then $Q_{ij} = lk(L_i, L_j)$, where L_i and L_j are components of the Kirby diagram. In particular, for the Hopf link with each component decorated with 0, we have $Q = \text{adiag}(1, 1)$.

Definition 2.5.4. A Morse function $f : M \to \mathbb{R}$ is called *perfect* if the number of critical points is the sum of Betti numbers. Equivalently, all Morse inequalities are in fact equalities.

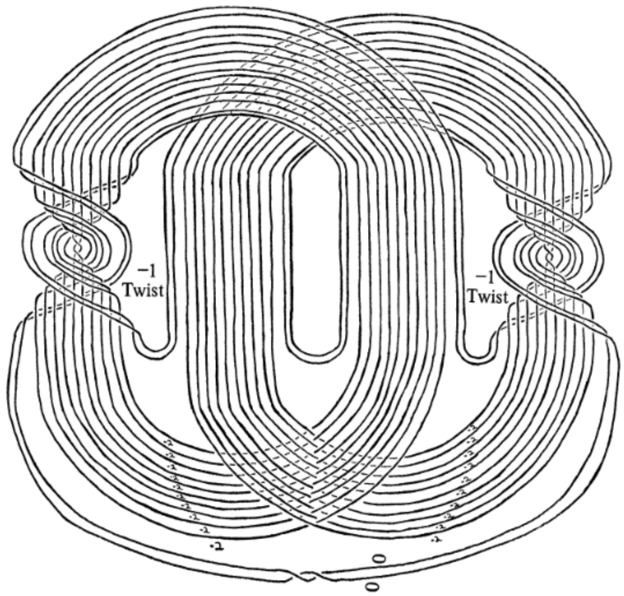


Figure 2.7: K3 surface Kirby diagram.

Open question. Suppose X^4 is simply connected, closed, and smooth. Does it admit a perfect Morse function? Equivalently, does it admit a handle decomposition consisting only of 2-handles (and one 0 handle, and one 4 handle)?

Remark. All unimodular bilinear forms arise as the intersection of some closed simply connected *topological* 4-manifold, not necessarily smooth. (If these were smooth, it would answer the above question.)

However, any such Q does arise as the intersection form of a smooth 4-manifold with boundary! It suffices to consider the 4 manifold corresponding to any Kirby diagram given by the link L with linking numbers $Q_{ij} = lk(L_i, L_j)$.

Example. Recall that E_8 corresponds to a non-smoothable simply connected closed topological 4-manifold in the Freedman sense. What is the smooth 4-manifold with boundary obtained from the link diagram?

The corresponding link diagram looks a bit like the Audi logo, with an unknot for every edge in the Dynkin diagram of E_8. The boundary of the corresponding 4-manifold is the *Poincaré homology sphere.* One can show that the boundary of the 4-manifold corresponding to the trefoil knot (labelled with integer 1) also has boundary the Poincaré homology sphere.

Definition 2.5.5. A *surgery diagram* for a 3-manifold Y^3 is a Kirby diagram for a four manifold X with $\partial X = Y$, consisting only of 2-handles.

Theorem 2.5.6 (Lickorish-Wallace)**.** *Every closed oriented 3-manifold admits a surgery diagram.*

2.6 Surgery diagrams (lecture 6)

Recall from the previous lecture the notion of surgery diagrams, and the Lickorish-Wallace theorem:

Definition 2.6.1. A *surgery diagram* for a 3-manifold Y^3 is a Kirby diagram for a four manifold X with $\partial X = Y$, consisting only of 2-handles.

Theorem 2.6.2 (Lickorish-Wallace)**.** *Every closed oriented 3-manifold admits a surgery diagram.*

Proof. By a theorem of Rokhlin, we know that every Y^3 arises as ∂X^4 for some compact smooth manifold X. Draw a Kirby diagram for X. Since $D^3 \times \mathbb{S}^1$ is boundary isomorphic to $\mathbb{S}^2 \times D^2$, we replace all 1-handles with 0-framed 2-handles to obtain a new four manifold which still has boundary Y^3. By "flipping the diagram upside down", any 3-handles correspond to 1-handles. By following the same procedure, we can eliminate all 3-handles. All that remains are 2-handles, as required. □

Example. Some examples of surgery diagrams are as follows:

- The empty diagram corresponds to $\mathbb{S}^3$.
- The 0-framed 2-handle is $\mathbb{S}^1 \times \mathbb{S}^2$.
- An n-framed 2-handle is the lens space $-L(n, 1)$.
- The 1-framed trefoil corresponds to the Poincaré sphere. The 0-framed Borromean rings corresponds to the torus T^3.

Remark. We can read off homology from the surgery diagram! We have a Kirby diagram for $X, \partial X = Y$, consisting of 2-handles. Thus $H_1(X) = H_3(X) = 0$. On the other hand, $H_2(X)$ is generated by 2-handles. This gives

$$H_2(Y) \to H_2(X) \to H_2(X, Y) \to H_1(Y) \to H_1(X) = 0,$$

where the map $H_2(X) \to H_2(X, Y)$ is $Q : \mathbb{Z}^r \to \mathbb{Z}^r$. Then $H_1(Y) = \operatorname{coker} Q$, and $H_2(Y) \cong H^1(Y)$ is the free part of $H_1(Y)$.

2.7 Kirby calculus

Recall *Cerf's theorem.* This applies to Kirby diagrams, to give the so called Kirby calculus.

Theorem 2.7.1 (Cerf's theorem)**.** *Any two handle decompositions are related by a sequence of handle slides, handle cancellations.creations, and isotopies.*

A corollary is that Kirby diagrams are related by *Kirby moves*:

Theorem 2.7.2. *Any two Kirby diagrams for X^4 are related by a sequence of the following moves:*

- *Isotopies of handles, i.e. Reidemeister moves of the Kirby diagram.*
- *Handle slides (which manifest differently for 1-handles and 2-handles).*
- *Handle creation and cancellation (which also manifests differently for 1-handle/2-handle pairs and 2-handle/3-handle pairs).*
- *A consequence of dotted notation is that there is one more move independent of Cerf's theorem corresponding to sliding a 2-handle over a 1-handle.*

We now describe each of the above moves. Handle slides of 1-handles are exactly as shown in 2.8. Handle slides of 2-handles are as shown in 2.9, but require some subtlety. The

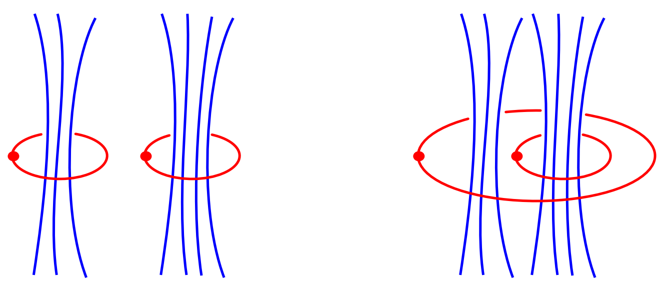


Figure 2.8: Kirby move: 1-handle handle slide.

idea is that the framing of the handle doing the sliding changes to a new integer. Suppose the knots K_1, K_2 representing the handles h_1, h_2 have framings n_1, n_2 respectively. Suppose the handle h_1 slides over h_2. Then the new framing for h_1 is given by

$$\text{framing } h_1' = \operatorname{lk}(K_1, K_1) = n_1 + n_2 \pm 2\operatorname{lk}(K_1, K_2).$$

The sign $\pm$ depends on whether or not the slide is orientation preserving. Note that the integer k in the figure is not necessarily equal to n_2, since the "flat diagram" is not the zero framing, but rather the framing given by the writhe of the diagram. Next we describe han-

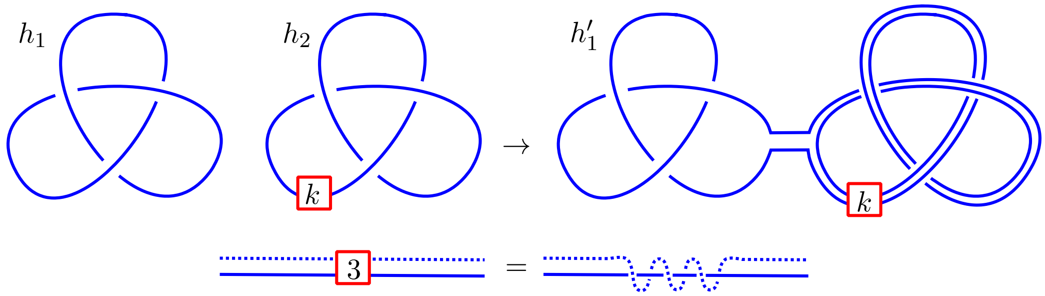


Figure 2.9: Kirby move: 2-handle handle slide.

dle creation and cancellation. Cancellation of 2-handles and 3-handles is diagrammatically simple, since 3-handles are not drawn in Kirby diagrams. Unknotted 2-handles labelled with a 0 can be removed from the diagram. On the other hand, cancellation of 1-handles and 2-handles is expressed in figure 2.10. K denotes a knot, and n is the framing. Finally

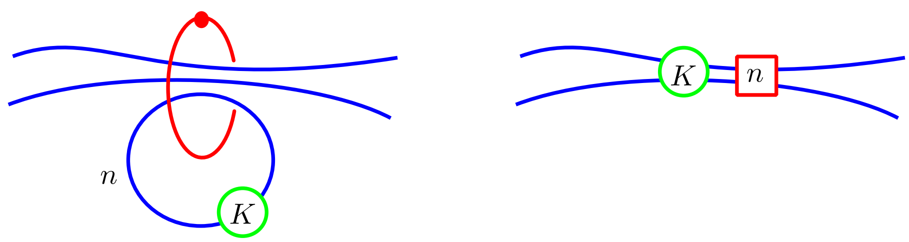


Figure 2.10: Kirby move: 1/2-handle creation/cancellation.

the last type of Kirby move is “sliding a 2-handle over a 1-handle”. By deconstructing what the dotted notation means, it is clear that the following holds (figure 2.11.)

Example. We now compute some examples. In figure 2.12, we attempt to understand the diagram on the left by a 2-handle *handle slide*. Suppose both components are given an anticlockwise orientation. Then the slide reverses the orientation, so the new framing of the sphere on the left is given by $2 + 0 - 2\,\mathrm{lk}(K_1, K_2) = 0$. Therefore our diagram on the right has the framings given. By Reidemeister moves, this is a Hopf link with framings 0 and 0, i.e. $\mathbb{S}^2 \times \mathbb{S}^2$.

Example. Next suppose we have a Hopf link, but with framings 0 and 1. (Figure 2.13) Then a handle slide as above gives a similar diagram, but with and extra “loop”. By Reidemeister moves, this produces two disjoint unknots! The new framings are 1 and -1. Therefore the corresponding manifold is $\overline{\mathbb{CP}^2} \# \mathbb{CP}^2$.

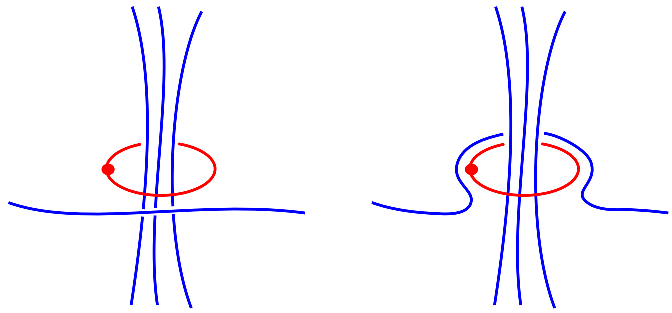


Figure 2.11: Sliding 2-handles over 1-handles.

Figure 2.12: Kirby calculus example

In general we find that a Hopf link with framings 0 and p represents $\overline{\mathbb{CP}^2}\#\mathbb{CP}^2$ if p is odd, and $\mathbb{S}^2 \times \mathbb{S}^2$ is p is even.

What about the case of Hopf links with framings p, q? This gives the intersection form

$$Q = \begin{pmatrix} p & 1 \\ 1 & p \end{pmatrix},$$

which has determinant $pq - 1$. This is usually not ± 1! In other words, it doesn't give a valid intersection form for a manifold without boundary. (Or with contrapositive phrasing, in general we obtain a 4-manifold with boundary.)

A similar theorem holds for surgery diagrams.

Theorem 2.7.3. *Two surgery diagrams represent the same 3-manifold if and only if they are related by Reidemeister moves, handle-slides, or blow-ups and blow-downs.*

Here a *blow-up* or *blow-down* refers to the fact that ± 1-framed unknots are boundary homeomorphic to the empty diagram. Therefore for surgery diagrams, it is completely valid to just drop them.

Example. Consider the Hopf-link with framing 0 and 1. Then by handle-sliding, we obtain an unlink with framing -1 and 1. By blow-downs, this corresponds to the empty diagram. Therefore the corresponding 3-manifold is $\mathbb{S}^3$.

Note that blow-ups and blow-downs can be generalised, as shown in figure 2.14.

Figure 2.13: Kirby calculus example 2

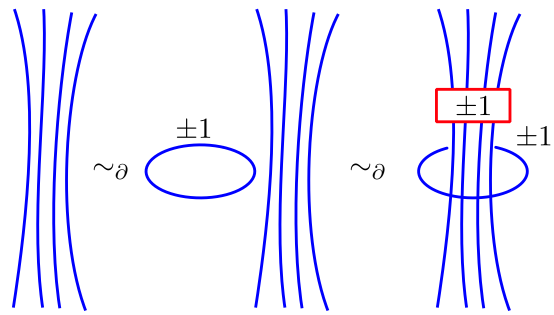


Figure 2.14: Blow-ups and blow-downs for surgery diagrams

2.8 Heegaard diagrams

Another method for representing 4-manifolds is the notion of a *trisection*. To define this, we first consider its analogue for 3-manifolds, namely Heegaard splittings. Recall from earlier lectures that a handle decomposition of a 3-manifold gives rise to a *Heegard splitting*.

Suppose Y is a 3-manifold. Then we can write

$$Y = \underbrace{Y_0 \mid Y_1}_{H_g} \mid \underbrace{Y_2 \mid Y_3}_{H'_g}.$$

Here H_g and H'_g are diffeomorphic handlebodies, the first consists of 1-handles, and the second of 2-handles. Moreover, the two pieces have a common boundary, namely the unique surface Σ_g.

This gives rise to the notion of a *Heegaard diagram*: this is a copy of Σ_g with a collection of $2g$ curves

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$$

on Σ_g. The $\alpha_1, \ldots, \alpha_g$ are attaching spheres for the 1-handles (viewed as 2-handles), and the $\beta_1, \ldots, \beta_g$ are attaching spheres for the 2-handles. The α_i should be linearly independent in $H_1(\Sigma)$, as should β_i. Then

$$Y = \Sigma_g \cup \bigcup_{i=1}^{g} D^2_{\alpha_i} \cup \bigcup_{i=1}^{g} D^2_{\beta_i} \cup B^3_\alpha \cup B^3_\beta.$$

Again by an application of Cerf's theorem, we obtain the following theorem:

Theorem 2.8.1. *Two Heegaard diagrams represent the same 3-manifold if and only if they differ by*

- *a sequence of handle-slides (α over α, β over β),*
- *isotopies (Reidemeister moves),*
- *stabilisation/unstabilisation (creation/cancellation of 1-handle/2-handle pairs). In other words,*

$$(\Sigma, \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g) \sim (\Sigma \# T^2, \alpha_1, \dots, \alpha_{g+1}, \beta_1, \dots, \beta_{g+1})$$

  where α_{g+1} and β_{g+1} intersect at a single point. (e.g. if they are a meridian and longitude of T^2.)

Remark. Stabilisation/unstabilisation shows that the genus of a Heegaard diagram is not fixed. Therefore a given Heegaard diagram is often called a *genus g* diagram (of Y).

Example.

- T^2 with α a meridian and β a longitude represents $\mathbb{S}^3$.
- T^2 with α and β both meridians represents $\mathbb{S}^1 \times \mathbb{S}^2$.

2.9 Trisections (lecture 7)

Definition 2.9.1. Let X be a closed smooth connected 4-manifold. Then for $0 \leq k \leq g$, a (g,k)-*trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

- For each i, there is a diffeomorphism $\varphi_i : X_i \to \natural^k(\mathbb{S}^1 \times B^3)$.
- The boundary of each X_i is $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$. Each of these has a Heegaard splitting

$$\partial X_i = \#^k(\mathbb{S}^1 \times \mathbb{S}^2) = Y^-_{k,g} \sqcup_{\Sigma_g} Y^+_{k,g}.$$

- Given any i, $\varphi_i(X_i \cap X_{i+1}) = Y^-_{k,g}$, and $\varphi_i(X_i \cap X_{i-1}) = Y^+_{k,g}$.

Definition 2.9.2. A *trisection diagram* is a set of three curves $\alpha_i, \beta_i, \gamma_i$ on Σ_g, with $i \in \{1, \dots, g\}$, such that any two subcollections is a Heegaard diagram for $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$ (and represents the splitting $\#^k(\mathbb{S}^1 \times \mathbb{S}^2) = Y^-_{k,g} \sqcup_{\Sigma_g} Y^+_{k,g}$).

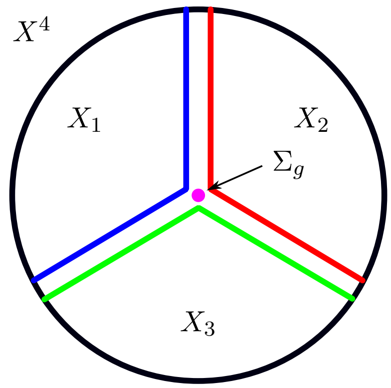


Figure 2.15: Anatomy of a trisection. Each coloured boundary ∂X_i has a Heegaard decomposition $Y^-_{k,g} \sqcup_{\Sigma_g} Y^+_{k,g}$.

The basic anatomy of a trisection is shown in figure 2.15.

Remark. One can show that $\chi(X) = 2 + g - 3k$, i.e. k is determined by g! On the other hand, g is fixed modulo 3 for any given X. Therefore we speak of "genus g trisections" of X.

Example. Suppose a trisection has $g = 0$ and $k = 0$. Filling out the figure 2.15 above, we find that $\Sigma_g = \mathbb{S}^2$, and each boundary component is $\mathbb{S}^3$ (with Heegaard decomposition $B^3 \sqcup_{\mathbb{S}^2} B^3$.) Then each X_i is a copy of B^4, and $X = \mathbb{S}^4$.

Example. Suppose a trisection has $g = 1$ and $k = 1$. The corresponding trisection diagram consists of a torus, with α, β, γ all meridians. This gives $X = \mathbb{S}^1 \times \mathbb{S}^3$.

Example. Consider $\mathbb{CP}^2$. This is a *toric variety*, with moment map $f : \mathbb{CP}^2 \to \mathbb{R}^2$ given by

$$f([z_0 : z_1 : z_2]) = \Big(\frac{|z_0|^2}{\sum |z_i|^2}, \frac{|z_1|^2}{\sum |z_i|^2}\Big).$$

The image of f is the triangle defined by sides $[0, 1] \times 0$ and $0 \times [0, 1]$. Given any p in the interior of the triangle, the preimage of p under f gives a torus. Consider the three regions of the triangle given by three orthogonal rays from p to the boundary of the triangle. Label each region Q_1, Q_2, Q_3. Then each $X_i = f^{-1}(Q_i)$ defines a trisection of $\mathbb{CP}^2$. $f^{-1}(Q_i \cap Q_j)$ is a solid torus.

This gives rise to a trisection with $g = 1$ and $k = 0$. The trisection diagram is a torus with three curves on it, namely $(0, 1), (1, 0), (1, 1)$ according to the canonical bijection between isotopy classes of simple closed curves on a torus and primitive elements of $\mathbb{Z}^2$.

Theorem 2.9.3 (Gay-Kirby)**.** *Every closed smooth connected oriented 4-manifold admits a trisection.*

Proof. We give a proof sketch. Choose a "2-valued Morse function" $f : X \to B^2$. The local models are

- generic points: f is a submersion $(t, x, y, z) \to (t, x)$
- folds: $(t, x, y, z) \to (t, \pm x^2 \pm y^2 \pm z^2)$
- cusps: $(t, x, y, z) \to (t, x^3 - tx \pm y^2 \pm z^2)$.

We now consider a family of functions $f_t : X \to \mathbb{R}$. Cusps occur when f_t experiences the birth or death of a singularity, and folds are curves of critical points. The image of X^4 in B^2 is called a *Cerf graphic*, and by massaging the Cerf graphic in analogous ways to handle-moves, the Cerf graphic can be arranged to form a trisection. $\square$

Theorem 2.9.4 (Gay-Kirby)**.** *Any two trisections of X^4 are related by a sequence of*

- *diffeomorphisms,*
- *α, β, or γ-handle slides,*
- *stabilizations, i.e. connected sums with a diagram representing $\mathbb{S}^4$. This diagram looks like a fidget spinner with extra loops, as shown in 2.16.*

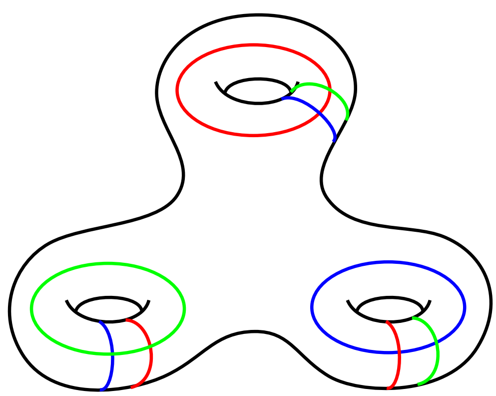


Figure 2.16: Trisection diagram for $\mathbb{S}^4$. Each colour represents α_i, β_i, or γ_i.

Chapter 3

Construction of Seiberg-Witten gauge theory

Some of the goals of this section are to prove the following results:

1. Prove Donaldson's diagonalisability theorem.
2. Show the existence of exotic smooth structures in dimension 4.
3. Prove the Thom conjecture (which concerns the genus of surfaces $\Sigma \subset \mathbb{CP}^2$).
4. Prove the Milnor conjecture (which concerns the genus of surfaces $\Sigma \subset T_{p,q}$).

To do this, we use the tools of Seiberg-Witten gauge theory. To state the Sieberg-Witten equations, we must first introduce the relevant definitions.

3.1 Clifford modules

Consider the Laplacian $\Delta = -\sum(\partial/\partial x_i)^2$. This is an operator $C^\infty(\mathbb{R}^n, \mathbb{C}^n) \to C^\infty(\mathbb{R}^n, \mathbb{C}^n)$. The Laplacian is *self-adjoint*; $\langle \Delta\varphi, \psi\rangle = \langle \varphi, \Delta\psi\rangle$. When does the Laplacian admit a square root? We want

$$D = \sum A_i \partial/\partial x_i, \quad \langle D\varphi, \psi\rangle = \langle \varphi, D\psi\rangle, \quad D^2 = \Delta.$$

Expanding what this means, we require $A_i^2 = -1$, $A_i^* = -A_i$, and $A_iA_j + A_jA_i = 0$ whenever $i \neq j$.

Definition 3.1.1. A *Clifford algebra* is a real algebra generated by elements A_i satisfying $A_i^2 = -1$ and $A_iA_j + A_jA_i = 0$ whenever $i \neq j$.

Definition 3.1.2. Let $\mathcal{H}$ denote an n-dimensional real inner product space. A *Clifford module* of $\mathcal{H}$ is a Hermitian complex vector space V equipped with a Clifford multiplication, i.e. a map $\gamma : \mathcal{H} \to \mathrm{End}(V)$ such that

1. If $\|e\| = 1$, then $\gamma(e)^2 = -1$.
2. If $e_1 \perp e_2$, then $\gamma(e_1)\gamma(e_2) + \gamma(e_2)\gamma(e_1) = 0$.
3. $\gamma(e)^* = -\gamma(e)$.

Thus a Clifford module is a skew-Hermitian representation of a Clifford algebra.

Theorem 3.1.3. *If $n = 2k$, then there exists a unique finite dimensional irreducible Clifford module (S, γ) up to isomorphism, with $\dim_{\mathbb{C}} S = 2^k$.*

If $n = 2k+1$, then there are exactly two finitely dimensional irreducible Clifford modules up to isomorphism; (S, γ) and $(S, -\gamma)$. These have $\dim_{\mathbb{C}} S = 2^k$.

Example. Suppose $\mathcal{H}$ has basis e_1, e_2, e_3. Let $S = \mathbb{C}^2$, and $\gamma(e_i) = B_i$, where the B_i are *Pauli matrices*:

$$B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then (S, γ) and $(S, -\gamma)$ are the two Clifford modules of $\mathcal{H}$, up to isomorphism.

Example. Suppose $\mathcal{H}$ has basis e_1, e_2, e_3, e_4. Let $S = \mathbb{C}^4 = S^+ \oplus S^{-1}$, and

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i \\ B_i & 0 \end{pmatrix},$$

where the B_i are as above, and $B_4 = I$. Then (S, γ) is the unique irreducible module of $\mathcal{H}$ up to isomorphism.

Definition 3.1.4. A *spin^c structure* on an n-dimensional oriented Riemannian manifold X is a Hermitian bundle $S \to X$ with bundle map $\rho : TX \to \mathrm{End}(S)$ such that for all x, $(S_x, \rho_x : T_xX \to \mathrm{End}(S_x))$ is isomorphic to an irreducible Clifford module for T_xX.

Example. If $n = 3$, a spin^c structure is a Hermitian bundle $S \to X$ of rank 2, with a map $\rho : TX \to \mathrm{End}(S)$ such that there exists an orthonormal basis e_i at each x for T_xX, and a Hermitian basis for S, with $\rho(e_i) = B_i$.

In fact, this can be thought of as $U(2)$-bundle together with a compatibility condition with TX.

Example. If $n = 4$, a spin^c structure corresponds to two Hermitian bundles S^+ and S^- of rank 2, with a map $\rho : TX \to \mathrm{Hom}(S^+, E^-)$, such that there is an orthonormal basis e_i and Hermitian basis for S^+, S^-, with $\rho(e_i) = B_i$.

Question from class. Is the category of Clifford algebras semisimple? What happens to the earlier theorems if we drop irreducibility?

Answer. Yes, every Clifford algebra is a direct sum of irreducible Clifford algebras. $\square$

3.2 Spin^c structure definitions (lecture 8)

Recall from the previous lecture that a *spin^c structure* on an oriented Riemannian 4-manifold X is a pair of rank 2 Hermitian bundles $S^+, S^- \to X$ with bundle map $\rho : TX \to \mathrm{End}(S)$ such that for all x, $(S_x, \rho_x : T_xX \to \mathrm{End}(S_x))$ is isomorphic to an irreducible Clifford module for T_xX.

More explicitly, we can refine ρ to a map $\gamma : TX \to \mathrm{Hom}(S^+, S^-) \subset \mathrm{End}(S)$. Then a spin^c structure is a map γ such that for each x, there exists an orthonormal basis e_i for TX and a unitary basis for $S^\pm$ such that

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix},$$

where each B_i is a Pauli matrix (with $B_4 = I$.)

$S = S^+ \oplus S^- \to X$ is called the *spinor bundle*.

Remark. Since the determinant of each B_i is 1, the determinant line bundles of S^+ and S^- are isomorphic. (To establish this isomorphism, it suffices to verify that $\rho(e) = 1$ whenever $\|e\| = 1$.)

Definition 3.2.1. The *class* of (S, γ) is defined to be

$$c_1(\det S^+) = c_1(S^+) = c_1(S^-) \in H^2(X; \mathbb{Z}).$$

Note that $c_1(S) = 2c_1(S^\pm)$.

Remark. Fix some $x \in X$. Then $\mathrm{Aut}(S_x, \gamma_x) = \mathbb{S}^1$. To see this, observe that any automorphism commuting with γ_x corresponds to a pair A^+ and A^- of automorphisms on S^+ and S^-, such that $B_iA^+ = A^-B_i$ for each i. Since $B_4 = I$, this implies that $A := A^+ = A^- \in U(2)$. On the other hand, $AB_iA^{-1} = B_i$ implies that A is central, so $A \in Z(U(2)) = \mathbb{S}^1$.

Another way to think about spin^c structures is via principal bundles. Recall that $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ is a 2:1 covering map. Then the statement that X^n is a Riemannian manifold is equivalent to saying that the frame bundle of its tangent bundle is a principal $\mathrm{SO}(n)$-bundle. A *spin structure* is then a lift of this structure to a $\mathrm{Spin}(n)$-bundle.

With this perspective, we can define

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}/2\mathbb{Z}} U(1) = \{(g, e^{i\theta}) \in \mathrm{Spin}(n) \times U(1)\}/\sim$$

where $\sim$ is the equivalence relation $(g, e^{i\theta}) = (\tau g, e^{-i\theta})$. Here τg denotes the unique element so that $\{g, \tau g\}$ is the fibre above a point in $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. With this definition of $\mathrm{Spin}^c(n)$, a spin^c structure is equivalently a lift of the frame bundle of TX to a $\mathrm{Spin}^c(n)$-bundle.

Example. Suppose $n = 3$. Then $\mathrm{Spin}(3) = \mathrm{SU}(2) = S(\mathbb{H})$, where $S(\mathbb{H})$ denotes the unit quaternions. There's a 2:1 covering map

$$S(\mathbb{H}) \to \mathrm{SO}(3), \quad h \mapsto (x \mapsto hxh^{-1}).$$

Then $\mathrm{Spin}^c(3) = \mathrm{SU}(2) \times_{\mathbb{Z}/2\mathbb{Z}} U(1) = U(2)$. To see this equality, for each $A \in U(2)$, consider the map

$$A \mapsto ((\det A)^{-1/2} A)((\det A)^{1/2} I).$$

The first component on the left belongs to $SU(2)$, and the second to $U(1)$. The failure of this map being well defined is that $(\det A)^{1/2}$ is only well defined up to sign, but this exactly accounted for in the fibre product with respect to $\mathbb{Z}/2\mathbb{Z}$. Therefore this map gives an isomorphism, so $\mathrm{Spin}^c(3) = U(2)$ as required. In summary we have the following result:

A spin^c structure on X^3 is a $U(2)$-bundle $S \to X$ with compatibility conditions given by $\rho : TX \to \mathrm{End}(S)$.

Example. Suppose $n = 4$. Then $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$. The covering map is given by

$$(h_1, h_2) \mapsto (x \mapsto h_1 x h_2^{-1}),$$

where we have again identified SU(2) with $S(\mathbb{H})$. With this interpretation,

$$\begin{aligned} \mathrm{Spin}^c(4) &= (\mathrm{SU}(2) \times \mathrm{SU}(2)) \times_{\mathbb{Z}/2\mathbb{Z}} U(1) \\ &\subset (\mathrm{SU}(2) \times 0 \times_{\mathbb{Z}/2\mathbb{Z}} U(1)) \times (0 \times \mathrm{SU}(2) \times_{\mathbb{Z}/2\mathbb{Z}} U(1)) = U(2) \times U(2). \end{aligned}$$

More explicitly, the subset is

$$\mathrm{Spin}^c(n) = \{(A, B) \in U(2) \times U(2) : \det A = \det B\} \subset U(2) \times U(2).$$

In summary, a four dimensional spin^c structure is given by $U(2)$-bundles $S^+, S^- \to X$ satisfying additional compatibility conditions via $\gamma : TX \to \mathrm{End}(S^+ \oplus S^-)$.

3.3 Spin^c structure existence and classification

Suppose X is a smooth simply connected oriented closed 4-manifold. Recall that X admits a spin structure if and only if $w_2(TX) = 0$, or equivalently if and only if Q_X is even. Existence of spin^c structures is less constrained:

Proposition 3.3.1. Any smooth simply connected oriented closed 4-manifold X admits a spin^c structure. The space of spin^c structures is an affine space, modelled on $H^2(X; \mathbb{Z})$. In other words, given any $s_0, s_1 \in \mathrm{Spin}^c(X)$, $s_0 - s_1$ is well defined in $H^2(X; \mathbb{Z})$. In particular, there are non-canonical isomorphisms $\mathrm{Spin}^c(X) \cong H^2(X; \mathbb{Z})$.

Remark. If $s \in \mathrm{Spin}^c(X)$, and $h \in H^2(X; \mathbb{Z})$ is given by $h = c_1(E)$ for a complex line bundle E, then $s + h \in \mathrm{Spin}^c(X)$ is $(S^+ \otimes E, S^{-1} \otimes E, \rho \otimes \mathrm{id})$.

Proof. We give a proof for the case where $\pi_1(X) = 1$. Suppose $\{U_\alpha\}_{\alpha\in A}$ is an atlas for X, such that each U_α gives a trivialisation $TX|_{U_\alpha} = \mathbb{R}^4 \times U_\alpha$, and $S_\alpha \coloneqq S|_{U_\alpha}$ is the standard Clifford module.

Choosing any α, β, we have trivialisations from each chart, with S_α corresponding to U_α, and S_β corresponding to U_β. Thus on the intersection $U_{\alpha\beta}$, the transition map gives an isomorphism $S_\alpha \to S_\beta$. Succinctly, we have a map

$$\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(S, \gamma) = \mathbb{S}^1.$$

Given any three charts with non-empty intersection, we can compose the above maps to obtain

$$\varphi_{\gamma\alpha} \circ \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} : U_{\alpha\beta\gamma} \to \mathbb{S}^1.$$

Our goal is to glue the maps over U_α to form a spin^c structure on X. For this we require the transition maps to satisfy the *cocycle condition*, in this case $\varphi_{\gamma\alpha} \circ \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = 1$. When is this true?

The obstruction to the above identity is given by Čech 2-cocycles $[\varphi] \in H^2(X, C^\infty\mathbb{S}^1)$ of the Sheaf cohomology, where

$$C^\infty\mathbb{S}^1(U) \coloneqq C^\infty(U, \mathbb{S}^1)$$

for each U. We now show that this particular cohomology group vanishes. Consider the long exact sequence

$$0 \to \mathbb{Z} \to C^\infty\mathbb{R} \xrightarrow{\exp} C^\infty\mathbb{S}^1 \to 0.$$

The corresponding sequence of cohomology is given by

$$H^2(X; C^\infty\mathbb{R}) \to H^2(X, C^\infty\mathbb{S}^1) \to H^3(X; \mathbb{Z}).$$

The cohomology group on the left vanishes because $C^\infty\mathbb{R}$ has partitions of unity, and the cohomology group on the right vanishes by Poincaré duality (since we assumed that $\pi_1(X) = 0$, so in particular $H_1(X; \mathbb{Z}) \cong H^3(X; \mathbb{Z})$ vanishes). It follows that the cohomology group in the middle must also vanish, as required. This shows that our obstruction vanishes, completing the proof of existence in the case $\pi_1(X) = 0$.

For the general case, it turns out that even if $\pi_1(X)$ doesn't vanish (so that $H^3(X; \mathbb{Z})$ is non-trivial), we can still look at the image of $H^2(X, C^\infty\mathbb{S}^1)$ under the induced map to conclude that the cohomology group is trivial. □

Question from class. Doesn't this imply spin structures exist?

Answer. No, recall that spin structures exist if and only if Q_X is even. The reason we don't have a contradiction here is because there is no map Spin$^c \to$ Spin (because of the "modulo 2"). We only have a natural map Spin$^c \to$ SO, which is perfectly fine since the latter is a Riemannian structure. □

Proposition 3.3.2. Classification: suppose (S, γ) and (S', γ') are two spin^c structures. On U_α, we obtain isomorphisms $\psi_\alpha : (S, \gamma)|_{U_\alpha} \to (S', \gamma')|_{U_\alpha}$, and on an intersection $U_{\alpha\beta}$, we have $\psi_\alpha \circ \psi_\beta^{-1} : U_{\alpha\beta} \to \mathrm{Aut}(S, \gamma) = \mathbb{S}^1$. This gives a 1-cocyle in $H^1(X; C^\infty \mathbb{S}^1)$. But this cohomology group is in fact isomorphic to $H^2(X; \mathbb{Z})$, from the exact sequence

$$0 = H^1(X; C^\infty \mathbb{R}) \to H^1(X; C^\infty \mathbb{S}^1) \to H^2(X; \mathbb{Z}) \to H^2(X; C^\infty \mathbb{R}) = 0.$$

Therefore spin^c structures exist, and are non-canonically isomorphic to $H^2(X; \mathbb{Z})$ (they are affine over $H^2(X; \mathbb{Z})$).

Remark. The above result depends on dimension 4: there exist 6-dimensional manifolds admitting no spin^c structures.

Recall that we defined the *class* of a spin^c structure (S, γ) to be the first Chern class $c_1(S^\pm) = c_1(\det S^\pm)$. For $h \in H^2(X; \mathbb{Z})$, we noted that $s + h \in \mathrm{Spin}^c(X)$ was given by the bundle $S^\pm \otimes E$, where E is a complex line bundle with $c_1(E) = h$. Therefore if c denotes the class of s, then the class of $s + h$ is given by

$$c_1(\mathbb{S}^+ \otimes E) = c_1(\det S^+ \otimes E^2) = c_1(S^+) + 2c_1(E) = c + 2h.$$

If $H^2(X; \mathbb{Z})$ has no 2-torsion, (e.g. $\pi_1(X) = 0$), then $s \in \mathrm{Spin}^c(X)$ is determined by its class, $c_1(s) = c_1(S^+)$. In other words, the map $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$ is an injection.

What is the image of $c_1 : \mathrm{Spin}^c(X) \to H^2(X; \mathbb{Z})$? One can show that these are exactly the *characteristic elements* of X;

$$\mathrm{im}\, c_1 = \{k \in H^2(X; \mathbb{Z}) : k \mod 2 = w_2(TX)\} = \mathrm{char}(X).$$

Note that $k \mod 2 = w_2(TX)$ means that $\langle k, a \rangle = \langle a, a \rangle$ modulo 2.

For example, if X is spin, then $w_2(TX)$ vanishes. Therefore $\mathrm{Spin}^c(X)$ is isomorphic to the characteristic elements of X, which are $2H^2(X; \mathbb{Z}) \subset H^2(X; \mathbb{Z})$. If $X = \mathbb{CP}^2$, then $H^2(X; \mathbb{Z}) = \mathbb{Z}$ and $Q_X = (1)$. The characteristic elements of $\mathbb{CP}^2$ are $2\mathbb{Z} + 1 \subset \mathbb{Z}$.

3.4 Hodge theory

Let V denote an inner product space over $\mathbb{R}$, equipped with an orientation. That is, a choice of volume form $\mathrm{vol} \in \Lambda^n V$. The *Hodge star* is the map

$$\star : \Lambda^k V \to \Lambda^{n-k} V$$

such that $\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mathrm{vol}$. Explicitly, if $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, then $v_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ gives a basis for $\Lambda^k V$, and $\star v_I = \pm v_{\overline{I}}$, where $\overline{I}$ is the complement of I in $\{1, \ldots, n\}$.

Example. When $k = 2$, and the dimension of V is 4, then

$$\dim \Lambda^2 V = \binom{4}{2} = 6,$$

with

$$\star : \Lambda^2 V \to \Lambda^2 V, \star^2 = 1, \quad \Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$$

where $\Lambda^{\pm} V$ are the $\pm$ eigenspaces of $\star$. What are the bases for the eigenspaces? For $\Lambda^+ V$, we have basis

$$e_1 \wedge e_2 + e_3 \wedge e_4, \quad e_1 \wedge e_3 - e_2 \wedge e_4, \quad e_1 \wedge e_4 + e_2 \wedge e_3.$$

For $\Lambda^- V$, we have a basis

$$e_1 \wedge e_2 - e_3 \wedge e_4, \quad e_1 \wedge e_3 + e_2 \wedge e_4, \quad e_1 \wedge e_4 - e_2 \wedge e_3.$$

Suppose X is an oriented Riemannian 4-manifold. Then

$$\Lambda^2 T^* X = \Lambda^+ T^* X \oplus \Lambda^- T^* X.$$

The first factor on the right consists of self-dual 2-forms, and the second factor, anti-self-dual 2-forms.

Definition 3.4.1. $\mathcal{H}^2(X; \mathbb{R})$ denotes the *harmonic 2-forms*,

$$\mathcal{H}^2(X; \mathbb{R}) = \{\omega \in \Omega^2(X) : d\omega = d^\star \omega = 0\}.$$

Here $d^\star = - \star d \star$. The action of $\star$ on harmonic 2-forms gives a decomposition

$$\mathcal{H}^2(X; \mathbb{R}) = \mathcal{H}^+ \oplus \mathcal{H}^-$$

of $\pm$ eigenspaces. The self-dual forms give a cohomology theory with $H = \mathcal{H}^+$.

3.5 Hodge meets spin^c (lecture 9)

Recall that the *Hodge star* is a map

$$\star : \Omega^k(M) \to \Omega^{n-k}(M),$$

where M is a closed oriented n-manifold. On the other hand, the exterior derivative is a map

$$d : \Omega^k(M) \to \Omega^{k+1}(M),$$

which gives rise to the de Rham complex, with cohomology $H^k(M; \mathbb{R}) = \ker d / \operatorname{im} d$. We also define the dual of the exterior derivative by

$$d^\star = \pm \star d \star : \Omega^{k+1}(M) \to \Omega^k(M).$$

Theorem 3.5.1. *The* Hodge decomposition theorem *states that*

$$\Omega^k(M) = \operatorname{im} d \oplus \mathcal{H}^k(M) \oplus \operatorname{im} d^\star,$$

where $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta\omega = 0\}$ *are the* Harmonic 2-forms*, and* $\Delta = dd^\star + d^\star d$*.*

Now suppose X is a 4-manifold, so $\star : \Omega^2(X) \to \Omega^2(X)$ is an involution ($\star^2 = 1$). Then

$$\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X),$$

where $\Omega^2_\pm(X)$ are the $\pm$ eigenspaces of $\star$. We can further define projection operators $\Pi^\pm : \Omega^2(X) \to \Omega^2_\pm(X)$ by

$$\Pi^\pm = \frac{1 \pm \star}{2},$$

so that $\Pi^+ + \Pi^- = 1$. We can further define a complex (analogously to the de Rham complex),

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d^+} \Omega^2_+(X), \quad d^+ = \Pi^+ \circ d.$$

Then the second cohomology H^2_+ of the above complex is exactly the $+$ eigenspace of $\star$ acting on $\mathcal{H}^2(X)$.

How does the Hodge star interact with spin^c structures? Suppose X is a 4-manifold with spin^c structure (S, γ). More explicitly, $\gamma : TX \to \operatorname{Hom}(S, S)$, with

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix},$$

so the $\gamma(e_i)$ anticommute. X is Riemannian, so we have a (musical) isomorphism $TX \cong T^*X$. Combining these two observations, γ extends to maps

$$\Lambda^k TX \otimes \mathbb{C} \to \operatorname{End}(S), \quad \gamma(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \gamma(e^{i_1}) \cdots \gamma(e^{i_k}).$$

Lemma 3.5.2. $\Lambda^2_+ \subset \Lambda^2 TX$ acts trivially on S^-, and $\gamma : \Lambda^2_+ \to \mathfrak{su}(S^+)$ is an isomorphism. Note that

$$\mathfrak{su}(S^+) = \{A \in \operatorname{End}(S^+) : A^* = -A, \operatorname{tr}(A) = 0\}.$$

Proof. Check in local bases. For example, $e_1 \wedge e_2 + e_3 \wedge e_4$ acts trivially on S^- since $B_1(-B_2^*) + B_3(-B_4^*) = 0$. $\square$

Corollary 3.5.3. *Fix* $\omega \in \Omega^2_+(X)$*. This gives a section* $\gamma(\omega) \in \Gamma(\mathfrak{su}(S))$*.*

3.6 Connections and curvature

Definition 3.6.1. An "E-valued k-form" is an element of

$$\Omega^k(X;E) = \Gamma(\Lambda^k T^*X \otimes E),$$

where $E \to X$ is a vector bundle. In particular, $\Omega^0(X;E) = \Gamma(E)$.

Definition 3.6.2. Let X be a smooth manifold, and $E \to X$ a vector bundle. A connection ∇_A on E is an operator

$$\nabla_A : \Omega^0(X;E) \to \Omega^1(X;E)$$

satisfying the Leibniz rule, i.e.

$$\nabla_A(fs) = df \otimes s + f\nabla_A s$$

for all smooth $f : X \to \mathbb{R}$ and $s \in \Gamma(E)$.

Remark. If ∇_A and ∇_B are connections, then

$$(\nabla_A - \nabla_B)(fs) = f(\nabla_A - \nabla_B)(s).$$

Therefore the difference of connections is not a connection, but rather it belongs to $\Gamma(\mathrm{Hom}(E, T^*X \otimes E)) = \Omega^1(X; \mathrm{End}(E))$. Hence the set $\{$ connections on $E\}$ is an affine space over $\Omega^1(X; \mathrm{End}(E))$.

Definition 3.6.3. Now let E be a Hermitian complex vector bundle. A connection ∇_A on E is *unitary* if

$$d\langle s,t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle$$

for all $s, t \in \Gamma(E)$.

Remark. If ∇_A and ∇_B are *unitary connections*, then their difference $\nabla = \nabla_A - \nabla_B$ satisfies

$$\langle \nabla s, t\rangle + \langle s, \nabla t\rangle = 0,$$

so $\nabla \in \Omega^1(X; \mathfrak{u}(E))$ where $\mathfrak{u}(E)$ is the unitary Lie algebra, and can be viewed as a subset of $\mathrm{End}(E)$. Hence the set $\{$ unitary connections on $E\}$ is an affine space over $\Omega^1(X; \mathfrak{u}(E))$.

Definition 3.6.4. Let X^4 be a manifold with spin^c structure (S, γ). A *spin^c connection* ∇_A on S is a unitary connection such that

$$\nabla_A(\gamma(v)s) = \gamma(v)\nabla_A s + \gamma(\nabla_{\mathrm{LC}} v)s$$

for all $v \in \Gamma(TX), s \in \Gamma(S)$. On the left side, $\gamma(v)s$ is Clifford multiplication, and ∇_{LC} is the Levi-Civita connection.

Remark. If ∇_A, ∇_B are spin^c connections, then $\nabla = \nabla_A - \nabla_B$ satisfies

$$\nabla(\gamma(v)s) = \gamma(v)\nabla s.$$

Therefore the set $\{$ spin^c connections on $S\}$ is an affine space over $\Omega^1(X; \mathrm{End}(S,\gamma) \cap \mathfrak{u}(S))$ On one hand, $\mathrm{End}(S,\gamma)$ consists of diagonal matrices zI, and on the other, $\mathfrak{u}(S)$ forces $z^* = -z$. Therefore we can make the identification

$$\Omega^1(X; \mathrm{End}(S,\gamma) \cap \mathfrak{u}(S)) = \Omega^1(X; i\mathbb{R}).$$

Hence $\{$ spin^c connections on $S\}$ is an affine space over $\Omega^1(X; i\mathbb{R})$.

Definition 3.6.5. The *curvature* of a connection ∇_A is simply given by $F_A = \nabla_A \circ \nabla_A$.

The idea is that ∇_A induces a connection on higher exterior powers, so the above is really a composition of maps

$$\Gamma(E) \to \Gamma(T^*X \otimes E) \to \Gamma(\Lambda^2 T^*X \otimes E).$$

Then for any $f : X \to \mathbb{R}$ and $s \in \Gamma(E)$, we have

$$F_A(fs) = \nabla_A(df \otimes s + f\nabla_A s) = d^2 f \otimes s + df \otimes \nabla_A s - df \otimes \nabla_A s + fF_A s = fF_A s.$$

This shows that $F_A \in \Omega^2(X; \mathrm{End}(E))$.

Remark. The curvature measures the failure of exactness of the "E-valued de Rham complex".

The curvature satisfies the following properties:

- Let ∇_A have curvature F_A. Then there is a canonical connection ∇_{A^τ} on $\det(E)$, with curvature $F_{A^\tau} = \mathrm{tr}(F_A)$.
- $c_1(E) = \left[\frac{1}{2\pi} F_{A^\tau}\right] \in \mathrm{im}(H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}))$.

Suppose X is a spin^c 4-manifold, with ∇_A a spin^c connection, $F_A \in \Omega^2(X; i\mathbb{R})$. In this case, $F_A = \frac{1}{2} F_{A^\tau}$. Further define $F_A^+ = \Pi^+ \circ F_A \in \Omega^2_+(X; i\mathbb{R}) \cong \Gamma(\mathfrak{su}(S^+))$.

For any $\Phi \in \Gamma(S)$, we obtain an element $(i\Phi\Phi^*)_0 \in \Gamma(\mathfrak{su}(S^+))$. The idea is that $h = i\Phi\Phi^* \in \Gamma(S \otimes S^*) = \Gamma(\mathrm{End}(S))$ is map

$$h : \psi \mapsto i\Phi\langle \Phi, \psi \rangle,$$

so that $h^* = -h$. To ensure that h lies in $\Gamma(\mathfrak{su}(S^+))$, we take the *trace free part*,

$$h_0 = h - \frac{1}{2}(\mathrm{tr} h) I.$$

With this in mind, we define

$$\sigma(\Phi) = \gamma^{-1}(\Phi\Phi^*)_0 \in \Omega^2_+(X; i\mathbb{R}).$$

In fact, we've now developed enough terminology and machinery to state one of the Seiberg-Witten equations:

$$F_A^+ = \sigma(\Phi).$$

Definition 3.6.6. Let X^4 be a 4-manifold with spin^c structure (S, γ) and spin^c connection ∇_A. The *Dirac operator* is defined by the composition

$$\Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S) \xrightarrow{g} \Gamma(TX \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

This is denoted by D_A (or sometimes $\not{D}_A$ or $\not{\partial}_A$). In fact, we can write

$$D_A = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix}, \quad D_A^+ : \Gamma(S^+) \to \Gamma(S^-).$$

Example. Suppose X denotes Euclidean 4-space, with $S^+ = S^- = \mathbb{C}^2$ the trivial bundle. A spin^c structure is given by

$$\gamma(e_i) = \begin{pmatrix} 0 & -B_i^* \\ B_i & 0 \end{pmatrix} = A_i.$$

Then a spin^c connection is given by the the exterior derivative by identifying $\mathbb{C}^2$ to with $\mathbb{R}^4$. Therefore the Dirac operator is given by

$$s \xmapsto{\nabla_A} ds \xmapsto{g} \sum_i e^i \frac{\partial s}{\partial x^i} \xmapsto{\gamma} \Big(\sum_i A_i \frac{\partial}{\partial x^i}\Big) s.$$

Therefore

$$D_A = \sum_i A_i \frac{\partial}{\partial x^i}.$$

But this is exactly a square root of the Laplacian! Recall that the A_i satisfies $A_i^2 = -1$, and $A_iA_j + A_jA_i = 0$ for $i \neq j$.

Example. Suppose $(X^4, g, (S, \gamma), \nabla_A)$ are arbitrary. Then D_A isn't necessarily a square root of the Laplacian - it will also have some curvature terms. Explicitly, this is given by the *Weitzenböck formula*:

$$D_A^2\Phi = \nabla_A^*\nabla_A\Phi + \frac{s}{4}\Phi + \frac{1}{2}\gamma(F_A)\Phi.$$

In the above, s denotes the *scalar curvature* of (X, g). The first term on the right is the Laplacian for Spinors; $\nabla_A^*\nabla_A$. The second term is a curvature term from g, and the third term is the curvature from ∇_A.

Question from class. Does the above formulation hold in pseudo-Riemannian settings?

Answer. Yes, the formulas above come from universal polynomial equations. □

Question from class. Do connections always exist?

Answer. Yes, you can locally trivialise the relevant bundles and locally define the trivial connection. Then by using partitions of unity, the local structures can be glued together. □

3.7 Seiberg-Witten equations

Definition 3.7.1. Let X be a spin^c 4-manifold, with spin^c connection ∇_A. Then the *Seiberg-Witten equations* are

$$D_A^+\Phi = 0, \quad F_A^+ = \sigma(\Phi)$$

where Φ is a positive spinor, i.e. $\Phi \in \Gamma(S^+)$, where $S = S^+ \oplus S^- \to X$ and $\gamma : TX \to \mathrm{End}(S)$ gives the spin^c structure. Recall that σ is a "squaring map", $\sigma(\Phi) = \gamma^{-1}((\Phi\Phi^*)_0)$.

What are some properties of the Seiberg-Witten equations? A very important property is *Gauge invariance*. Let $\mathcal{G} = \Gamma(\mathrm{Aut}(S, \gamma)) = C^\infty(X, \mathbb{S}^1)$. This is called the *Gauge group*. For any $u \in \mathcal{G}$, $u \cdot \Phi \in \Gamma(S^+)$, and we can define $u \cdot \nabla_A$ to be

$$u \cdot \nabla_A = \nabla_A - u^{-1}du.$$

This comes from the Leibniz rule, by computing $u(\nabla_A(u^{-1}\Phi))$. Note that $du \in \Omega^1(X; i\mathbb{R})$, so if $u = e^f$ with $f : X \to i\mathbb{R}$, then $u^{-1}du = df$.

Proposition 3.7.2. The Seiberg-Witten equations are gauge invariant. That is, for any $u \in \mathcal{G}$, if ∇_A and Φ satisfy the Seiberg-Witten equations, so do $u \cdot \nabla_A$ and $u \cdot \Phi$.

This is a non-trivial result in the sense that $\mathcal{G}$ is very large! It is an infinite dimensional group.

Definition 3.7.3. Let ∇_{A_0} be a fixed spin^c connection. We say that A is in *Coulomb gauge* with respect to A_0 if $d^\star(A - A_0) = 0$.

Everything can be put in Coulomb gauge by applying some $u \in \mathcal{G}$. In particular, solutions to the Seiberg-Witten equations modulo $\mathcal{G}$ are equivalent to solutions to Seiberg-Witten in the Coulomb gauge modulo $H^1(X; \mathbb{Z}) \times \mathbb{S}^1$.

3.8 Seiberg-Witten moduli space (lecture 10)

We consider X a closed smooth oriented Riemannian 4-manifold with metric g, and spin^c structure $s = (S, \gamma) \in \mathrm{Spin}^c(X)$, $S = S^+ \oplus S^-$. For simplicity, for the remainder of this lecture we assume X is simply connected ($\pi_1(X) = 0$).

Recall the Seiberg-Witten equations:

$$D_A^+\Phi = 0, \quad F_A^+ = \gamma^{-1}((\Phi\Phi^*)_0),$$

for A a spin^c connection, and $\Phi \in \Gamma(S^+)$. The *Gauge group* is $\mathcal{G} = \Gamma(\mathrm{Aut}(S, \gamma)) = C^\infty(X, \mathbb{S}^1)$. Then the *Seiberg Witten moduli space* is

$$\mathcal{M}_{SW} = \{(A, \Phi) \text{ satisfying SW}\}/\mathcal{G} = \{(A, \Phi) \text{ satisfying SW in Coulomb gauge}\}/\mathbb{S}^1.$$

Recall that a spin^c connection A is in *Coulomb gauge* with respect to A_0 if $d^\star(A - A_0) = 0$. Our goal is to count solutions to the Seiberg-Witten equations. To make sure this is well defined, we begin by proving that $\mathcal{M}_{SW}$ is compact.

Theorem 3.8.1. *$\mathcal{M}_{SW}$ is compact.*

Proof. Suppose (A, Φ) is a solution to SW. By Weitzenböck,

$$D_A^2\Phi = \nabla_A^*\nabla_A\Phi + \frac{s}{4}\Phi + \frac{1}{2}\gamma(F_A)\Phi.$$

On the other hand,

$$d\langle\Phi, \Phi\rangle = \langle\nabla_A\Phi, \Phi\rangle + \langle\Phi, \nabla_A\Phi\rangle = 2\mathrm{Re}\langle\nabla_A\Phi, \Phi\rangle.$$

It follows that

$$\frac{1}{2}\Delta|\Phi|^2 = \frac{1}{2}d^\star d\langle\Phi, \Phi\rangle = d^\star(\mathrm{Re}\langle\nabla_A\Phi, \Phi\rangle) \le \langle\nabla_A^*\nabla_A\Phi, \Phi\rangle.$$

But now by applying the Weitzenböck formula, we have

$$\nabla_A^*\nabla_A\Phi, \Phi\rangle = \langle D_A^2\Phi, \Phi\rangle - \frac{s}{4}\langle\Phi, \Phi\rangle - \frac{1}{2}\langle\gamma(F_A)\Phi, \Phi\rangle.$$

Since (A, Φ) solves SW, $D_A^2\Phi = 0$, so the first term on the right vanishes. Moreover,

$$\gamma(F_A^+) = \gamma(\gamma^{-1}(\Phi\Phi^*)_0).$$

Since Ω_-^2 acts trivially on S^+, $\gamma(F_A)\Phi = (\Phi\Phi^*)_0\Phi$. In summary we have

$$\langle\nabla_A^*\nabla_A\Phi, \Phi\rangle = -\frac{s}{4}|\Phi|^2 - \frac{1}{2}\langle(\Phi\Phi^*)_0\Phi, \Phi\rangle.$$

In a unitary basis at some $x \in X$ we can write $\Phi = (t, 0)$, where $t = |\Phi|$. Then

$$\Phi\Phi^* = \begin{pmatrix} |t|^2 & 0 \\ 0 & 0 \end{pmatrix}, \text{ so that } (\Phi\Phi^*)_0 = \frac{1}{2}\begin{pmatrix} |t|^2 & 0 \\ 0 & -|t|^2 \end{pmatrix}.$$

But then $\langle (\Phi\Phi^*)_0\Phi, \Phi\rangle = \frac{1}{2}|\Phi|^4$. Overall, this gives

$$\frac{1}{2}\Delta|\Phi|^2 \leq \langle \nabla_A^*\nabla_A\Phi, \Phi\rangle \leq -\frac{s}{4}|\Phi|^2 - \frac{1}{4}|\Phi|^4.$$

Since X is compact, we can choose $x \in X$ which maximises $|\Phi|$. Then $0 \leq \Delta|\Phi|^2$, so either $\Phi = 0$ or $|\Phi|^2 \leq -s$. In particular, if $s \geq 0$, then $\Phi = 0$. These pointwise bounds on Φ give L^p bounds on Φ. This induces bounds on $F_A^+ = \gamma^{-1}((\Phi\Phi^*)_0)$.

Write $A = A_0 + ia$, where $a \in \Omega^1(X;\mathbb{R})$. Then $F_A^+ = F_{A_0}^+ + id^+a$. By the Coulomb condition, we have $-id^\star(A - A_0) = d^\star a = 0$. Now considering the homology of

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega_2^+,$$

we have $H^1(X;\mathbb{R}) = \{a : d^+a = d^\star a = 0\}$. It follows that

$$d^+ + d^\star : \Omega^1 \to \Omega_2^+ \oplus (\Omega_0/\mathbb{R})$$

is a linear elliptic injective Fredholm operator. This provides an elliptic estimate. Namely, a bound on $(d^+ + d^\star)a$ induces a bound on a in some Sobolev norm. By elliptic bootstrapping, this gives C^∞ bounds on a and Φ, which then gives compactness. $\square$

3.9 Counting solutions to SW

Solutions to the Seiberg-Witten equations are classified in the following two ways:

$$\begin{cases} \text{Reducible, } \Phi = 0 & \text{fixed points of } \mathbb{S}^1 \text{ action.} \\ \text{Irreducible, } \Phi \neq 0 & \mathbb{S}^1 \text{ action is free: } e^{i\theta} : (A, \Phi) \mapsto (A, e^{i\theta}\Phi). \end{cases}$$

We first investigate reducible solutions. These satisfy the following properties:

$$\Phi = 0, \quad D_A\Phi = 0, F_A^+ = 0, F_{A_0}^+ + d^+a = 0,$$

where $a = A - A_0$. Therefore we have an injective map $d^+ + d^\star$ on the collection of a satisfying $d^\star a = 0$ and $d^+a = -F_{A_0}^+$. We conclude that there is either one solution or no solutions satisfying these properties.

Proposition 3.9.1. The Seiberg-Witten equations have 0 or 1 reducible solutions.

Our next goal is to count irreducible solutions. Define

$$\widetilde{SW} : \text{Connections} \oplus \Gamma(S^+) \to \Omega^2_+(X; i\mathbb{R}) \oplus \Gamma(S^-) \oplus (\Omega^0(X)/\mathbb{R})$$

by

$$(\nabla_A, \Phi) \mapsto (F_A^+ - \gamma^{-1}((\Phi\Phi^*)_0), D_A^+\Phi, d^\star(A - A_0)).$$

We claim that $\mathcal{M}_{SW} = \widetilde{SW}^{-1}(0)/\mathbb{S}^1$. (Again we are assuming for simplicity that our four manifold is simply connected.) For this to be true, we require that 0 is a regular value, so that the preimage is really a submanifold. Therefore we compute the derivative:

$$d\widetilde{SW}_{(A,\Phi)} = (d^+ + \langle \Phi, \ldots \rangle, D_{A_0}^+ + \cdots, d^\star).$$

This is a linear elliptic Fredholm operator! Therefore the *index* defined by

$$\operatorname{ind}(d\widetilde{SW}) = \dim\ker d\widetilde{SW} - \dim\operatorname{coker} d\widetilde{SW} \in \mathbb{Z}.$$

This is invariant under deformations. By the famous Aityah-Singer index theorem, we can actually compute this:

$$\operatorname{ind}(d\widetilde{SW}) = \frac{c_1(s)^2 - \sigma(X)}{4} - b_2^+(X) + b_1(X).$$

(In our case, $b_1(X) = 0$.) Here $\sigma(X)$ is the signature of the intersection form on X. Since the index is invariant under deformations, we consider perturbed Seiberg-Witten equations:

$$\widetilde{SW}(A, \Phi) = (\eta, 0, 0), \quad \eta \in \Omega^2_+(X; i\mathbb{R}).$$

Then $\widetilde{SW}^{-1}(\eta, 0, 0)$ is still compact, and we define $\mathcal{M}_{\widetilde{SW},\eta} = \widetilde{SW}^{-1}(\eta, 0, 0)/\mathbb{S}^1$. By the transversality theorem, for generic η, $\widetilde{SW}^{-1}(\eta, 0, 0)$ is a smooth manifold of dimension the index of $d\widetilde{SW}$. In summary we have established the following:

Proposition 3.9.2. For generic η, $\widetilde{SW}^{-1}(\eta, 0, 0)$ is a smooth manifold of dimension $\operatorname{ind}(d\widetilde{SW})$.

To conclude that the *quotient* $\widetilde{SW}^{-1}(\eta, 0, 0)/\mathbb{S}^1$ is a smooth manifold, we want the $\mathbb{S}^1$ action to be free. But in general the action isn't quite free, since reducible solutions to the Seiberg-Witten equations might exist. We now determine some conditions to eliminate any reducible solutions. Recall that a reducible solution satisfies

$$\widetilde{SW}(A, \Phi) = (\eta, 0, 0), \quad \Phi = 0, \quad F_A^+ = F_{A_0}^+ + d^+a = \eta, \quad d^\star a = 0.$$

Inspecting the sequence

$$\Omega^0 \to \Omega^1 \xrightarrow{d^+} \Omega^2_+,$$

we find that reducible solutions exist if and only if $\eta - F_{A_0}^+$ lies in the image of d^+. This is a codimension $b_+^2(X)$ condition, so to guarantee that there are no reducible solutions for generic η, it suffices to assume that $b_+^2(X) > 0$.

Finally recall that the signature and Euler characteristic of a 4-manifold is given by

$$\sigma(X) = b_2^+ - b_2^-, \quad \chi(X) = 2 - 2b_1 + b_2^+ + b_2^-.$$

Proposition 3.9.3. If $b_+^2(X) > 0$, then for generic η,

$$\mathcal{M}_{SW,\eta} = \mathcal{M}_{SW}(X, s, g, \eta) = \widetilde{SW}^{-1}(0)/\mathbb{S}^1$$

is a smooth compact manifold of dimension

$$d = \frac{c_1(s)^2 - \sigma(X)}{4} - b_2^+ + b_1 - 1 = \frac{c_1(s)^2 - (3\sigma(X) + 2\chi(X))}{4}.$$

In particular, when $d = 0$, $\mathcal{M}_{SW,\eta}$ is a finite collection of points.

3.10 The Seiberg-Witten invariant

In summary we have shown that "counting solutions" is well defined, given a perturbation and $b_+^2(X) > 0$.

Definition 3.10.1. The *Seiberg-Witten invariant* $SW_X(s, g, \eta)$ of X is the signed count of points in $\mathcal{M}_{SW,\eta}$, for some fixed choice of orientations. More precisely, we fix a "homology orientation", i.e. we orient $H^0(X) \oplus H^1(X) \oplus H_+^2(X)$.

Remark. The above definition implicitly assumes that $d = 0$. However, the definition can be extended to any even $d > 0$. But in all known cases, it turns out that $SW_X = 0$ for positive d.

Remark. A 4-manifold X is of *simple type* if $SW_X(s) = 0$ for all s with $d > 0$. *Witten's conjecture* is that all 4-manifolds are of simple type. This is known to be true for symplectic 4-manifolds (e.g. all complex projective surfaces).

Theorem 3.10.2. *If $b_2^+(X) \geq 2$, then $SW_X(s, g, \eta)$ is independent of generic η and g.*

Proof. Consider $(g_0, \eta_0), (g_1, \eta_1)$ in metrics $\times\, \Omega_+^2$. We interpolate by a family (g_t, η_t). Then $M = \bigcup_{t \in [0,1]} \mathcal{M}_{SW,(g_t,\eta_t)}$ is a smooth manifold of dimension $d + 1 = 1$. We can avoid reducibles in a one-parameter family: to do this, we require $b_2^+ \geq 2$, since existence of reducibles are a codimension b_2^+ condition as mentioned earlier. Now M is a 1-manifold with boundary $-\mathcal{M}_{SW,(g_0,\eta_0)} \sqcup \mathcal{M}_{SW,(g_1,\eta_1)}$. But the signed count of boundary points of any 1-manifold is always zero! $\square$

Hereafter we write $SW_X(s)$, for $s \in \mathrm{Spin}^c(X)$, given the assumption $b_2^+ \geq 2$ and $d = 0$.

Remark. When $b_2^+ = 1$, there is a wall of perturbations in which reducibles exist. This wall partitions the space metrics $\times\, \Omega_+^2$ into two chambers. We denote the chambers by $\pm$, and the Seiberg-Witten invariant takes two values $SW_X^+(s)$ and $SW_X^-(s)$, which differ by ± 1.

Recall that $\mathrm{Spin}^c(X)$ is an affine space over $H^2(X;\mathbb{Z})$. When $\pi_1(X) = 1$, we have an injective map

$$c_1 : \mathrm{Spin}^c(X) \to H^2(X;\mathbb{Z}).$$

The image consists of characteristic elements:

$$\mathrm{Spin}^c(X) \cong \{k \in H^2(X;\mathbb{Z}) : \langle k, a\rangle = \langle a, a\rangle \mod 2 \text{ for all } a\} = \mathrm{Char}(X).$$

With this identification established, we can give the final juicy definition of the Seiberg-Witten invariant:

Definition 3.10.3. The *Seiberg-Witten invariant* of a four manifold X is the map

$$SW_X : \mathrm{Char}(X) \to \mathbb{Z}$$

defined by $SW_X(k) = SW_X(s)$ for a Spin^c structure s satisfying $c_1(s) = k$.

We explore the properties of the Seiberg-Witten invariant in the following lecture, and look at some applications.

Chapter 4

Applications of Seiberg-Witten theory

4.1 Basic properties of SW (lecture 11)

Recall that the *Seiberg-Witten invariant* was defined as a map

$$SW_X : \mathrm{Char}(X) \to \mathbb{Z}.$$

We will see that $SW_X(k)$ often vanishes.

Definition 4.1.1. A characteristic element $k \in \mathrm{Char}(X)$ is called a *basic class* if $SW_X(k) \neq 0$.

Eight basic properties of the Seiberg-Witten invariant are as follows:

1. $SW_X(k) = 0$ for all but finitely many k. In other words, there are only finitely many basic classes.

2. If X admits a metric of positive scalar curvature, then SW_X vanishes identically.

3. The Seiberg-Witten invariant satisfies a notion of symmetry:

   $$SW_X(-k) = (-1)^{b_2^+(X)-b_1(X)+1} SW_X(k).$$

   (In our case we only established the existence of the Seiberg-Witten invariant for simply connected 4-manifolds so that $b_1(X) = 0$. However, the Seiberg-Witten invariants exist more generally, in which case the above formula is the correct generalisation.)

4. Another important vanishing property of the Seiberg-Witten invariant is the following: suppose $X = X_1 \# X_2$, where $b_2^+(X_i) \geq 1$ for both $i \in \{1, 2\}$. Then SW_X identically vanishes.

5. A related property is called the *blow up formula*. Suppose X is of simple type, with basic classes $k_1, \dots, k_s$. Then
$$X' = X \# \overline{\mathbb{CP}^2}$$
has basic classes $\{k_i \pm E : i = 1, \dots, s\}$, where $E \in H^2(\overline{CP^2}; \mathbb{Z})$. (This is called the *exceptional class*, and represents $\overline{\mathbb{CP}^1}$ in $\overline{\mathbb{CP}^2}$.) Moreover,
$$SW_{X'}(k_i \pm E) = \pm SW_X(k_i).$$

6. So far we have only shown that SW vanishes in various cases, but have yet to exhibit that SW is ever non-trivial. This is remedied here.

   Suppose X is a complex projective surface. Then $\pm c_1(TX)$ are characteristic elements, and
$$SW_X(\pm c_1(TX)) = \pm 1.$$

7. The following is a generalisation of the previous property, due to Taubes: let X be symplectic, and J a compatible almost-complex structure. Then
$$SW_X(\pm c_1(TX, J)) = \pm 1.$$

8. Adjunction inequality.

   (a) Let $\Sigma \subset X$ be an embedded oriented closed surface, with self intersection number $[\Sigma]^2$ at least 0, and $[\Sigma] \neq 0$. Then for any basic class k on X,
$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |k \cdot [\Sigma]|$$
(where $k \cdot [\Sigma]$ is again the intersection number).

   (b) If X is of simple type, and $g(\Sigma) \neq 0$, the above result also holds in the case where $[\Sigma]^2 < 0$.

4.2 Basic applications of SW

Using these properties, one huge result we can prove is the existence of exotic structures on four manifolds.

Example. Define the following 4-manifolds:
$$X_1 = K3 \# \overline{\mathbb{CP}^2}, \quad X_2 = (\#^3 \mathbb{CP}^2) \# (\#^{20} \overline{\mathbb{CP}^2}).$$

Then X_1 and X_2 are connected sums of simply connected manifolds, and hence simply connected. Hence by Freedman's theorem, if X_1 and X_2 have equivalent intersection forms, they are homeomorphic. Explicitly, we have
$$Q_{X_1} = 2(-E_8) \oplus 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-1), \quad Q_{X_2} = 3(1) \oplus 20(-1).$$

One way of seeing these are equivalent is to use the classification of unimodular bilinear forms. Specifically, we see that $Q_{X_1} = 2(-E_8) \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-1)$ is indefinite and odd. Therefore it is equivalent to $m(1) \oplus n(-1)$ for unique m, n. By inspection, we see that

$$m + n = 2 \times 8 + 3 \times 2 + 1 = 23, \quad m - n = 2 \times (-8) + 0 - 1 = -17.$$

Therefore $m = 3$ and $n = 20$, so Q_{X_1} is equivalent to Q_{X_2}. It follows that X_1 and X_2 are homeomorphic.

To see that they are not diffeomorphic, we show that they have different Seiberg-Witten invariants. Since a K3 surface is a complex projective variety, by property 6, $SW_{K3}(c_1(TK3)) = \pm 1$. But we also know that $K3$ surfaces are Calabi-Yau, so in particular $c_1(TK3) = 0$. (One way to see this is to recall that $Z_d \subset \mathbb{CP}^3$ can be defined as the zero set of a homogeneous degree d polynomial in 4 variables, and then $c_1(TZ_d) = (4-d)h$ for some h. In particular, $K3 = Z_4$, giving us the desired result. This calculation was given in the section on characteristic classes earlier in the notes.)

Therefore $SW_{K3}(0) = \pm 1$. Next by property 5 (the blow up formula), the exceptional class $E \in H^2(\overline{\mathbb{CP}^2}; \mathbb{Z})$ satisfies

$$SW_{X_1}(E) = \pm 1.$$

On the other hand, we can write

$$X_2 = (\#^2\mathbb{CP}^2)\#(\mathbb{CP}^2\#(\#^{20}\overline{\mathbb{CP}^2}).$$

This expresses X_2 as a connected sum of two 4-manifolds each with $b_2^+ \geq 1$. Therefore by property 4, SW_{X_2} identically vanishes. In particular, $SW_{X_1} \neq SW_{X_2}$, so X_1 and X_2 are non-diffeomorphic smooth manifolds.

Finally we note that the vanishing result can also be proven geometrically, using property 2. By a theorem of Schoen and Yau, if M_1, M_2 are manifolds of dimension n at least 3, and M_1 and M_2 each admit metrics with positive scalar curvature, then so does the connected sum. Therefore by noting that the Fubini-Study metric on $\mathbb{CP}^2, \overline{\mathbb{CP}^2}$ has positive curvature, we conclude that X_2 admits a metric of positive scalar curvature as required. Thus $SW_{X_2} = 0$.

Example. Next consider $X_1 = Z_5 \subset \mathbb{CP}^3$. This is a "surface of general type", obtained as the zero set of a quintic homogeneous polynomial in four variables. We also consider $X_2 = (\#^9\mathbb{CP}^2)\#(\#^{44}\overline{\mathbb{CP}^2})$. Both X_1 and X_2 can be shown to be simply connected, with

$$Q_{X_1} = 9(1) \oplus 44(-1) = Q_{X_2}.$$

Therefore by Freedman's theorem, they are homeomorphic. By property 6, X_1 has nontrivial Seiberg-Witten invariant, but X_2 has everywhere vanishing Seiberg-Witten invariant by property 2 or 4. Therefore X_1 and X_2 are not diffeomorphic.

Other applications are the following theorems:

Theorem 4.2.1. *Let X be a symplectic simply connected closed 4-manifold. Then there is no decomposition $X = X_1 \# X_2$, such that $b_2^+(X_i) > 0$ for both $i \in \{1,2\}$.*

Proof. Suppose X has a decomposition $X = X_1 \# X_2$, such that $b_2^+(X_i) > 0$ for both $i \in \{1,2\}$. Then SW_X vanishes identically by property 4. However, If X is symplectic, property 7 ensures that SW_X does not vanish everywhere. Therefore X cannot have a decomposition as above. □

Theorem 4.2.2. *There exist simply connected almost-complex 4-manifolds that are not symplectic.*

Proof. Recall that an *almost complex structure* on X is a complex structure $J \in \mathrm{End}(TX)$, $J^2 = -1$ on TX. That is, (TX, J) is a complex bundle over X.

By algebraic topology, if X is a simply connected 4-manifold, it admits an almost complex structure if and only if $b_2^+(X)$ is odd. But now we can take $X = \#^3\mathbb{CP}^2$, and this is almost complex since $b_2^+ = 3$. However, it cannot be symplectic by the previous theorem, since

$$X = (\mathbb{CP}^2)\#(\#^2\mathbb{CP}^2),$$

and then both factors have positive b_2^+. Therefore by the previous theorem, X cannot be symplectic. □

Question from class. So far all of the SW invariants seem to be 0 or 1, do we ever get large numbers?

Answer. Yes, we can get 200, or 300, or, well, any integer you want! □

4.3 Proofs of the basic properties

Property 1, finiteness. There are only finitely many basic classes. This follows from compactness of the moduli space. □

Property 2, curvature. SW vanishes identically on any 4-manifold admitting a metric with positive scalar curvature. This follows from the Weitzenböck formula. □

Property 3, symmetry. $SW_X(-k) = (-1)^{b_2^+(X) - b_2(X)+1} SW_X(-k)$. Let $(S, \gamma : TX \to \mathrm{End}(S))$ be a spin^c structure. This has a *conjugate structure*, (S^*, γ^*), where $S^* \cong \overline{S}$ via the Hermitian metric. Then $c_1(S^+) = -c_1(\overline{S^+})$. Finally the result follows from the one-to-one correspondence

$$\{\text{Solutions of SW for } (S,\gamma)\} \leftrightarrow \{\text{Solutions of SW for } (S^*,\gamma^*)\}.$$

□

Property 4, connected sum. If $X = X_1 \# X_2$, with $b_2^+(X_i) \geq 1$, then $SW_X = 0$. For our proof outline, we assume X is of simple type. $X_1 \# X_2$ can be decomposed as X_1 and X_2 glued together with a cylinder $\mathbb{S}^3 \times [-T, T]$ between them. The process of *neck stretching* sends T to ∞. Then studying the limit, we can show that

$$\widetilde{\mathcal{M}}_{SW}(X) \cong \widetilde{\mathcal{M}}_{SW}(X_1) \times \widetilde{\mathcal{M}}_{SW}(X_2),$$

where $\widetilde{\mathcal{M}}_{SW}(X)$ consists of (A, Φ) satisfying the Seiberg-Witten equations, and $d^\star(A_0 - A) = 0$ for some A_0. Thus $\mathcal{M}_{SW} = \widetilde{\mathcal{M}}_{SW}/\mathbb{S}^1$.

But now taking their products, the additional factor of $\mathbb{S}^1$ manifests in that $\mathcal{M}_{SW}(X)$ is an $\mathbb{S}^1$-bundle over $\mathcal{M}_{SW}(X_1) \times \mathcal{M}_{SW}(X_2)$. Here the fibre of $\mathbb{S}^1$ corresponds to a gluing parameter. Less formally, we have

$$\{\text{Solutions to SW on } X\} \leftrightarrow \{\text{solutions on } X_1, \text{solutions on } X_2, \text{gluing parameter}\}.$$

But now if $d(X, s)$ denotes the "expected dimension of $\mathcal{M}_{SW}$", namely

$$\frac{c_1(s)^2 - \sigma}{4} - b_2^+ + b_1 - 1,$$

then we must have

$$0 = d(X, s) = d(X_1, s_1) + d(X_2, s_2) + 1.$$

Therefore one of $d(X_i, s_i)$ must be negative, so there are no irreducible solutions for such and (X_i, s_i). Hence $SW_X = 0$. □

Property 5, blow-up formula. If $X' = X \# \overline{\mathbb{CP}^2}$, and X is of simple type, the basic classes of X' are exactly $k \pm E$ where k is a basic class of X, and E is the exceptional class of $\overline{\mathbb{CP}^2}$. Since $b_2^+(\overline{\mathbb{CP}^2}) = 0$, reducible solutions exist generically. Therefore we obtain exactly one reducible solution on $(\overline{\mathbb{CP}^2}, \pm E)$, even though $d(\overline{\mathbb{CP}^2}, \pm E) = -1$. Pairing this with irreducibles on (X, s) gives irreducibles on $(X \# \overline{CP^2}, s \pm E)$. □

Property 6, non-triviality. SW_X is non-trivial on complex projective surfaces. The idea is to interpret solutions to SW on complex surfaces as *divisors*. Then $s = c_1(TX)$ corresponds to the empty curve, from which we conclude that $SW_X(\pm c_1(TX)) = \pm 1$. □

Property 7, non-triviality 2. If X is symplectic and J is a compatible almost complex structure, then $SW_X(\pm c_1(TX, J)) = \pm 1$. This is a huge result proven by Taubes using hard analysis. The result is called "SW = GW", in which it was shown that the Gromov-Witten invariants and Seiberg-Witten invariants are equal (when they are both defined). While SW counts solutions to the SW equations, GW counts the number of J-holomorphic curves. □

Property 8, adjunction inequality, part (a). Let $\Sigma \subset X$ be embedded oriented closed, with non-negative self-intersection number. Then $2g(\Sigma) - 2 \geq [\Sigma]^2 + |k \cdot [\Sigma]|$ for any basic class k.

The first step in this proof is to reduce the inequality to a slightly easier one. By property 3 (symmetry), we can replace k with $-k$, so that it suffices to prove $k \cdot [\Sigma] + [\Sigma]^2 \leq 2g(\Sigma) - 2$ without the absolute value.

The second step in this proof (which is all we can prove in the remainder of this lecture) is to reduce to the case where $[\Sigma]^2 = 0$. We do this via the blow up formula: suppose $\widetilde{X} = X \# \overline{\mathbb{CP}^2}$. Then this contains $\widetilde{\Sigma} = \Sigma \# \overline{\mathbb{CP}^1}$ as an embedded oriented closed surface of the same genus as Σ. Moreover, k is basic in X if and only if $k - E$ is basic in $\widetilde{X}$. Computing intersection numbers, we have

$$[\widetilde{\Sigma}]^2 = [\Sigma]^2 + [\overline{\mathbb{CP}^1}]^2 = [\Sigma]^2 - 1.$$

This gives a way of reducing intersection numbers. Next to verify that the reduction is valid, we need to ensure that the adjunction formula holds for Σ if it holds for $\widetilde{\Sigma}$. This is indeed the case:

$$\begin{aligned} 2g(\widetilde{\Sigma}) &\geq [\widetilde{\Sigma}]^2 + (k - E) \cdot [\widetilde{E}] \\ \Rightarrow 2g(\Sigma) - 2 &\geq [\Sigma]^2 - 1 + k \cdot [E] - E[\overline{\mathbb{CP}^1}] = [\Sigma]^2 k \cdot [E]. \end{aligned}$$

Therefore we can inductively blow up until the self intersection number is zero. We now proceed with the main proof.

Claim: If $[\Sigma]^2 = 0$, then $k \cdot [\Sigma] \leq 2g(\Sigma) - 2$. Since $[\Sigma]^2 = 0$, it has a neighbourhood diffeomorphic to $\Sigma \times D^2 \subset X$. Recalling the result from the previous lecture concerning metric independence (for generic metrics), whenever $SW_X(k) \neq 0$, it must be the case that $\mathcal{M}_{SW}(X, s, g) \neq 0$ for all metrics g. This is because being non-zero is an open condition. Therefore there exists a solution (A, Φ) to the Seiberg-Witten equations on X for any metric g, with no perturbation.

Which metric will we choose? We again do some neck-stretching: we can write X as

$$(D^2 \times \Sigma) \cup ([0, R] \times \mathbb{S}^1 \times \Sigma) \cup (X - (D^2 \times \Sigma)).$$

For each R, will consider g on X to restrict to the product metric on the cylinder $[0, R] \times \mathbb{S}^1 \times \Sigma$. On the first two factors the metric is canonical, and on Σ we choose g to be the constant curvature metric with volume equal to 1. (Typically Σ is hyperbolic, but may also be a torus or a sphere.) We will study how (A, Φ) behaves as R goes to infinity. This proof is continued in the following lecture. $\square$

4.4 Proof of the adjunction inequality for SW (lecture 12)

In the previous lecture, we started a proof of the *adjunction inequality*, which is one of the properties of the Seiberg-Witten invariant. Here we complete the proof. First let us restate the result.

Proposition 4.4.1. Let X be a smooth 4-manifold with $b_2^+ \geq 2$. Let $\Sigma \subset X$ be a smoothly embedded oriented connected surface. Assume $[\Sigma]^2 \geq 0$, with $[\Sigma] \neq 0$. Then for any basic class k,

$$2g(\Sigma) - 2 \geq [\Sigma]^2 + |k \cdot [\Sigma]|.$$

Moreover, if X is of simple type, and $g(\Sigma) \neq 0$, this result holds for all Σ.

The second part of the theorem is due to Ozsváth and Szabó, and we do not give a proof. For the first part, we continue the proof from where we left in the previous lecture.

Proof. In the previous lecture, we showed that we can reduce the problem to showing that

$$2g(\Sigma) - 2 \geq k \cdot [\Sigma]$$

in the case where $[\Sigma]^2 = 0$. In this setting, we will consider metrics on X and study some bounds on solutions to the SW equations.

Lemma 4.4.2. Let (A, Φ) be a solution to the Seiberg-Witten equations on X. Then if s denotes the scalar curvature of X,

$$2\sqrt{2}\|F_A^+\| \leq \|s\|$$

where $\|\cdot\|$ is the L^2 norm on X.

Proof. We use some bounds established in earlier lectures using the Weitzenböck formula. In particular, we established that

$$\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle = -\frac{s}{4}|\Phi|^2 - \frac{1}{4}|\Phi|^4.$$

Integrating each term of X, we have

$$\int |\nabla_A \Phi|^2 + \frac{1}{4}\int |\Phi|^4 = \frac{1}{4}\int (-s)|\Phi|^2.$$

Applying the Cauchy-Schwarz inequality to the right hand side gives

$$\frac{1}{4}\int (-s)|\Phi|^2 \leq \frac{1}{4}\Big(\int s^2\Big)^{1/2}\Big(\int |\Phi|^4\Big)^{1/2}.$$

Combining this with the above equation, we have

$$\frac{1}{4}\int |\Phi|^4 \leq \int |\nabla_A \Phi|^2 + \frac{1}{4}\int |\Phi|^4 \leq \frac{1}{4}\Big(\int s^2\Big)^{1/2}\Big(\int |\Phi|^4\Big)^{1/2}.$$

If Φ vanishes identically the desired inequality holds trivially. Otherwise we divide through by $(\int |\Phi|^2)^{1/2}/4$ to obtain

$$\||\Phi|^2\| = \Big(\int |\Phi|^4\Big)^{1/2} \leq \Big(\int s^2\Big)^{1/2} \leq \|s\|.$$

Since $F_A^+ = \gamma^{-1}((\Phi\Phi^*)_0)$, writing $\Phi = (t, 0)$, we can compute (as in an earlier lecture) that $|\Phi| = t$, while

$$(\Phi\Phi^*)_0 = \frac{1}{2}\begin{pmatrix} t^2 & 0 \\ 0 & -t^2 \end{pmatrix}.$$

But now $|(\Phi\Phi^*)_0| = \frac{1}{2}t^2$. Next we note that γ changes this norm by a factor of square-root-2, so $|F_A^+| = |\gamma^{-1}(\Phi\Phi^*)_0| = \frac{1}{2\sqrt{2}}t^2 = |\Phi|^2$. These pointwise equalities give an L^2 equality,

$$8\|F_A^+\|^2 = 8\int |F_A^+|^2 = \int |\Phi|^4 = \||\Phi|^2\|.$$

Taking the root of each side and equating with the previous inequality gives

$$2\sqrt{2}\|F_A^+\| \leq \|s\|$$

as desired. □

Our goal is to establish the adjunction inequality by combining this result with the Gauss-Bonnet theorem. Before we do this we need one more lemma:

Lemma 4.4.3. Let $\alpha \in \Omega^2(X^4)$ be closed. Then $[\alpha]^2 = \|\alpha^+\|^2 - \|\alpha^-\|^2$ (where the norms are again in L^2).

Proof. This is a direct calculation:

$$\begin{aligned}
[\alpha]^2 &= \int_X \alpha \wedge \alpha \\
&= \int (\alpha^+ + \alpha^-) \wedge (\alpha^+ + \alpha^-) \\
&= \int \alpha^+ \wedge \alpha^+ + \int \alpha^- \wedge \alpha^- + 2\int \alpha^- \wedge \alpha^+ \\
&= \int \alpha^+ \wedge \star\alpha^+ - \int \alpha^- \wedge \star\alpha^- - 2\int \alpha^+ \wedge \star\alpha^- \\
&= \|\alpha^+\|^2 - \|\alpha^-\|^2 + 2\langle \alpha^+, \alpha^- \rangle = \|\alpha^+\|^2 - \|\alpha^-\|^2.
\end{aligned}$$

□

We now apply the two previous lemmas to bound $\|F_A\|$. From Chern-Weil theory, we can write

$$c_1(S^+) = [\frac{i}{2\pi} F_A].$$

Therefore by the most recent lemma,

$$0 = \|F_A^+\|^2 - \|F_A^-\|^2 - 4\pi^2 c_1(S^+)^2.$$

On the other hand, we also have $\|F_A\|^2 = \|F_A^+\|^2 + \|F_A^-\|^2$. Adding the two expressions gives

$$\begin{aligned} \|F_A\|^2 &= 2\|F_A^+\|^2 - 4\pi^2 c_1(S^+)^2 \\ &\leq \frac{1}{4}\|s\|^2 + \text{constant}. \end{aligned}$$

We now restrict ourselves to a certain family of metrics to control the scalar curvature term. We write

$$X = (D^2 \times \Sigma) \cup ([0, R] \times \Sigma \times \mathbb{S}^1) \cup (X - D^2 \times \Sigma).$$

where the "cylinder" is embedded between the two pieces on the ends. We choose metrics g_R which restrict to the product metric on the cylinder, and are the constant-curvature metric with volume 1 on Σ. We will later consider the limit as R goes to infinity, which is called *neck stretching*. We write $X = C \cup (X - C)$.

Since g_R is fixed on the non-cylinder parts of X, we have

$$\int s^2 = \int_{X-C} s^2 + \int_C s^2 = \text{constant} + R \int_\Sigma s_\Sigma^2.$$

By the Gauss-Bonnet theorem, since we chose our metric to be the volume-one constant curvature metric on Σ, we have

$$\int_\Sigma \kappa = 2\pi(2g - 2) \Rightarrow s_\Sigma = 2\kappa = 4\pi(2g - 2).$$

Combining this with the previous inequality, we obtain an almost-topological bound on $\|F_A\|$, namely

$$\|F_A\|^2 \leq R(2\pi(2g - 2))^2 \text{constant}.$$

From Chern-Weil theory, the left side is in fact a bound for $k \cdot [\Sigma]$. That is, for any R,

$$R(2\pi\langle c_1(S^+), [\Sigma]\rangle)^2 = R\Big(\int_\Sigma F_A\Big)^2 \leq \int_C |F_A|^2 \leq \int_X |F_A|^2 = \|F_A\|^2.$$

Therefore for any R, we have

$$R(2\pi\langle c_1(S^+), [\Sigma]\rangle)^2 \leq R(2\pi(2g - 2))^2 \text{constant}$$

where the constant is independent of R. It follows that

$$k \cdot [\Sigma] = \langle c_1(S^+), [\Sigma]\rangle \leq 2g - 2$$

as required. $\square$

4.5 Resolving the Thom conjecture using SW

The *Thom conjecture* is a lower bound on the genus of a smoothly embedded projective surface representing an algebraic curve. To establish these bounds, we pair the *adjunction inequality* with the *adjunction formula* (which we now state).

Theorem 4.5.1. *Let X^4 be equipped with an almost complex structure J. Suppose $\Sigma \subset X$ is J-holomorphic, with $J^*(T\Sigma) = T\Sigma$. Then*

$$2g(\Sigma) - 2 = [\Sigma]^2 - c_1(TX, J) \cdot [\Sigma].$$

Proof. We decompose the tangent bundle as

$$TX|_\Sigma = T\Sigma \oplus N\Sigma.$$

Then

$$c_1(TX) \cdot [\Sigma] = c_1(T\Sigma) \cdot [\Sigma] + c_1(N\Sigma) \cdot [\Sigma] = \chi(\Sigma) + [\Sigma]^2.$$

□

For example, this formula applies to complex curves in complex projective surfaces, or J-holomorphic curves in symplectic 4-manifolds.

Theorem 4.5.2 (Symplectic Thom conjecture, Osváth-Szabó 1998)**.** *Let (X, ω) be a symplectic 4-manifold, and $\Sigma \subset X$ a symplectic surface with volume form $\omega|_\Sigma$. Then Σ is genus minimising in its homology class.*

By genus minimising, we mean any other embedded surface representing the same homology class (not necessarily symplectic) has genus at least that of Σ.

Proof. We note that $[\omega]^2 = [\omega \wedge \omega] > 0$, so $b_2^+(X) \geq 1$. We give a proof assuming $b_2^+(X) \geq 2$, and describe how to prove the general case later.

If $\Sigma = \mathbb{S}^2$, the result holds trivially. Therefore we assume Σ has positive genus, and $b_2^+(X) \geq 2$. Since X is symplectic, it is of simple type. By a result of Taubes, since X is symplectic, $k = -c_1(TX)$ is a basic class. Choose J compatible with ω such that Σ is J-holomorphic.

Let $S \subset X$ be any other surface with $[S] = [\Sigma]$. Then

$$2g(S) - 2 \geq [S]^2 + k \cdot [S] = [\Sigma]^2 + k \cdot [\Sigma] = 2(\Sigma) - 2,$$

where the first equality uses the adjunction inequality, and the second uses the adjunction formula. It follows that $g(S) \geq g(\Sigma)$, so Σ is genus minimising.

If $b_2^+(X) = 1$ (which is the remaining case), then SW takes values which change by ± 1 passing between each of the two chambers. By carefully working through the details, the same proof holds. □

Theorem 4.5.3 (Thom conjecture, Kronheimer-Mrowka 1994)**.** *Let $S \subset \mathbb{CP}^2$ be a smoothly embedded surface, representing the homology class $[C]$ of a degree d complex projective curve C. Then*

$$g(S) \geq \frac{(d-1)(d-2)}{2} = g(C).$$

Proof. First by the symplectic Thom conjecture, $g(S) \geq g(C)$. It remains to show that $\frac{(d-1)(d-2)}{2} = g(C)$. This follows from the adjunction formula:

$$2g(C) - 2 = [C]^2 + k \cdot [C] = d^2 - 3d.$$

Therefore $g(C) = (d-1)(d-2)/2$. □

Corollary 4.5.4 (Local Thom conjecture)**.** *Let $\Sigma \subset \mathbb{C}^2$ be an affine algebraic smooth curve. Then Σ is locally genus minimising.*

By locally genus minimising, we mean that if $B \subset \mathbb{C}^2$ is a ball, $\partial B \pitchfork \Sigma$, and $S \subset B$ is a surface such that $\partial S = S \cap \partial B = \Sigma \cap \partial B$, then

$$g(S|_B) \geq g(\Sigma|_B).$$

Proof. This follows from the Thom conjecture. Compactify Σ to obtain $\overline{\Sigma}$. Perturb to ensure smoothness. Similarly perturb S so that $S \cap \partial B = \overline{\Sigma} \cap \partial B$. By the Thom conjecture,

$$g(S \cup (\overline{\Sigma} - B)) \geq g(\overline{\Sigma}).$$

It follows that $g(S|_B) \geq g(\overline{\Sigma}|_B) = g(\Sigma|_B)$ as required. □

4.6 Resolving the Milnor conjecture using SW

The Milnor conjecture concerns the value of the slice genus of torus knots. Notions of knot genus are generally hard to compute since they consider minimums over large families of objects. Nevertheless, the slice genus of a torus knot can be computed using the local Thom conjecture.

Definition 4.6.1. Let $K \subset \mathbb{S}^3$ be a knot. The *slice genus* of K is

$$g_s(K) = \min\{g(S) : S \subset B^4, \partial S = S \cap \partial B = K\}.$$

Definition 4.6.2. A *slice knot* is a knot with slice genus 0. In other words, any knot that bounds a smoothly embedded disk in the four-ball.

Corollary 4.6.3. *If K arises as a transverse intersection $\partial B \cap S$, with S an affine algebraic curve, then $g_s(K) = g(S|_B)$.*

Proof. This is immediate from the local Thom conjecture. □

Corollary 4.6.4 (Milnor conjecture, Kronheimer-Mrowka 1993)**.** *Let $T_{p,q}$ denote the p,q-torus knot. Then*

$$g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

Proof. By the previous corollary, it remains to show that the torus knot $T_{p,q}$ arises as the intersection of an affine algebraic curve Σ and ∂B^4, where the genus of $\Sigma \cap B$ is $(p-1)(q-1)/2$.

Let $p, q \geq 1$, with $\gcd(p,q) = 1$. Then wrapping p strands on a torus longitudinally, with q twists around the meridian, we obtain the torus knot $T_{p,q}$. Consider the surface

$$\Sigma = \{x^p - y^q = 0\} \subset \mathbb{C}^2.$$

Then it can be shown that $T_{p,q} = \Sigma \cap \partial B(\sqrt{2})$ as follows: first observe that $\partial B(\sqrt{2}) = \{|x|^2 + |y|^2 = 2\}$ contains the torus $T^2 = \{|x| = 1\} \times \{|y| \times 1\}$. Then the parametrisation $x = e^{iq\theta}, y = e^{ip\theta}$ realises the knot, for $\theta \in [0, 2\pi]$.

Unfortunately, our Σ is not quite smooth! We therefore deform to $\Sigma_\varepsilon = \{x^p - y^q - \varepsilon\}$. This is now smooth, and $\Sigma_\varepsilon \cap \partial B(\sqrt{2})$ is isotopic to $\Sigma \cap \partial B(\sqrt{2})$.

Next we determine the genus of $\Sigma_\varepsilon \cap B(\sqrt{2})$. Consider the projection map $(x, y) \in \Sigma_\varepsilon \mapsto x \in \mathbb{C}$. This is a $q : 1$ covering map, branched over the points with $x^p = \varepsilon$. To compute the genus of Σ_ε, we can therefore use the Riemann-Hurwitz formula:

$$1 - 2g(\Sigma_\varepsilon \cap B) = \chi(\Sigma_\varepsilon \cap B) = q\chi(D^2) - p(q-1) = p + q - pq.$$

Therefore

$$g(\Sigma_\varepsilon \cap B) = \frac{(p-1)(q-1)}{2}$$

as required. □

In the next lecture, we will see how this generalises to *quasi-positive knots*.

4.7 Knots bounding affine algebraic curves (lecture 13)

We continue studying some of the implications of Seiberg-Witten theory on knots. One of the "goals" of knot theory is to determine unknotting numbers - this is the most intuitive invariant measuring the complexity of a knot, but is in general difficult to determine.

Definition 4.7.1. Let K be a knot. The *unknotting number* of K, denoted $u(K)$, is the minimum number of crossing changes required to turn K into the unknot, where the minimum is taken over all diagrams.

Example. For the unknot, the unknotting number is clearly 0. On the other hand, any knot that isn't the unknot has unknotting number at least 1.

Since the trefoil 3_1 is distinct from the unknot, $u(3_1) \geq 1$. On the other hand, uncrossing any of the crossings in the standard diagram of the trefoil gives the unknot, so $u(3_1) \leq 1$. Therefore $u(3_1) = 1$.

One of the uses of the slice genus is that it gives a lower bound for the unknotting number.

Proposition 4.7.2. Let K be a knot. Then

$$g_s(K) \leq u(K).$$

Proof. This comes from Morse theory. The idea is that every crossing change corresponds to the addition or removal of the genus of an oriented surface bound by the knot. Passing an index 1 critical point of a Morse function on a slice surface corresponds to changing un-crossings as in figure 4.1. Changing a *crossing* can be achieved by two such moves, so each crossing change adds or removes genus. It follows that at least $g_s(K)$ crossing changes are required to obtain a genus-0 surface, so $g_s(K) \leq u(K)$. ☐

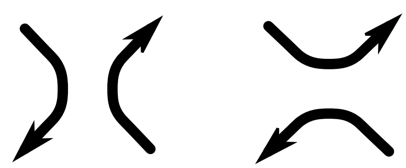


Figure 4.1: Change in level sets for index 1 critical points

Corollary 4.7.3. *The unknotting number of the torus knot $T_{p,q}$ is $(p-1)(q-1)/2$.*

Proof. Since we've already established the slice genus, it remains to show that a diagram of the torus knot can be unknotted with $(p-1)(q-1)/2$ changes - indeed, the standard diagram can be unknotted with this many changes. ☐

In the proof of the Milnor conjecture, the property of torus knots being employed was that they arise as $K = S \cap \partial B^4$, where $S \subset \mathbb{C}^2$ is an affine algebraic curve transverse to ∂B^4. To see how the proof might extend to other knots, it is natural to ask which knots arise as such intersections. This is answered by studying *braid groups.*

Definition 4.7.4. The *configuration space* is defined to be the collection of points

$$\mathrm{Conf}_n(\mathbb{R}^m) = \{\overline{x} \subset \mathbb{R}^n : |\overline{x}| = n\}.$$

(Thus a "point" in $\mathrm{Conf}_n(\mathbb{R}^m)$ is really a collection of n distinct points in $\mathbb{R}^m$.)

Definition 4.7.5. An *n-braid* is a closed loop in $\mathrm{Conf}_n(\mathbb{R}^2)$ which starts and ends at $\{(1,0),\ldots,(n,0)\}$.

Example. A braid can be visualised in $\mathbb{R}^2 \times [0,1]$. An example is given on the left side of figure 4.2. In fact, by inspecting this figure, we see that it can instead be represented as

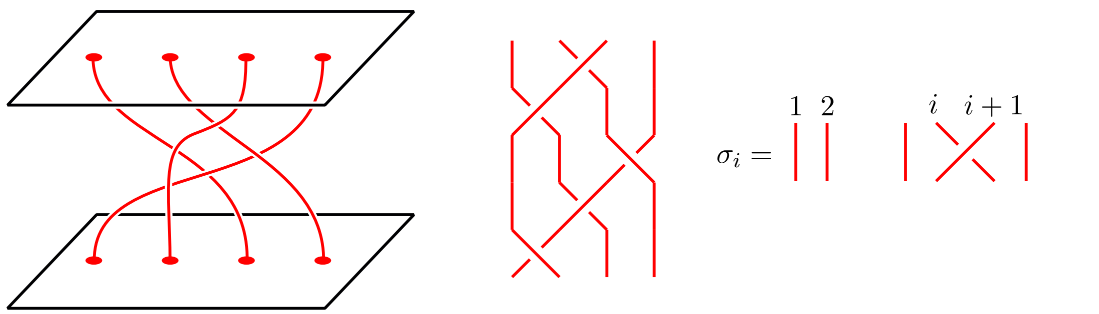


Figure 4.2: Example of a 4-braid and its braid diagram.

a diagram in the plane comprised of arcs with "crossing data" (in much the same way as a link diagram). For this particular example, it translates to the diagram in the middle of figure 4.2. Therefore we see that the braid group is generated by the elements σ_i as shown on the right.

Definition 4.7.6. The *braid group* on n braids is the collection of equivalence classes of braids in $\mathrm{Conf}_n(\mathbb{R}^2)$, where two braids are equivalent if they can be homotoped from one to the other. Therefore

$$B_n \coloneqq \pi_1(\mathrm{Conf}_n(\mathbb{R}^2)).$$

By figure 4.2, we see that

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle.$$

Definition 4.7.7. A braid is *positive* if it can be expressed as a product of σ_is and no inverses. A braid is *quasi-positive* if it is of the form

$$\prod_{k=1}^{m}(w_k\sigma_{i_k}w_k^{-1})$$

where w_k is any word in the braid group.

Definition 4.7.8. Given any braid, its *closure* is the link formed by gluing the start points of each braid to the corresponding end points.

Theorem 4.7.9. *Every link is the closure of a braid.*

Definition 4.7.10. A knot (or link) is called *braid positive* if it is the closure of a positive braid. Similarly a knot (or link) is *quasi-positive* if it is the closure of a quasi-positive braid.

Remark. We don't call braid positive links "positive", since they refer to oriented links with only positive crossings - a different notion.

Theorem 4.7.11 (Rudolph, Boileau-Orevkov)**.** *A knot $K \subset \mathbb{S}^3$ arises as $K = S \cap \partial B$, with $S \subset \mathbb{C}^2$ an affine algebraic curve transverse to ∂B, if and only if K is quasi-positive. Moreover, in this instance, K bounds a complex curve of genus*

$$g(S \cap B) = \frac{m - n + 1}{2},$$

provided that $K = \bar{b}$, for b a quasi-positive n-braid of the form

$$b = \prod_{k=1}^{m} w_k \sigma_{i_k} w_k^{-1}.$$

Corollary 4.7.12. *The slice genus of a quasi-positive knot K is $g_s(K) = (m - n + 1)/2$, with m and n as above.*

Remark. The m and n above are not unique, but $m - n$ is always fixed for any quasi-positive knot K.

Remark. For braid positive knots, it can be shown that $u(K) = g_s(K)$. However, for quasi-positive knots, this is not generally true. A counter example is given by the knot 8_{20}.

4.8 Donaldson diagonalisability theorem

We now give a proof of Donaldson's diagonalisability theorem, which was instrumental in classifying homeomorphism classes of 4-manifolds in terms of intersection forms. Recall the following theorem from chapter 1 of the notes:

Theorem 4.8.1. *Let X be a simply connected closed smooth 4-manifold. Then the homeomorphism class of X is determined uniquely by*

$$\sigma(Q_X), \quad \textit{parity}(Q_X), \quad \chi(X).$$

The ingredients of this theorem are the classification of symmetric unimodular Z-bilinear forms, Freedman's theorem, and Donaldson's diagonalisability theorem.

Theorem 4.8.2 (Donaldson diagonalisability)**.** *Let X be a simply connected closed smooth 4-manifold. Then if Q_X is definite, it is diagonal.*

The original proof, in 1982, used Yang-Mills theory. Here we use Seiberg-Witten theory to give a proof outline.

Proof. Suppose Q_X is negative definite. (Otherwise we can reverse the orientation of X.) Then $b_2^+ = 0$, so we cannot avoid reducibles among the solutions to the Seiberg-Witten equations. Choose a metric and spin^c-structure, and let $c_1(S^+) = k$ be a characteristic element of $H^2(X)$. Then we write

$$\widetilde{\mathcal{M}}_{SW} = \{(A,\Phi) \text{ solutions to SW}, d^\star(A - A_0) = 0\}, \quad \mathcal{M}_{SW} = \widetilde{\mathcal{M}}_{SW}/\mathbb{S}^1.$$

By transversality, for a generic perturbation η, $\widetilde{\mathcal{M}}_{SW,\eta}$ is a smooth manifold of dimension

$$d + 1 = \frac{k^2 - \sigma}{4} - b_2^+ + b_1 = \frac{k^2 + b_2}{4}.$$

The second equality comes from the fact that Q_X is negative definite, so $b_2(X) = b_2^-(X) = -\sigma(X)$. The d is the *expected dimension* of $\mathcal{M}_{SW}$.

Now we count the number of reducible solutions, by studying (A, Φ) such that $\Phi = 0$, $F_A^+ = \eta$, $F_{A_0}^+ + d^+(A - A_0) = \eta$. To solve for $d^+a = \eta - F_{A_0}^+$, we inspect the sequence

$$\Omega^0 \to \Omega^1 \to \Omega^2_+.$$

In this case $H^1 = H^2_+ = 0$, so the equation $d^+a = \eta - F_{A_0}^+$ has a unique solution modulo gauge ($d^\star = 0$). This means there is a unique reducible solution x.

Now we study the geometry of $\mathcal{M}_{SW}$. away from the reducible solution, it's a smooth manifold of dimension $d = 2m - 1$. At the reducible solution, the local model is a cone on $\mathbb{CP}^{m-1}$ (since the reducible is a fixed point of the $\mathbb{S}^1$ action). Thus consider the smooth manifold $\mathcal{M}^*_{SW} = \mathcal{M}_{SW} - \{x\}$. The $\mathbb{S}^1$ gives a line bundle $L \to \mathcal{M}^*_{SW}$ where

$$L_{[(A,\Phi)]} = \{(A, z\Phi) : z \in \mathbb{C}\}.$$

Then $L|_{\mathbb{CP}}$ is the tautological bundle. But this almost gives a contradiction! Write $c_1(L) = u \in H^2(\mathcal{M}^*_{SW})$. Then u restricted to $\mathbb{CP}^{m-1}$ is a generator of $H^2(\mathbb{CP}^{m-1})$. Therefore $u^{m-1} \cdot [\mathbb{CP}^{m-1}] = 1$. But this is nonsense, because $\mathbb{CP}^{m-1}$ bounds a cycle in $\mathcal{M}^*_{SW}$ (namely the complement of a neighbourhood of the reducible solution x). This means that $[\mathbb{CP}^{m-1}] = 0 \in H^2(\mathcal{M}^*_{SW})$, which is a contradiction! (Almost!)

This contradiction holds assuming m is positive. But m could be non-positive. In this case, $\mathcal{M}^*_{SW}$ is empty, so $\widetilde{M}_{SW} = \{x\}$.

In summary it follows that m is necessarily negative, so that

$$m = \frac{k^2 + b_2}{8} \leq 0.$$

This rearranges to the requirement that $k^2 \leq -b_2$. We now reduce to algebra: we have a symmetric unimodular $\mathbb{Z}$-bilinear form

$$Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}, \quad k^2 + r \leq \text{ for all } k \in \text{Char}.$$

But now the proof reduces to a theorem of Elkies:

Theorem 4.8.3 (Elkies)**.** *If Q is a symmetric unimodular bilinear form with the above property, then Q is diagonal.*

It follows that any simply connected closed smooth 4-manifold with a *definite* intersection form has a *diagonalisable* intersection form. □

4.9 Infinitude of smooth structures on $K3$.

We use *Fintushel-Stern knot surgery* to prove that the homeomorphism class of $K3$ surfaces admit infinitely many smooth structures.

Definition 4.9.1. *Fintushel-Stern knot surgery* is the following procedure:

- Fix a four manifold X, and a knot K. Suppose there is an embedding of a torus $T^2 \hookrightarrow X$ with "elliptic fibre", that is to say $[T^2]^2 = 0$ and $[T^2] \neq 0$.
- Let $N(T^2)$ denote a regular neighbourhood of T^2, which can be written as $N(T^2) = T^2 \times D^2 \subset X$. Then $\partial N(T^2) = T^2 \times \mathbb{S}^1 = T^3$.
- Our knot K also determines a *knot complement*, which is $\mathbb{S}^3 - N(K)$, for a regular neighbourhood $N(K) = K \times D^2$ of K. But then $\mathbb{S}^1 \times (\mathbb{S}^3 - N(K))$ has boundary $\mathbb{S}^1 \times \partial(\mathbb{S}^3 - N(K)) = \mathbb{S}^1 \times \partial N(K) = T^3$.
- Since $N(T^2)$ and $\mathbb{S}^1 \times (\mathbb{S}^3 - N(K))$ have the same boundaries, we define the *Fintushel-Stern knot surgery* to be

  $$X_K = (X - N(T^2)) \sqcup_{T^3} (\mathbb{S}^1 \times (\mathbb{S}^3 - N(K))).$$

  There is a choice in the gluing: we send a meridian of K to $*\partial D^2$.

Theorem 4.9.2. *If X_K is obtained from a simply connected smooth manifold X by Fintushel-Stern knot surgery, then X_K is homeomorphic to X.*

Proof. Suppose X is simply connected. By Mayer-Vietoris and Seifert Van-Kampen, one can show that for any K, $\pi_1(X_K) = \pi_1(X) = 1$, and $Q_{X_K} = Q_X$. Therefore by Freedman's theorem, X is homeomorphic to X_K. □

Definition 4.9.3. Let X be a 4-manifold. The *Seiberg-Witten series* of X is the formal power series

$$\mathcal{SW}_X = \sum_{k \in \mathrm{Char}(X)} SW_X(k) e^k$$

where SW_X is the Seiberg-Witten invariant of X.

Example. Let X be a $K3$ surface. Then $c_1(X) = 0$, so $SW_X(0) = 1$. In fact, $\mathcal{SW}_X = 1$.

Theorem 4.9.4 (Fintushel-Stern)**.** *Let X_K be the smooth manifold obtained by Fintushel-Stern surgery along the knot K and torus T. Then*

$$\mathcal{SW}_{X_K} = \mathcal{SW}_X \Delta_K(t), \quad t = e^{2[T]}.$$

In the above theorem, $\Delta_K(t)$ denotes the Alexander polynomial with the "symmetric normalisation" $\Delta(t) = \Delta(t^{-1})$. Recall that the Alexander polynomial is characterised by skein relations:

- $\Delta_0(t) = 1$.
- $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$.

Here L_+, L_-, and L_0 correspond to the same link with a single crossing modified: L_+ has the positive oriented crossing, L_- the negative crossing, and L_0 the un-crossing.

Example. The Alexander polynomial of the trefoil knot is $t - 1 - t^{-1}$. The Alexander polynomial of Borromean rings is $(t-1)^4$, multiplied by some $\pm t^k$ so that it becomes "symmetric" in the sense mentioned above.

Since the trefoil knot has Alexander polynomial $t - 1 - t^{-1}$, if X_K is obtained from $K3$ via Fintushel-Stern surgery along $K = 3_1$, then

$$\mathcal{SW}_{X_K} = e^{2[T]} - 1 + e^{-2[T]}.$$

I.e.

$$SW_{X_K}(s) = \begin{cases} 1 & \text{if } s = \pm 2[T] \\ -1 & \text{if } s = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.9.5. The Alexander polynomial satisfies the following properties:

- $\Delta_K(t) = \Delta_K(-t)$
- $\Delta_K(1) = 1$
- All polynomials satisfying the above arise as $\Delta_K(t)$ for some K.

Corollary 4.9.6. *If K_1, K_2 are knots with distinct Alexander polynomials, then X_{K_1} and X_{K_2} are homeomorphic but not diffeomorphic.*

Corollary 4.9.7. *By the previous theorem, for a knot K, if X_K is obtained by Fintushel-Stern knot surgery from $K3$ along K, then*

$$\mathcal{SW}_{X_K} = \Delta_K(t).$$

By the previous proposition there are infinitely many distinct Alexander polynomials, so there are infinitely many smoothly distinct X_K. But these are all homeomorphic to $K3$, so the homeomorphism class of $K3$ admits infinitely many smooth structures.

Open question. Do there exist knots K_1, K_2 such that $\Delta_{K_1}(t) = \Delta_{K_2}(t)$, but X_{K_1} is not diffeomorphic to X_{K_2}?

One easy way to obtain distinct knots with the same Alexander polynomial is to reflect the knot. (e.g. the trefoil knot is chiral; i.e. not equivalent to its reflection.) However, in this case the following results are known:

Proposition 4.9.8. Let K, Q be knots. Denote the reflection of K by $\overline{K}$. Then for any X, the manifolds $X_{Q\#K}$ and $X_{Q\#\overline{K}}$ are diffeomorphic.

Note that the above holds when Q is the unknot, so $X_K \cong X_{\overline{K}}$ is a special case.

The Alexander polynomial is determined by the knot complement (which we can see in the way that the Seiberg-Witten series is determined by the Alexander polynomial, which itself comes from the knot complement). Moreover, one can show that the knot complement is an Eilenberg Maclane space,

$$\mathbb{S}^3 - N(K) = K(\pi_1(\mathbb{S}^3 - K), 1).$$

Therefore the Seiberg-Witten invariant is determined by the fundamental group of the knot complement. This motivates the following open question:

Open question. Does $\pi_1(\mathbb{S}^3 - K)$ determine the diffeomorphism type of X_K?

If this open question holds true, then diffeomorphism types of a given homeomorphism class of a smooth manifold are at least as complicated as the fundamental groups of knot complements. Moreover, the following theorem holds:

Theorem 4.9.9. *If K_1, K_2 are prime and have the same knot groups (i.e. the fundamental groups of their knot complements are isomorphic), then $K_1 = K_2$ or $K_1 = \overline{K_2}$.*

This means that fundamental groups of knot complements are at least as complicated as knots. Therefore if the previous open question holds true, we would have *simply connected smooth 4-manifolds are at least as complicated as knots.* That would be unfortunate from a classification viewpoint!

4.10 Furuta's 10/8-theorem (lecture 14)

Suppose X^4 is a smooth, closed, simply connected spin 4-manifold. (Recall that X admits a spin structure if and only if Q_X is even.) Then

$$b_2(X) \geq \frac{11}{8}|\sigma(X)|.$$

Equivalently,

$$Q_X = m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2p(-E_8),$$

with $m \geq 2p$.

Proof. We give a proof sketch. The main ideas are as follows:

1. Study the Pin(2) symmetries of the solution space to SW. Construct a Pin(2)-equivariant "SW map".
2. Extend the analysis of this map to Sobolev completions (which are easier to work with in this case).
3. Construct and study finite dimensional approximations to the infinite dimensional map.
4. Use Furuta's theorem on local properties of the approximations to more closely study the SW map, to conclude the result.

1. Since Q_X is even, 0 is a characteristic element of X. Define a spin^c structure s on X to be a lift of the underlying spin structure. Then $c_1(s) = 0$.

It follows that $s = \overline{s}$, so the space of solutions to the SW equations is invariant under conjugation:

$$j : (A, \Phi) \mapsto (-A, \overline{\Phi})$$

sends solutions to solutions. On the other hand, Gauge invariance ensures that the solution space is also invariant under

$$(A, \Phi) \mapsto (A, e^{i\theta}\Phi).$$

Combining these symmetries gives Pin(2) invariance of solutions to SW:

$$\mathrm{Pin}(2) = \mathbb{S}^1 \cup j \cdot \mathbb{S}^1 \subset \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}.$$

Recall the map

$$\widetilde{SW} : \Gamma(S^+) \oplus \text{Connections} \to \Gamma(S^-) \oplus \Omega^2_+ \oplus (\Omega^0/\mathbb{R})$$

defined by

$$\widetilde{SW}(\Phi, A) = (\not{D}_A\Phi, F_A^+ - \gamma^{-1}(\Phi\Phi^*)_0, d^\star(A - A_0)).$$

This is Pin(2)-equivariant. We write this as

$$\widetilde{SW} : \mathbb{H}^\infty \oplus \widetilde{R}^\infty \to \mathbb{H}^\infty \oplus \widetilde{R}^\infty,$$

where the Pin(2) is given by left multiplication on $\mathbb{H}^\infty$, and its action on $\widetilde{\mathbb{R}}$ is given by

$$\mathbb{S}^1 \cdot x = x, \quad j \cdot x = -x.$$

2. To study this map, we want to work in the completions of the domain and codomain so that functional analysis can be applied. We begin by introducing some terminology. Let $E \to M$ be a vector bundle over a Riemannian manifold, where E is equipped with an inner product and connection. We denote the space of smooth sections by $C^\infty(E)$, and the kth Sobolev completion by $L^2_k(E)$. The kth Sobolev completion is the space of all φ such that

$$\nabla^i \varphi \in L^2, i \in \{0, \ldots, k\}.$$

The L^2_k norm is

$$\|\varphi\|^2_{L^2_k} = \|\varphi\|^2 + \cdots + \|\nabla^k \varphi\|^2.$$

Note that different metrics and connections give rise to equivalent norms. In our case, we obtain a map

$$\widetilde{SW} : \mathcal{H} = L^2_k(S^+ \oplus T^*X) \to \mathcal{H}' = L^2_{k-1}(S \oplus \Lambda^2_+ T^*X \oplus \underline{\mathbb{R}})/\mathbb{R}.$$

We can decompose this map into two pieces:

$$\widetilde{SW} = \ell + c, \quad \ell = d_{A_0}\widetilde{SW} = (\not{D}_{A_0}, d^+ + d^\star).$$

Then ℓ is the linear part of $\widetilde{SW}$, and c consists of the constant and quadratic terms. From Fredholm theory, we have

$$\operatorname{ind} \ell = \dim \ker \ell - \dim \operatorname{coker} \ell = \frac{c_1(s)^2 - \sigma}{4} - b_2^+.$$

This is realised by

$$\operatorname{ind} \not{D}_{A_0} = \frac{c_1(s)^2 - \sigma}{4} = -\frac{\sigma}{4}, \quad \operatorname{ind}(d^+ + d^\star) = -b_2^+.$$

3. We now investigate finite dimensional approximations of $\widetilde{SW} : \mathcal{H} \to \mathcal{H}'$. Choose a sequence of finite dimensional subspaces

$$\operatorname{coker} \ell \subset V_k \subset V_{k+1} \subset \cdots \subset \mathcal{H}',$$

such that $\bigcup_n V_n$ is dense in $\mathcal{H}'$. (The subscript denotes the dimension.) Let $U_n = \ell^{-1}(V_n)$. The Sobolev completions are Hilbert spaces, so we have access to orthogonal projections. For each n, define

$$\widetilde{SW}_n = \ell + \operatorname{proj}_{V_n} c : U_n \to V_n.$$

This is the "finite dimensional approximation" to $\widetilde{SW}$.

4. An important lemma by Furuta concerning the local structure of solutions is the following:

Lemma 4.10.1. There exists $R, \varepsilon > 0$ such that for all sufficiently large n, if $x \in U_n$ satisfies $\|x\| < 2R$, and $|\widetilde{SW}_n(x)| < \varepsilon$, then $\|x\| < R$.

The main idea is that $\widetilde{SW}_n$ is a good approximation to $\widetilde{SW}$ on bounded sets. From this theorem, we obtain a map

$$\widetilde{SW}_n^+ : B(2R)/\partial B(2R) \to B(\varepsilon)/\partial B(\varepsilon)$$

defined by

$$x \mapsto \begin{cases} \widetilde{SW}_n(x) & \text{if } |\widetilde{SW}(x)| < \varepsilon \\ * & \text{otherwise.} \end{cases}$$

Up to homotopy, this is a Pin(2)-equivariant map $\widetilde{SW}_n^+ : U_n^+ \to V_n^+$. As Pin(2) representations, we can write $V_n = \mathbb{H}^a \oplus \widetilde{R}^b$ for some a and b. But now using finite dimensionality and the index of ℓ, we see that $U_n = \mathbb{H}^{a-\sigma/16} \oplus \widetilde{\mathbb{R}}^{b-b_2^+}$. Explicitly, this is because

$$\operatorname{ind} \ell = \dim \ker \ell - \dim \operatorname{coker} \ell = \dim U_n - \dim V_n,$$

so writing $Q_X = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2p(-E_8)$ gives $\sigma = -16p$ and $b_2^+ = m$. In summary, we obtain a Pin(2)-equivariant map

$$f : (\mathbb{H}^{a+p} \oplus \widetilde{\mathbb{R}}^{b-m})^+ \to (\mathbb{H}^a \oplus \widetilde{R}^b)^+.$$

The map has natural restrictions on the direct summands. Restriction onto the second summand gives a map

$$f : (0, a) \mapsto (0, (d^+ + d^\star)a), \quad a = A - A_0.$$

This is a linear map corresponding to the inclusion $(\widetilde{\mathbb{R}}^{b-m})^+ \hookrightarrow (\widetilde{R}^b)^+$. The existence of such a Pin(2)-equivariant map forces, essentially by the Borsuk Ulam theorem, that $m \geq 2p + 1$. This implies the 10/8-theorem. $\square$

Recently (2018) the theorem was improved:

Theorem 4.10.2 (Hopkins, Lin, Shi, Xu)**.** *If X is a simply connected closed smooth 4-manifold, with $n = 2p \geq 4$, then*

$$m \geq \begin{cases} 2p+2 & p \equiv 1, 2, 5, 6 \\ 2p+3 & p \equiv 3, 4, 7 \\ 2p+4 & p \equiv 0 \end{cases} \mod 8.$$

More precisely, it was shown using Pin(2)-equivariant stable homotopy theory that any map as constructed at the end of the above proof exists *if and only if*

$$n \geq \begin{cases} 2p+2 & p \equiv 1,2,5,6 \\ 2p+3 & p \equiv 3,4,7 \\ 2p+4 & p \equiv 0 \end{cases} \qquad \bmod 8.$$

Since this is a complete characterisation, it shows that this is the best possible result that can be obtained using Sieberg-Witten theory.

Open question. Is the "11/8"-theorem true? That is, if

$$Q_X = m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2p(-E_8),$$

then is $m \geq 3p$?

If so, this would complete the classification of homeomorphism classes of simply connected smooth 4-manifolds.

4.11 Exotic smooth structures on $\mathbb{R}^4$

In this section, we prove the existence of an exotic smooth structure on $\mathbb{R}^4$. Later we give a proof outline that there are infact uncountably many distinct smooth structures on $\mathbb{R}^4$.

Our first example with arise from

$$X = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}.$$

Its intersection form is

$$Q_X = (1) \oplus 9(-1) \cong (-E_8) \oplus (-1) \oplus (1).$$

Denote by α an element of $H_2(X;\mathbb{Z})$ that spans the (1) term of Q_X. Suppose for a contradiction that α is represented by a *smoothly embedded* 2-sphere $\mathbb{S}^2$. We can then choose a D^2-bundle over $\mathbb{S}^2$ with Euler number 1, so that the boundary of the neighbourhood is $\mathbb{S}^3$. Then this boundary realises a smooth connected sum

$$X = X' \# \mathbb{CP}^2, \quad Q_{X'} = (-E_8) \oplus (-1).$$

But one can show that $(-E_8) \oplus (-1)$ is not diagonalisable, despite being definite. This contradicts Donaldson's diagonalisability theorem, so X' cannot be smooth.

While α cannot be represented by a smoothly embedded sphere, it can actually be represented by a topologically embedded sphere. This follows from Freedman's theory: by gluing Casson handles onto disks, one can construct $\Sigma \subset X$ such that Σ is a topological sphere representing α, and moreover Σ has a neighbourhood which smoothly embeds in $\mathbb{CP}^2$. Namely,

$$U \cong \mathbb{CP}^2 - B^4, \quad U \text{ a neighbourhood of } \Sigma.$$

Proposition 4.11.1. $Z = \mathbb{CP}^2 - \Sigma$ is homeomorphic to $\mathbb{R}^4$, but not diffeomorphic.

Proof. We first establish that Z is not homeomorphic to $\mathbb{R}^4$ by using Freedman's theorem for open 4-manifolds. One can verify that $\pi_1(Z) = 1$, and $H_*(Z) = 0$, using Seifert-van Kampen and Mayer-Vietoris. In particular, Z is contractible.

Next note that Z is "simply connected at infinity". I.e. for all $C \subset Z$ compact, there exists $D \subset Z$ compact with $C \subset D$ such that $\pi_1(Z - D) \to \pi_1(Z - C)$ is trivial. In our case, $Z \cap U \cong \mathbb{CP}^2 - B^4 - \mathbb{CP}^1 = \mathbb{S}^3 \times (0, 1)$. This is the "end" of Z. This is simply connected as required.

By Freedman's theorem for open 4-manifolds, it follows that Z is homeomorphic to $\mathbb{R}^4$.

$\sim$

Next we show that Z is *not* diffeomorphic to $\mathbb{R}^4$. Assume it is. Any compact subset of $\mathbb{R}^4$ can be enclosed in a smoothly embedded 3-sphere (i.e. given any $C \subset \mathbb{R}^4$ compact, we can write $\mathbb{R}^4 = (\mathbb{R}^4 - B^4) \cup B^4$, with $C \subset B^4$.) In our case, let $K = \mathbb{CP}^2 - U$, where U is the neighbourhood defined earlier. Then K is compact. Suppose for a contradiction that K is enclosed in a smooth 3-sphere. Then

$$X'' = (X - \mathrm{nbhd}(\Sigma)) \cup_{\mathbb{S}^3} B^4$$

is a smooth simply connected manifold, with

$$Q_{X''} = (-E_8) \oplus (-1).$$

But by Donaldson's theorem, this is impossible! □

Theorem 4.11.2. *There uncountably many exotic $\mathbb{R}^4$s.*

Proof. We give a proof outline. Above we verified the existence of one exotic $\mathbb{R}^4$, which we denote by $E\mathbb{R}^4$. Let $h : \mathbb{R}^4 \to E\mathbb{R}^4$ be a homeomorphism. Then write

$$h(B^4(\rho)) = E\mathbb{R}^4_\rho.$$

The refined theorem is the following:

Theorem 4.11.3. *There exists $\rho_0 > 0$ such that for all $s > t > \rho_0$, $E\mathbb{R}^4_s$ is not diffeomorphic to $E\mathbb{R}^4_t$.*

Therefore our one example earlier provides an uncountable family. A proof sketch of the refined result is as follows: suppose $\varphi : E\mathbb{R}^4_t \to E\mathbb{R}^4_s$ is a diffeomorphism. Let $h(\mathbb{S}^3(\rho)) = Y_\rho$. Next choose $x \in (t, s)$ and consider $\varphi^{-1}(Y_x)$. This gives copies of Y_x on either side of Y_t. By repeatedly gluing, we obtain a smooth 4-manifold "with periodic end". One can conclude that the intersection form Q of this smooth manifold with periodic end is actually just $Q_{X-U} = (-E_8) \oplus (-1)$.

However, Taubes proved a version of Donaldson's theorem for smooth manifolds with periodic ends from which it follows that Q must be diagonalisable. This is a contradiction, so φ cannot exist. □

Remark. The proof of Donaldson's theorem for smooth manifolds with periodic ends uses Yang-Mills theory. There is no known proof using Seiberg-Witten theory.

Question from class. Does the space of all smooth structures on $\mathbb{R}^4$ have any meaningful structure? topological or algebraic?

Answer. No, the above only gives a small family, in general we know nothing about what the space of *all* smooth structures looks like. $\square$

Chapter 5

Khovanov homology

The Khovanov homology is an intrinsically "combinatorial" invariant of knots. Using this, we will obtain the following:

- A new proof of the Milnor conjecture (Rasmussen)
- A new proof of the Thom conjecture (Lambert-Cole)
- Existence of exotic $\mathbb{R}^4$s (Rasmussen-Gompf)
- A possible approach to disprove the smooth 4-dimensional Poincaré conjecture.

5.1 Definition of Khovanov homology (lecture 15)

Today we explore the definition and proof of invariance. We work with an oriented link $L \subset \mathbb{S}^3$, with planar diagram $D \subset \mathbb{R}^2$. Recall that Reidemeister moves of link diagrams characterise isotopy of links.

Proposition 5.1.1. The outline of Khovanov homology is as follows:

1. For each link diagram D, there is a corresponding cochain complex

$$D \rightsquigarrow C(D) = \bigoplus_{i,j \in \mathbb{Z}} C^{i,j}(D).$$

   This is equipped with boundary maps

$$d : C^{i,j}(D) \to C^{i+1,j}(D), \quad d^2 = 0.$$

2. We see that the index i gives the homological grading. On the other hand, the index j defines the "quantum" or "Jones" grading.

3. The Khovanov homology is defined by

$$\mathrm{Kh}^{\bullet,\bullet}(L) = H^{\bullet,\bullet}(C(D)) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(L).$$

   We show that this is invariant under Reidemeister moves, and hence an invariant of L.

Question from class. Are there any maps between the different Jones gradings?

Answer. Yes, we can define maps, but for our purposes we do not do this (yet). □

Remark. In Russian, Khovanov is pronounced a little more like Hovanov. (Technically the kh is a voiceless velar fricative.) On the other hand, we see above that our theory should really be called a *cohomology* theory rather than a *homology* theory. Therefore it would be more correct for our theory to be

Hovanov Khomology.

Why do we call the j index the "Jones" index? Given a chain complex, its Euler characteristic is defined to be

$$\chi(H^\bullet(C)) = \sum_i (-1)^i \operatorname{rk} H^i(C).$$

For a bigraded complex, we modify this definition to obtain a Laurent polynomial. In particular, for the Khovanov homology,

$$\chi(\mathrm{Kh}^{\bullet,\bullet}(L)) = \sum_{i,j} (-1)^i q^j \operatorname{rk} \mathrm{Kh}^{i,j}(L) = \widetilde{J}_L(q) \in \mathbb{Z}[q, q^{-1}].$$

Remarkably, this Euler characteristic is an "unnormalised Jones polynomial":

$$\widetilde{J}_L(q) = (q + q^{-1}) J_L(q^2), \qquad \text{for } J_L(t) \text{ the Jones polynomial.}$$

Definition 5.1.2. Recall that the Jones polynomial is the polynomial invariant that transformed knot theory, characterised by the following skein relations:

- $J_0(t) = 1$.
- $t^{-1} J_{L_+}(t) - t J_{L_-}(t) = (t^{1/2} - t^{-1/2}) J_{L_0}(t)$.

Here L_+, L_-, and L_0 correspond to the same link with a single crossing modified: L_+ has the positive oriented crossing, L_- the negative crossing, and L_0 the un-crossing.

Example. For example, the trefoil knot has Jones polynomial

$$J_{3_1}(t) = t + t^3 - t^4.$$

Therefore $\widetilde{J}_{3_1}(q) = (q + q^{-1})(q^2 + q^6 - q^8) = q + q^3 + q^5 - q^9$.

It turns out the Khovanov homology of the trefoil can be described as in the following table:

$j \backslash i$	0	1	2	3	χ
9				$\mathbb{Z}$	-1
7				$\mathbb{Z}/2\mathbb{Z}$	0
5			$\mathbb{Z}$		1
3	$\mathbb{Z}$				1
1	$\mathbb{Z}$				1

reading the Euler characteristic off the table, it is clear that we recover

$$\chi(\mathrm{Kh}^{\bullet,\bullet}(3_1)) = q + q^3 + q^5 - q^9$$

as required.

Remark. We soon observe that links with an odd number of components only have non-trivial homology in the odd Jones degrees, while links with even components have non-trivial homology in the even degrees.

Before proceeding further, we establish some notation. Hereafter M will denote a *graded* abelian group. (Think: Jones grading.) To shift the grading up by ℓ, we write $M\{\ell\}$.

Now consider a cochain complex $C^0 \to C^1 \to C^2 \to \cdots$. (Think: homological grading.) Then $C[s]$ corresponds to shifting this grading up by s. That is,

$$C[s]^k = C^{k-s}.$$

Note that this convention is the opposite of some sources. We follow *Bar-Natan, On Khovanov's categorification of the Jones polynomial.*

Definition 5.1.3. We now define the modules in the Khovanov complex. (The boundary maps will come later.)

1. Let D be an oriented link diagram, with n crossings. Then each crossing is either positive or negative - we write $n = n_+ + n_-$ where n_+ is the number of positive crossings, and n_- the negative crossings.

2. Regardless of orientation, any crossing can be resolved in exactly two ways:

The resolutions are labelled 0 or 1 depending on the choice. Our diagram D can have all crossings resolved in 2^n ways, each resolution corresponding to some $\alpha \in \{0,1\}^n$. This is called the *cube of resolutions*. The resolution of D corresponding to α is denoted by D_α.

3. Any two resolutions that differ by one choice (e.g. $(0,0,1,0,1)$ and $(0,0,0,0,1)$) have an edge between them. These are formally $\xi \in \{0,1,*\}^n$, with

$$\xi = (\xi_1, \ldots, \xi_n), \quad \xi_j = * \text{ for a unique } j.$$

In the above example, the edge would be

$$\xi = (0,0,*,0,1).$$

4. Define $V = \mathbb{Z} \oplus \mathbb{Z}$, spanned by v_+ and v_-. Any $\alpha \in \{0,1\}^n$ determines a module,

$$V_\alpha(D) = V^{\otimes k}\{|\alpha|\}, \quad |\alpha| = \sum_{\alpha_i}, k = \# \text{ circles in } D_\alpha.$$

Moreover, each $v_\pm$ has Jones grading ± 1. (Thus $v_+ \otimes v_+$ has Jones grading 2, and so on.)

5. A pre-shifted complex is defined by $[|D|]^r = \bigoplus_{\alpha, |\alpha| = r} V_\alpha(D)$. The Khovanov complex is defined by shifting this complex:

$$C^{\bullet,\bullet}(D) = ([|D|]^\bullet[-n_-]\{n_+ - 2n_-\}, d).$$

(Of course we have yet to define the boundary map d.)

Definition 5.1.4. Now with the "objects" of the Khovanov complex defined, we define the maps.

1. Every edge in the cube of resolutions (oriented from $|\alpha|$ to $|\alpha|+1$) joins two resolutions whose number of components differs by 1. If the number of components *decreases*, the map is of type m:

$$m : \begin{cases} v_+ \otimes v_+ \mapsto v_+ \\ v_+ \otimes v_- \mapsto v_- \\ v_- \otimes v_+ \mapsto v_- \\ v_- \otimes v_- \mapsto 0. \end{cases}$$

If the number of components *increases*, the map is of type Δ:

$$m : \begin{cases} v_+ \mapsto v_- \otimes v_+ + v_+ \otimes v_- \\ v_- \mapsto v_- \otimes v_-. \end{cases}$$

This defines the boundary map on two components, and on the rest the map is defined to be the identity. This gives d_ξ for each edge ξ.

2. Define $(-1)^\xi = (-1)^{\sum_{i<j} \xi_i}$, where j is the location of $*$ in ξ. For example, $*00 \rightsquigarrow 1$, $1*1 \rightsquigarrow -1$.

3. The differential d^r of the complex is defined by

$$d^r = \sum_{\xi \text{ starts at } \alpha, |\alpha|=r} (-1)^\xi d_\xi.$$

5.2 Khovanov example: the right-handed trefoil

Example. As an example, we work through the trefoil knot. We first determine the cube of resolutions in terms of diagrams (figure 5.1) and then the actual maps (figure 5.2).

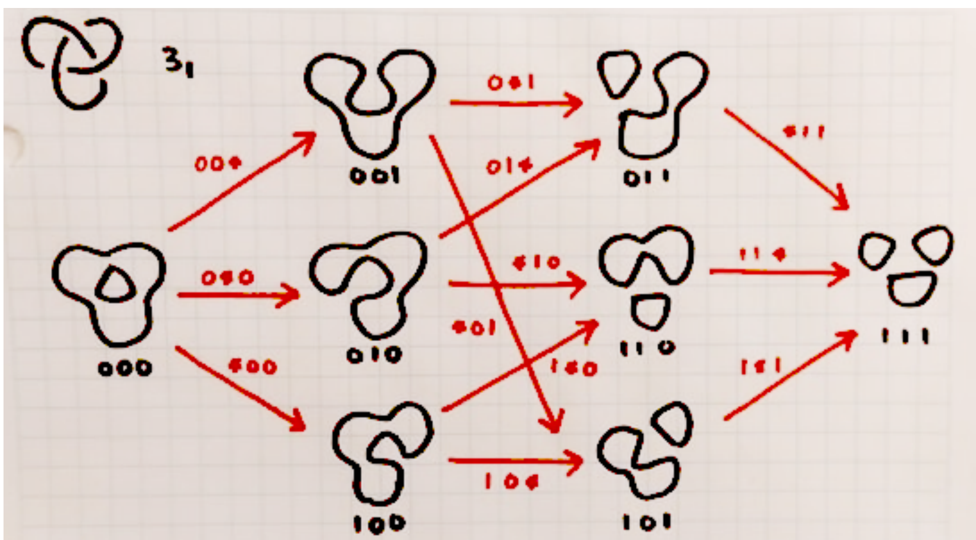


Figure 5.1: Cube of resolutions of the trefoil in terms of diagrams.

Based on this information, the bigraded complex forms the following table.

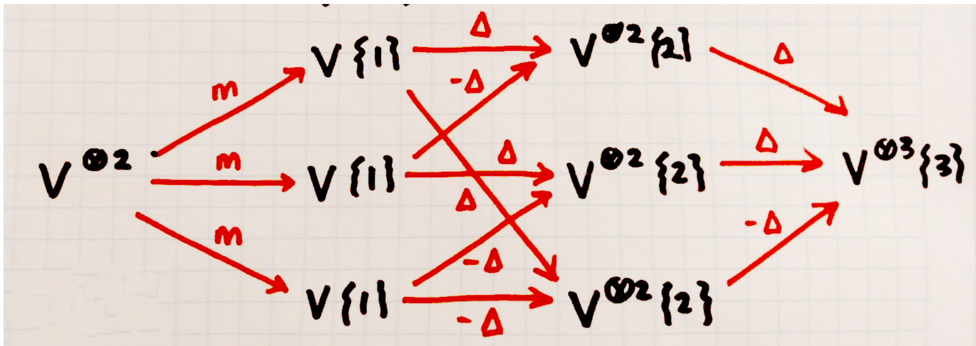


Figure 5.2: Khovanov complex of the trefoil.

$j \backslash i$	0	1	2	3
9				$v_+ \otimes v_+ \otimes v_+$
7			$v_+ \otimes v_+$ $v'_+ \otimes v'_+$ $v''_+ \otimes v''_+$	$v_+ \otimes v_+ \otimes v_-$ $v_+ \otimes v_- \otimes v_+$ $v_- \otimes v_+ \otimes v_+$
5	$v_+ \otimes v_+$	v_+ v'_+ v''_+	$v_+ \otimes v_-, v_- \otimes v_+$ $v'_+ \otimes v'_-, v'_- \otimes v'_+$ $v''_+ \otimes v''_-, v''_- \otimes v''_+$	$v_+ \otimes v_- \otimes v_-$ $v_- \otimes v_+ \otimes v_-$ $v_- \otimes v_- \otimes v_+$
3	$v_+ \otimes v_-,\ v_- \otimes v_+$	v_- v'_- v''_-	$v_- \otimes v_-$ $v'_- \otimes v'_-$ $v''_- \otimes v''_-$	$v_- \otimes v_- \otimes v_-$
1	$v_- \otimes v_-$			

Based on the above table and maps, we can compute homology groups. For example,

$$\mathrm{Kh}^{3,9}(3_1) = \mathrm{Kh}^{0,1}(3_1) = \mathbb{Z}, \quad \mathrm{Kh}^{s,9}(3_1) = \mathrm{Kh}^{t,1}(3_1) = 0, s \neq 3, t \neq 0.$$

These are immediate, since all boundary maps in the $j = 1$ and $j = 9$ gradings are trivial. We do not provide all calculations here, but we now determine the homology for the $j = 7$ grading. The potentially non-trivial homology occurs in the $(2, 7)$ and $(3, 7)$ cells, where we have a sequence isomorphic to

$$\cdots \to 0 \to \mathbb{Z}^3 \xrightarrow{d} \mathbb{Z}^3 \to 0 \to \cdots .$$

To determine the map d, we refer back to figure 5.2. Since each map is Δ, by also referring to the signs, we find the following:

$$\begin{aligned}
v_+ \otimes v_+ &\mapsto v_+ \otimes v_+ \otimes v_- + v_+ \otimes v_- \otimes v_+ \\
v'_+ \otimes v'_+ &\mapsto v_+ \otimes v_+ \otimes v_- + v_- \otimes v_+ \otimes v_+ \\
v''_+ \otimes v''_+ &\mapsto -v_+ \otimes v_- \otimes v_+ - v_- \otimes v_+ \otimes v_+.
\end{aligned}$$

Expressing this as a matrix, we have

$$d = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The second matrix is the Smith normal form of the matrix representing d. Using this change of basis, we have a sequence

$$\cdots \to 0 \to \mathbb{Z}^2 \oplus \mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}^2} \oplus 2} \mathbb{Z}^2 \oplus \mathbb{Z} \to 0 \to \cdots .$$

Therefore the homology can be read off as

$$\mathrm{Kh}^{2,7}(3_1) = 0, \quad \mathrm{Kh}^{3,7}(3_1) = \mathbb{Z}/2\mathbb{Z}.$$

Computing the rest of the table, we find that the Khovanov homology of the trefoil is as follows.

Khovanov homology of 3_1

$j \backslash i$	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}/2\mathbb{Z}$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

Proposition 5.2.1. The Khovanov complex is genuinely a complex, that is, $d^2 = 0$.

Proof. This follows from a case-by-case analysis. $\square$

5.3 Isotopy invariance of Khovanov homology

We have established that the Khovanov homology is truly a homology theory, but it has not yet been shown to be independent of the choice of diagram (of a given link). We must show that it is invariant under Reidemeister moves. We make use of the following lemma extensively (but first we need some definitions).

Definition 5.3.1. Let (C, d) be a complex, and $C' \subset C$ a subcomplex. This means that $d(C') \subset C'$. This also gives rise to a quotient complex, C/C'. We then obtain a short exact sequence

$$0 \to C' \to C \to C/C' \to 0$$

of complexes, which induces the usual long exact sequence on (co)homology

$$\cdots \to H^i(C') \to H^i(C) \to H^i(C/C') \to H^{i+1}(C') \to \cdots.$$

Lemma 5.3.2. If C' is acyclic, i.e. if $H^*(C') = 0$, then $H^*(C) \cong H^*(C/C')$. Similarly if $H^*(C/C') = 0$, then $H^*(C') \cong H^*(C)$.

This this notation established, we are ready to prove invariance under Reidemeister moves. Invariance under those of types 2 and 3 are left as an exercise, but we prove invariance of Khovanov homology under type 1 Reidemeister moves.

Proposition 5.3.3. Khovanov homology is invariant under type 1 Reidemeister moves.

Proof. Let D be a diagram with a crossing x that can be removed by a type 1 Reidemeister move. Write $[|D|]$ to denote the pre-shifted Khovanov complex of D. This factors as

$$C = [|D_0|] \xrightarrow{m} [|D_1|]\{1\}$$

where $[|D_0|]$ is a subcomplex which consists of all diagrams where x has a 0 resolution, and $[|D_1|]$ the subcomplex corresponding to x having the 1 resolution. Note that each diagram (vertex) in $[|D_0|]$ has an additional component L coming from the 0 resolution of x. On the other hand, the 1 resolution at x corresponds exactly to the type 1 Reidemeister move at x, so that $[|D_1|]$ is exactly the pre-shifted complex of D after applying a type 1 Reidemeister move.

The component L contributes two free elements v_+ and v_-. Consider the subcomplex C' of C, where the space associated to L is restricted to the span of v_+. Since the map m is defined by

$$m : v_+ \otimes w \mapsto w,$$

we have an isomorphism

$$C' = [|D_0|]_{v_+ \text{ at } L} \xrightarrow{m, \cong} [|D_1|]\{1\}.$$

The quotient complex C/C' is then given by

$$C/C' = [|D_0|]_{v_- \text{ at } L} \xrightarrow{m} 0.$$

But $[|D_0|]_{v_- \text{ at } L}$ is isomorphic to $[|D'|]\{-1\}$, where D' is D after the type 1 Reidemeister move has been applied. The shift $\{-1\}$ is to cancel the change in grading due to D' having one fewer crossing. But now by the previous lemma,

$$\mathrm{Kh}(D) = [|D|]\{n_+ - 2n_-\} = [|D_0|]_{v_- \text{ at } L}\{n_+ - 2n_-\} = [|D'|]\{n_+ - 2n_- - 1\} = \mathrm{Kh}(D').$$

$\square$

Proposition 5.3.4. The Khovanov homology is invariant under type 2 and type 3 Reidemeister moves.

Proof. These follow a similar argument. Details can be found in Dror Bar-Natan's paper (which is available on the ArXiV). $\square$

5.4 Generalising Khovanov homology: TQFTs (lecture 16)

Recall that any crossing in a link diagram can be *resolved* in two ways, giving either the 0 resolution or 1 resolution. If a link diagram is *oriented*, there is a unique way to resolve each crossing so that it agrees with the orientation.

Given a link diagram D with c components, there are 2^c possible orientations $\mathcal{O}$, each with a unique resolution $D_{\mathcal{O}}$.

In the first section of today's lecture, we explore the core of the invariance proofs of the previous lecture to better understand Khovanov homology. A seemingly arbitrary choice was that each component of a resolution was associated to $\mathbb{Z} \oplus \mathbb{Z}$, and the maps m and Δ were not motivated either.

We now attempt to better understand the underlying ingredients of Khovanov homology, independent of the choices.

1. The spaces were direct sums and tensor products of $V = \mathbb{Z} \oplus \mathbb{Z}$. These had maps $m : V \otimes V \to V$, and $\Delta : V \to V \otimes V$.

2. $1 \in V$ is a unit for m, and $\varepsilon : V \to \mathbb{Z}$ defined by $\varepsilon(v_+) = 0$ and $\varepsilon(v_-) = 1$ is a counit for Δ.

3. The map m itself is a commutative associative multiplication. Δ is a cocommutative coassociative comultiplication.

4. The maps satisfy the *Frobenius law*, $\Delta \circ m = (m \otimes 1) \circ (1 \otimes \Delta)$.

These are exactly the ingredients of a commutative Frobenius algebra.

Proposition 5.4.1. To obtain a homological invariant of knots like Khovanov homology, we need V a commutative Frobenius algebra, free of rank 2.

The easiest way to think about commutative Frobenius algebras is to consider (1+1)-dimensional topological quantum field theories (TQFTs).

Theorem 5.4.2. *There is an equivalence of groupoids*

$$\{\mathit{TQFTs}\ 2\mathbf{Cob} \to \mathbf{Vect}_k\} \longleftrightarrow \mathbf{comFrob}_k.$$

We do not give a formal proof, but describe (1+1) dimensional TQFTs (i.e. functors $2\mathbf{Cob} \to \mathbf{Vect}_k$), and give examples of how they correspond to commutative frobenius algebras.

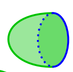
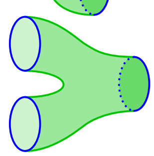
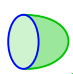
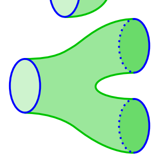
Remark. Here we describe TQFTs as functors into vector spaces, but in our context they are abelian groups.

Definition 5.4.3. The category $2\mathbf{Cob}$ consists of (1+1) dimensional cobordisms. That is, the objects are closed one-manifolds (disjoint unions of circles), and the morphisms are cobordisms between them.

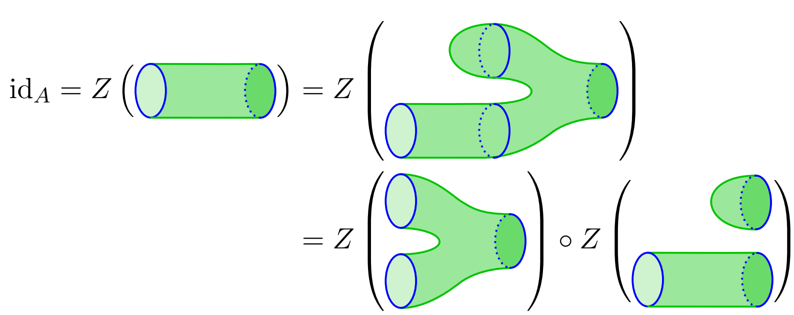
Definition 5.4.4. $\mathbf{Vect}_k$ is the category of vector spaces over a field k. A (1+1) dimensional TQFT is a functor that sends a 1-manifold to a vector space, and a cobordism to a homomorphism between them. Moreover, these respect the monoidal (tensor product) structure: for X, Y 1-manifolds,

$$Z(X \sqcup Y) = Z(X) \otimes Z(Y).$$

The following table describes the four generators of $2\mathbf{Cob}$, and how they correspond to maps in a Frobenius algebra.

M	$Z(M)$	Interpretation
	$1 : k \to A$	unit
	$m : A \otimes A \to A$	multiplication
	$\varepsilon : A \to k$	counit
	$\Delta : A \to A \otimes A$	comultiplication

Properties such as associativity, commutativity, and the Frobenius law can all be verified by using the classification of surfaces. We give one example here:



$$= m \circ (1 \otimes \mathrm{id}_A).$$

Example. Khovanov homology can be expressed in a perhaps more intuitive form by using the perspective of Frobenius algebras. Write

$$V = \mathbb{Z}[x]/(x^2).$$

Define $m : V \otimes V \to V$ to be the usual product on $\mathbb{Z}[x]/(x^2)$. 1 is of course a unit. The map $\Delta : V \to V \otimes V$ defined by

$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x$$

is a comultiplication, and $\varepsilon : V \to \mathbb{Z}$ defined by

$$\varepsilon(1) = 0, \quad \varepsilon(x) = 1$$

is a counit. This defines the Khovanov homology with the symbols $v_+ = 1$ and $v_- = x$.

Example. We can consider a *deformation*

$$V = \mathbb{Z}[x]/(x^2 - t),$$

over the ring $\mathbb{Z}[t]$. Let 1 and ε be as above, and m the usual multiplication on V. We define a modified comultiplication maps as follows:

$$\Delta(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x + t(1 \otimes 1).$$

This also defines a Frobenius algebra. With the notation v_+, v_-, the multiplication and comultiplication maps can be written as

$$m : \begin{cases} v_+ \otimes v_+ & \mapsto v_+ \\ v_+ \otimes v_-, v_- \otimes v_+ & \mapsto v_- \\ v_- \otimes v_- & \mapsto tv_+ \end{cases}$$

$$\Delta : \begin{cases} v_+ & \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- & \mapsto v_- \otimes v_- + tv_+ \otimes v_+. \end{cases}$$

This gives rise to a complex $C'(D)$ of $\mathbb{Z}[t]$-modules. When $t = 0$ this is the Khovanov complex. When $t = 1$, this is the *Lee complex*, which we denote by $C_{\mathrm{Lee}}(D)$.

The corresponding integral homology theories are denoted by $\mathrm{Kh}(K)$ and $\mathrm{Lee}(K)$, called the Khovanov and Lee homologies respectively. We write $\mathrm{Kh}'(K)$ to represent the Khovanov-Lee homology over $\mathbb{Z}[t]$.

5.5 Lee homology and spectral sequences

At the end of the previous section we introduced the Khovanov-Lee homology $\mathrm{Kh}'(K)$, which is valued in $\mathbb{Z}[t]$. Evaluation at 0 gives the Khovanov homology, and evaluation at 1 the Lee homology.

If $C'(D)$ is the Khovanov-Lee complex, the boundary maps can be written as

$$d + t\Phi : C^i(D) \to C^{i+1}(D)$$

where the $t\Phi$ term can be read off the modified definitions of m and Δ. Here d is the usual Khovanov differential, which changes (i, j) by $(1, 0)$. On the other hand, Φ changes (i, j) by $(1, 4)$. We have not only that $d^2 = 0$, but also $(d + \Phi)^2 = 0$.

Observe that for any j, $C^{q \geq j}$ is closed under the action of $(d + \Phi)$. Therefore the Khovanov complex has a filtration

$$\cdots C^{q \geq j} \supset C^{q \geq j+1} \supset \cdots .$$

A filtered complex is exactly what gives rise to a *Spectral sequence*.

Definition 5.5.1. A *spectral sequence* is a collection of pages. I.e. a collection of complexes (E^r, d^r), where $d^r \circ d^r = 0$, and $E^{r+1} = H^\bullet(E^r, d^r)$.

Example. In our context, the filtration of the Lee complex gives a spectral sequence with

$$\begin{aligned} E^1 &= (C^\bullet, d), \\ E^2 &= (H^\bullet(E^1), \Phi^*) = (\mathrm{Kh}(K), \Phi^*), \\ \Rightarrow E^\infty &= H^\bullet(C, d + \Phi) = \mathrm{Lee}(K). \end{aligned}$$

The important result being used is that every filtered complex gives a spectral sequence which converges to the homology of the original complex.

Example. Write $\mathrm{Kh}(K; \mathbb{Q})$ to denote $\mathrm{Kh}(K) \otimes_{\mathbb{Z}} \mathbb{Q}$. We write out some of the pages of the rational spectral sequence corresponding to the trefoil knot.

E^1 page for 3_1

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7			$\mathbb{Q}^3$	$\mathbb{Q}^3$
5	$\mathbb{Q}$	$\mathbb{Q}^3$	$\mathbb{Q}^6$	$\mathbb{Q}^3$
3	$\mathbb{Q}^2$	$\mathbb{Q}^3$	$\mathbb{Q}^3$	$\mathbb{Q}$
1	$\mathbb{Q}$			

E^2 page for 3_1

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7				
5			$\mathbb{Q}$	
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

E^3 page for 3_1

$j \backslash i$	0	1	2	3
9				
7				
5				
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

E^∞ page for 3_1

$j \backslash i$	0	1	2	3
9				
7				
5				
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

Observe that $\mathrm{Lee}(3_1) \cong \mathbb{Q}^2 = \mathbb{Q}^{2^c}$ where c is the number of components of the trefoil knot. This is a general result.

Theorem 5.5.2. $\mathrm{Lee}(L;\mathbb{Q}) \cong \mathbb{Q}^{2^c}$, *where c is the number of components of L.*

Proof. To prove this, we define a new basis for V. Specifically, define a and b by

$$a = v_+ + v_-, \quad b = v_- - v_+.$$

The Lee complex boundary maps are then induced by

$$m : \begin{cases} a \otimes a & \mapsto 2a \\ a \otimes b, b \otimes a & \mapsto 0 \\ b \otimes b & \mapsto -2b \end{cases}, \quad \Delta : \begin{cases} a & \mapsto a \otimes a \\ b & \mapsto b \otimes b. \end{cases}$$

Claim: $\mathrm{Lee}(L)$ is generated by the "canonical generators" which we now construct.

1. Let $\mathcal{O}$ be an orientation of a diagram D of L. (There are 2^c choices of orientation).
2. There is a unique resolution $D_{\mathcal{O}}$ of D which is compatible with the orientation. This is a disjoint union of circles.
3. Let $C \in D_{\mathcal{O}}$. Define $\tau(C) \in \mathbb{Z}/2\mathbb{Z}$ to be the number of circles separating C from infinity, plus 1 if C is oriented clockwise.
4. Define $g_C = a$ if $\tau(C) = 0$, and $g_C = b$ if $\tau(C) = 1$. Define

$$S_{\mathcal{O}} = \bigotimes_{C \in D_{\mathcal{O}}} g_C.$$

The claim is that the $S_{\mathcal{O}}$ (of which there are exactly 2^c) are generators of $\mathrm{Lee}(L)$. We break this proof into two pieces.

Lemma 5.5.3. The collection of $S_{\mathcal{O}}$ forms an orthonormal set in $\mathrm{Lee}(D)$, so that $\dim \mathrm{Lee}(D) \geq 2^c$.

We first note that if any two circles have the same label (either a or b) then they *cannot* meet at a resolved vertex. It follows that each $S_{\mathcal{O}}$ is a cycle, i.e. $(d+\Phi)S_{\mathcal{O}} = 0$. Therefore $[S_{\mathcal{O}}] \in \mathrm{Lee}(D)$.

Now that it has been established that these are all elements of $\mathrm{Lee}(D)$, we equip $C_{Lee}(D)$ with an inner product by declaring that the $\{a \otimes a \otimes b \otimes \cdots\}$ is an orthonormal basis. The map $d + \Phi$ has an adjoint with respect to the inner product, namely

$$(d+\Phi)^* : \begin{cases} a \otimes a & \mapsto a \\ b \otimes b & \mapsto b \\ a & \mapsto 2a \otimes a \\ b & \mapsto -2b \otimes b \\ \text{rest} & \mapsto 0. \end{cases}$$

Then one can show that $(d + \Phi)^* S_{\mathcal{O}} = 0$. But this implies that each $S_{\mathcal{O}}$ descends to an element of $\mathrm{Lee}(D)$ while preserving pairwise orthogonality, since

$$\mathrm{Lee}(D) = H^*(S_{\mathrm{Lee}}(D)) = \ker(d+\Phi)/\operatorname{im}(d+\Phi) \cong \ker(d+\Phi) \cap \ker(d+\Phi)^*.$$

In summary this proves that $\dim \mathrm{Lee}(D) \geq 2^c$.

Lemma 5.5.4. In fact, $\dim \mathrm{Lee}(D) = 2^c$.

To see this, it remains to prove that $\dim \mathrm{Lee}(D) \leq 2^c$. This follows from an induction on the number of crossings of D. Let D_0 and D_1 be 0 and 1 resolutions of a single crossing x in D. Then $C_{\mathrm{Lee}}(D_1) \subset C_{\mathrm{Lee}}(D)$ is a subcomplex. This gives rise to a long exact sequence

$$\cdots \to \mathrm{Lee}(D_1) \to \mathrm{Lee}(D) \to \mathrm{Lee}(D_0) \to \mathrm{Lee}(D_1) \to \cdots.$$

There are two cases to consider. First suppose the two strands crossing at x belong to distinct components of D. Then D_0 and D_1 each have $c-1$ components each. By the inductive hypothesis,

$$\dim \mathrm{Lee}(D_0) = \dim \mathrm{Lee}(D_1) = 2^{c-1}.$$

By the long exact sequence,

$$\dim \mathrm{Lee}(D) \leq \dim \mathrm{Lee}(D_0) + \dim \mathrm{Lee}(D_1) = 2^{c-1} + 2^{c-1} = 2^c.$$

This proves the first case. For the second case, suppose the strands meeting at x belong to the same component. Then one of D_0, D_1 has c components, and the other $c+1$ components. (Assume without loss of generality that D_0 has c components, and D_1 has $c+1$ components.) The induced map

$$\mathrm{Lee}(D_0) \xrightarrow{i} \mathrm{Lee}(D_1)$$

is then injective. Therefore $\dim \mathrm{Lee}(D) = \dim \operatorname{coker} i = 2^c$. (The size of the cokernel can be verified by showing that the canonical generators of $\mathrm{Lee}(D_0)$ map to half of those of $\mathrm{Lee}(D_1)$.) The other case is formally dual, with a surjective map and so on.

This completes the proof that $\dim \mathrm{Lee}(D) = 2^c$. Therefore $\mathrm{Lee}(D) = \mathbb{Q}^{2^c}$. $\square$

5.6 Rasmussen's s-invariant (lecture 17)

Let $K \subset \mathbb{S}^3$ be a knot. Recall that the *slice genus* is the minimal genus of a surface bound by K in a 4-ball:

$$g_s(K) \coloneqq \min\{g(\Sigma) : \Sigma \subset B^4 \text{ properly smoothly embedded}, \partial\Sigma = K.\}$$

Recall the Milnor conjecture (now a theorem), which we proved earlier using Seiberg-Witten gauge theory:

Theorem 5.6.1 (Milnor conjecture)**.** *Let K denote the p,q-torus knot, for p, q coprime. Then $g_s(K) = (p-1)(q-1)/2$.*

- The original proof, due to Kronheimer and Mrowka in 1993, used Yang-Mills gauge theory.
- Several years later, Kronheimer and Mrowka proved the result using Seiberg-Witten gauge theory.
- In 2004, Rasmussen gave a "combinatorial" proof. This is what we'll start discussing today.

Recall that the slice genus is a lower bound for the unknotting number, and the p,q-torus knot K can be unknotted in $(p-1)(q-1)/2$ moves. Therefore

$$g_s(K) \le u(K) \le \frac{(p-1)(q-1)}{2}.$$

On the other hand, today we introduce Rasmussen's s-invariant $s \in 2\mathbb{Z}$. We show that

1. $|s(K)| \le 2g_s(K)$.
2. $s(K) = (p-1)(q-1)$.

Therefore by combining 1 and 2,

$$\frac{(p-1)(q-1)}{2} = \frac{s(K)}{2} \le g_s(K).$$

This will prove the Milnor conjecture.

To give a definition of the s-invariant, we consider Khovanov and Lee homology with rational coefficients. Recall that a diagram D for an arbitrary knot K determines a complex $(C(D), d)$ called the *Khovanov complex*. This in turn determines a homology theory which is invariant under Reidemeister moves, which we call the *Khovanov homology* $\mathrm{Kh}(K)$. By perturbing the boundary maps, we obtain a different complex $(C_{\text{Lee}}(D), d+\Phi)$ called the *Lee complex*, and this also gives an invariant homology theory $\text{Lee}(K)$. Moreover,

$$\text{Kh}(K) \Rightarrow \text{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}.$$

Although $\text{Lee}(K)$ is almost trivial, the two surviving copies of $\mathbb{Q}$ have Jones (q) gradings. Let $s_{\max} \geq s_{\min}$ be the Jones gradings of the two copies. Since K is a knot, $s_{\max}, s_{\min}$ are both odd. Moreover, the isomorphism type of the spectral sequence is an invariant of K, so $s_{\max}$ and $s_{\min}$ are also invariants. It turns out that $s_{\max} = s_{\min} + 2$, so we define the *Rasmussen invariant* to be

$$s(K) = s_{\max}(K) - 1 = s_{\min}(K) + 1 \in 2\mathbb{Z}.$$

While this is the idea, we now give a formal definition of $s_{\max}(K)$ and $s_{\min}(K)$.

Definition 5.6.2. Let D be a diagram of a knot K. Then $C_{\text{Lee}}(D)$ has a filtration

$$C_{\text{Lee}}(D) \supset \cdots \supset C_{\text{Lee}}^{q\geq j}(D) \supset C_{\text{Lee}}^{q\geq j+1}(D) \supset \cdots \supset 0,$$

since the map $d+\Phi$ changes the bidegree (i,j) by $(1,0)$ (by d) and by $(1,4)$ (by Φ). For each j, we define

$$I_j = \text{im}(H^*(C_{\text{Lee}}^{q\geq j}(D)) \hookrightarrow H^*(C_{\text{Lee}}(D))) \subset \text{Lee}(D).$$

Note that there exists some N so that we need only consider $-N \leq j \leq N$ for j as above. Then

$$\text{Lee}(D) = I_{-N} \supset I_{-N+1} \supset \cdots \supset I_N = 0.$$

This induces a grading on $\text{Lee}(D)$, by

$$\text{Lee}(D) = \bigoplus_j I_j / I_{j+1}.$$

Now any class $[x]$ in $\text{Lee}(D)$ has a *grading*, namely

$$q([x]) = \max\{j : q(x) = j, x \in [x]\}, \quad q(x) = \max\{j : x \in C_{\text{Lee}}^{q\geq j}(D)\}.$$

In particular, we define

$$s_{\max}(K) = \max\{q([x]) : [x] \in \text{Lee}(K), [x] \neq 0\}, \quad s_{\min}(K) = \min\{q([x]) : [x] \in \text{Lee}(K), [x] \neq 0\}.$$

Given these formal definitions of the invariants $s_{\min}$ and $s_{\max}$, the definition of the Rasmussen invariant rests on the following result:

Proposition 5.6.3. Let K be a knot. Then $s_{\max}(K) = s_{\min}(K) + 2$.

Note that this justifies the definition of the Rasmussen invariant to be $s(K) = s_{\max} - 1 = s_{\min} + 1$.

Proof. The main idea of the proof is to study the two canonical generators $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$ of $\mathrm{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}$ introduced in the previous lecture. We use combinations of these to first show that $s_{\max} - s_{\min} \equiv 2 \mod 4$. (In particular, they differ by at least 2.) Whe they show that they differ by at most 2, to obtain the desired equality.

First note that for a knot K, we already know that C_{Lee} is supported only in odd quantum gradings. Define

$$\begin{aligned} C_{\mathrm{Lee,even}}(D) = & \text{ generated by elements with } q = 1 \mod 4 \\ C_{\mathrm{Lee,odd}}(D) = & \text{ generated by elements with } q = 3 \mod 4 \end{aligned}$$

Note that d preserves the q grading while Φ changes it by 4, so $d + \Phi$ preserves q modulo 4. In particular, $C_{\mathrm{Lee}}(D) = C_{\mathrm{Lee,even}}(D) \oplus C_{\mathrm{Lee,odd}}(D)$, where the direct summands are preserved by $d + \Phi$. It follows that

$$\mathrm{Lee}(K) = \mathrm{Lee}_{\mathrm{even}}(K) \oplus \mathrm{Lee}_{\mathrm{odd}}(K).$$

We now make use of this direct summand structure. Define $\iota : C_{\mathrm{Lee}}(D) \to C_{\mathrm{Lee}}(D)$ to act by 1 on $C_{\mathrm{Lee,even}}$, and -1 on $C_{\mathrm{Lee,odd}}$. Then any $x \in C_{\mathrm{Lee}}(D)$ decomposes as

$$x = \frac{x + \iota(x)}{2} + \frac{x - \iota(x)}{2},$$

where the first term lives in $C_{\mathrm{Lee,even}}$, and the second in $C_{\mathrm{Lee,odd}}$. We further define $i : V \to V$ by $i(v_-) = v_-$ and $i(v_+) = -v_+$. Then $\iota = \pm i^{\otimes n}$. Moreover, setting $a = v_- + v_+$ and $b = v_- - v_+$ as an alternative basis, we have $i(a) = b$ and $i(b) = a$.

We now analyse $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$ more closely. These actually arise from the same diagram! Switching all orientations in a diagram and then resolving gives rise to the same resolution, but with all orientations switched. Therefore

$$i([S_{\mathcal{O}}]) = \pm[S_{\overline{\mathcal{O}}}].$$

It follows that the canonical even/odd decomposition is given by

$$[S_{\mathcal{O}}] = \frac{[S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]}{2} + \frac{[S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}]}{2}.$$

This proves that the two copies of $\mathbb{Q}$ in $\mathrm{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}$ live in different gradings mod 4, as required. That is,

$$s_{\max} - s_{\min} \equiv 2 \mod 4.$$

In particular, $s_{\max}$ is at least $s_{\min} + 2$.

Finally we show that $s_{\max}$ is at most $s_{\min}+2$. This follows from a similar calculation as showing that the Khovanov homology is invariant under Reidemeister moves. Let D' denote the diagram of K obtained by adding a crossing via a type 1 move. Then

$$C_{\text{Lee}}(D') = \Big(C_{\text{Lee}}(D \sqcup 0) \to C_{\text{Lee}}(D)\Big).$$

This can expressed as the short exact sequence

$$0 \to C_{\text{Lee}}(D) \to C_{\text{Lee}}(D') \to C_{\text{Lee}}(D \sqcup 0_1) \to 0$$

which induces a long exact sequence in homology

$$\cdots \to \text{Lee}(K) \to \text{Lee}(K) \to \text{Lee}(K \sqcup 0_1) \xrightarrow{\partial} \text{Lee}(K) \to \cdots$$

where $\text{Lee}(K \sqcup 0_1) \cong \text{Lee}(K) \otimes V$. Depending on labels near the crossing x of D' obtained from the type 1 move, we denote the two canonical generators of $C_{\text{Lee}}(D)$ by s_a and s_b. Without loss of generality, $q(s_a - s_b) = s_{\max}$, and $q(s_a + s_b) = s_{\min}$. One can verify that

$$\partial([s_a - s_b] \otimes [a]) = [s_a],$$

from which it follows that

$$s_{\max} - 1 = q([s_a - s_b] \otimes [a]) \leq q([s_a]) + 1 = s_{\min} + 1.$$

Therefore $s_{\max} \leq s_{\min} + 2$ as required. Earlier we established that $s_{\max} \geq s_{\min} + 2$, so this completes the proof that $s_{\max} = s_{\min} + 2$. □

In summary the Rasumussen s-invariant is well defined.

5.7 The s-invariant bounds the slice genus

Recall that the proof strategy for proving Milnor's conjecture is two establish the following two facts:

1. $|s(K)| \leq 2g_s(K)$.
2. $s(K) = (p-1)(q-1)$.

We now prove the first of these.

Proposition 5.7.1. For a knot K, $|s(K)| \leq 2g_s(K)$.

Proof. The idea is to use the functoriality of Khovanov-homology under link cobordisms. Let L_0 and L_1 be links, with $\Sigma \subset \mathbb{R}^3 \times [0,1]$ a cobordism between them. We induce maps $F_\Sigma : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$, and $F_{\Sigma,\mathrm{Lee}} : \mathrm{Lee}(L_0) \to \mathrm{Lee}(L_1)$ and use their properties.

By Morse theory, Σ splits into building blocks with one critical point each, of indices 0,1, or 2. (These are with respect to the height function $\pi : \Sigma \to [0,1]$.) If D_0 is a diagram for L_0, and D_1 a diagram for L_1, then D_0 and D_1 must be related by a sequence of Reidemeister moves and *Morse moves*. By a Morse move, we mean the change in level set as we pass a critical point. Explicitly,

- Passing an index 0 critical point corresponds to taking a disjoint union with an unknot.
- Passing an index 1 critical point corresponds to locally swapping two horizontal arcs with two vertical arcs or vice versa.
- Passing an index 2 critical point corresponds to destroying a disjoint unknot.

Therefore to define a map $F_\Sigma : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$ we must define maps corresponding to each Reidemeister or Morse move, and glue them together. We must then verify that the map F_Σ is an invariant of Σ, that is, it must not depend on the choice of Morse function/decomposition.

Explicitly, to each move, we associate the following maps:

- For each Reidemeister move D_i to D_{i+1}, there is a canonical isomorphism $F_i : \mathrm{Kh}(D_i) \to \mathrm{Kh}(D_{i+1})$ as used in the proof of the well-definedness of Khovanov homology.
- For an index 0 Morse move D_i to $D_{i+1} = D_i \sqcup 0_1$, define $F_i : \mathrm{Kh}(D_i) \to \mathrm{Kh}(D_{i+1})$ to send $1 \mapsto v_+$ on the 0_1 component, and the identity elsewhere.
- For an index 1 Morse move D_i to D_{i+1}, define F_i to be m or Δ at the location of the move depending on the change in the number of components, and the identity elsewhere.
- For an index 2 Morse move D_i to D_{i+1}, define F_i to send v_- to 1 and v_+ to 0 at the location of the move, and the identity elsewhere.

If $D_1, \ldots, D_n$ are a sequence of diagrams from L_0 to L_1, the composition of the F_i defines the map $F_\Sigma : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1)$. We claim without proof that the map F_Σ is well defined up to sign as an invariant of Σ. That is, the map does not depend on the decomposition of Σ. (This is a theorem of Khovanov and Jacobsson.) Note that this fact is not actually needed for the proof!

A similar construction works for the Lee homology! We obtain maps $F_{\Sigma,\mathrm{Lee}} : \mathrm{Lee}(L_0) \to \mathrm{Lee}(L_1)$ as well.

Suppose Σ is an oriented cobordism from L_0 to L_1, such that every component of Σ has a boundary component on L_0. Then by verifying each Reidemeister and Morse move, one can show that $F_{\Sigma,\mathrm{Lee}}([S_{\mathcal{O}|_{L_0}}])$ is a non-zero multiple of $[S_{\mathcal{O}|_{L_1}}]$, where $\mathcal{O}$ is an orientation of Σ. This means that if Σ is a connected cobordism between knots K_0 and K_1, then $F_{\Sigma,\mathrm{Lee}} : \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$ is an isomorphism.

Suppose Σ has genus $g = g_s(K)$ for a knot K. Then removing a disk D, $\Sigma' = \Sigma - D$ is a genus g cobordism from K to the unknot. Now F_Σ and $F_{\Sigma,\mathrm{Lee}}$ are maps from Khovanov and Lee homologies of K to that of the unknot. How do they change the quantum gradings? Observe that Reidemeister moves leaves q invariant, while Morse moves of index 0 and 2 change q by $+1$, and Morse moves of index 1 change q by -1. Therefore F_Σ changes q by $\chi(\Sigma')$, and $F_{\Sigma,\mathrm{Lee}}$ by at least $\chi(\Sigma')$.

Let $x \in \mathrm{Lee}(K) - \{0\}$ be a class attaining $q(x) = s_{\max} = s + 1$. Then

$$1 \geq q(F_{\Sigma'}(x)) \geq q(x) + \chi(\Sigma') = s + 1 - 2g_s(K).$$

The first inequality is because $F_{\Sigma'}(x)$ lives in $\mathrm{Lee}(0_1)$. Therefore

$$s \leq 2g_s(K)$$

as required.

Finally for the general result, consider the mirror $\overline{K}$ of K. This bounds a surface $\overline{\Sigma}$ with the same genus as Σ. But now $s(\overline{K}) = -s(K)$, so

$$-s(K) \leq 2g = 2g_s(K).$$

Combining this with the previous result, we can bound $g_s(K)$ below by $|s(K)|/2$ as required. $\square$

5.8 Combinatorial proof of Milnor's conjecture (lecture 18)

In the previous lecture we defined the Rasmussen s-invariant for knots, and showed that it satisfies

$$|s(K)| \leq 2g_s(K).$$

Today we show that $s(T_{p,q}) = (p-1)(q-1)$. This will be a special case of the calculation of s for *positive knots*.

Definition 5.8.1. A knot K is *positive* if it has an oriented diagram with only positive crossings.

For example, a torus knot is a positive knot.

Remark. This notion is distinct from that of braid-positivity we introduced several lectures ago.

If D is a positive diagram of a positive knot, then its oriented resolution D_0 is in fact the zero resolution! Our final result needed to prove the Milnor conjecture is the following:

Proposition 5.8.2. If K has a positive diagram D with n crossings, and D_0 consists of k circles (components), then $s(K) = n + 1 - k$.

Proof. Recall from the previous lecture that the s-invariant has the explicit formula

$$s(K) = s = \frac{q([S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]) + q([S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}])}{2}.$$

Here one of $[S_{\mathcal{O}}] \pm [S_{\overline{\mathcal{O}}}]$ has degree $s+1$, and the other has degree $s-1$. Moreover,

$$q([S_{\mathcal{O}}]) = q([S_{\overline{\mathcal{O}}}]) = s - 1.$$

Explicitly, the left side is defined to be

$$q([S_{\mathcal{O}}]) = \max\{q(x) : x \text{ is homologous to } S_{\mathcal{O}}\} = \max\{q(x) : x = S_{\mathcal{O}} + d\alpha\}.$$

But $S_{\mathcal{O}}$ lives in the lowest homological grading (since our resolution D_0 is the zero resolution). Therefore there is no non-trivial α that can map to $d\alpha$, i.e. there is a unique class homologous to $S_{\mathcal{O}}$. Hence

$$q([S_{\mathcal{O}}]) = q(S_{\mathcal{O}}), \quad S_{\mathcal{O}} = (v_+ \pm v_-) \otimes (v_+ \pm v_-) \otimes \cdots .$$

The expression on the right has k factors. But this necessarily lies in the same quantum grading as $\otimes^k v_-$. Therefore by the definition of the Khovanov homology,

$$q(S_{\mathcal{O}}) = -k + (n_+ - 2n_-) = n - k = s - 1.$$

The claimed result follows. $\square$

Example. The standard diagram of the torus knot $T_{p,q}$ consists of $p(q-1)$ positive crossings, and its 0 resolution consists of q circles. Therefore $s(T_{p,q}) = p(q-1) - q + 1 = (p-1)(q-1)$.

We can now pull together a proof of Milnor's conjecture using just Rasmussen's s-invariant.

Proposition 5.8.3. The slice genus of the torus knot $T_{p,q}$ is

$$g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

Proof. The standard diagram can me unknotted in $(p-1)(q-1)/2$ moves, giving

$$g_s(T_{p,q}) \leq u(T_{p,q}) \leq \frac{(p-1)(q-1)}{2}.$$

Conversely, the Rasmussen s-invariant gives

$$\frac{(p-1)(q-1)}{2} = \frac{s(T_{p,q})}{2} \leq \frac{2g_s(T_{p,q})}{2} = g_s(T_{p,q}).$$

Therefore we have equality as required. $\square$

5.9 Combinatorial proof of the existence of exotic $\mathbb{R}^4$s

Another application of Khovanov homology is that it gives a novel proof of the existence of exotic smooth structures on $\mathbb{R}^4$, without requiring any gauge theory. More concretely, our proof outline is as follows:

1. Use Rasmussen's s invariant together with a result of Freedman to find knots that are *topologicall slice* but not slice.
2. Introduce the *trace embedding lemma.*
3. Use the trace embedding lemma with manifolds obtained from a knot as in 1 to construct an open manifold which is homeomorphic to $\mathbb{R}^4$ but cannot be diffeomorphic to it. A result of Freedman states that all open 4-manifolds admit admit smooth structures, so it must then be an exotic $\mathbb{R}^4$.

We now carry out the details. First we introduce relevant definitions and results to establish point 1.

Definition 5.9.1. A knot K is *slice* (or *smoothly slice*) if $g_s(K) = 0$. That is, if there exists a smooth properly embedded disk $D \subset B^4$ such that $\partial D = K \in \mathbb{S}^3$.

By replacing the notion of a smooth embedding with a topological embedding, we obtain a weaker condition.

Definition 5.9.2. A knot K is *topologically slice* if there exists a locally flat topologically embedded disk $D \subset B^4$ such that $\partial D = K \in \mathbb{S}^3$. This means that there is a topological embedding $\varphi : (D^2 \times D^2, \partial D^2 \times D^2) \to (B^4, \partial B^4 = \mathbb{S}^3)$ such that $\varphi(\partial D^2 \times 0) = K$. Then $\varphi(D^2 \times 0)$ is a topologically embedded disk which is *locally flat.*

Remark. The local flatness condition is necessary to obtain an "interesting" definition: without this assumption, all knots would be topologically slice by taking the embedded disk to be a cone over the knot.

We now use the following theorem of Freedman to establish the existence of topologically slice knots which aren't slice:

Theorem 5.9.3. *If $\Delta_K(t) = 1$, then K is topologically slice.*

Here Δ is the Alexander polynomial. One method of proliferating knots with trivial Alexander polynomials is to take the *Whitehead double* $\mathrm{Wh}(K)$ of a knot K. In particular,

$$\Delta_{\mathrm{Wh}(T_{2,3})} = 1.$$

However, we can also compute the s-invariant for any given knot - this particular knot satisfies $s(\mathrm{Wh}(T_{2,3})) = 2$. Since $s/2$ is a lower bound for the slice genus, we know that $g_s(\mathrm{Wh}(T_{2,3})) \geq 1$. Therefore $\mathrm{Wh}(T_{2,3})$ is not slice, despite being topologically slice!

Remark. In fact, $\mathrm{Wh}(\overline{T_{2,3}}) = 0$. This is because the "clasp" in the Whitehead double is not mirrored, i.e. the Mirror of a Whitehead double is not the Whitehead double of a mirror. In general, it is known that all Whitehead doubles of torus knots are not slice, but such a result is not known for mirrors of torus knots!

The next ingredient in our proof of the existence of exotic smooth structures on $\mathbb{R}^4$ is the trace embedding lemma. This relates the properties of being slice (or topologically slice) to embeddings of "traces of 0 surgeries of knots".

We establish some notation. Let K be a knot, and $\mathbb{S}^3_n(K)$ the manifold obtained by n-surgery along $K \subset \mathbb{S}^3$. Let $X_n(K)$ be the manifold obtained from B^4 by attaching an n-framed 2-handle along K. Then $X_n(K)$ is called the *trace* of the n-surgery along K, and satisfies $\partial X_n(K) = \mathbb{S}^3_n(K)$. Alternatively $X_n(K)$ can be thought of a cobordism from $\mathbb{S}^3$ to $\mathbb{S}^3_0(K)$ (with the $\mathbb{S}^3$ end capped).

Example. If K is the unknot, then $\mathbb{S}^3_0(K) = \mathbb{S}^1 \times \mathbb{S}^2$, and $X_0(K) = (D^2 \times \mathbb{S}^2) - B^4$.

The *trace embedding lemma* takes two forms for each notion of sliceness:

Proposition 5.9.4. $K \subset \mathbb{S}^3$ is (topologically) slice if and only if $X_0(K)$ embeds smoothly (locally flat topologically) in $\mathbb{S}^4$.

We only prove the smooth case, as the locally flat case is similar.

Proof. $\Rightarrow$. If K is slice, it bounds a disk D smoothly embedded in B^4. One can verify that

$$\mathbb{S}^4 = X_0(K) \sqcup_{\mathbb{S}^3_0(K)} (B^4 - \mathrm{int}(\mathrm{nbhd}(D))).$$

In particular, $X_0(K)$ embeds smoothly in $\mathbb{S}^4$.

$\Leftarrow$. We start by constructing an embedding $F : \mathbb{S}^2 \to X_0(K)$, so that $F(\mathbb{S}^2)$ is of the form $D \sqcup_K C$ where D is a smooth disk (and the core of the 2-handle of $X_0(K)$) and C has a single cone point. By assumption, there is a smooth embedding $i : X_0(K) \to \mathbb{S}^4$. Therefore we have an embedding $i \circ F : \mathbb{S}^2 \to \mathbb{S}^4$ which is smooth away from the cone point. Removing a small ball around the cone point, the image of $i \circ F$ restricts to a smoothly embedded disk in B^4, whose boundary is K. $\square$

The final step is to combine this result with the previous example of a non-slice topologically slice knot to construct an exotic $\mathbb{R}^4$.

Theorem 5.9.5. *There exist exotic $\mathbb{R}^4$s.*

Proof. Let K be a topologically slice knot which is not slice. Write

$$\mathbb{S}^4 = X_0(K) \cup (B^4 - \mathrm{nbhd}(D))$$

where D is a topologically flat disk, with boundary K. Define

$$Z = \mathbb{S}^4 - \{x\} - \mathrm{int}(X_0(K)) = \mathbb{R}^4 - \mathrm{int}(X_0(K)).$$

This is an *open* topological 4-manifold with boundary. A theorem of Freedman states that all open 4-manifolds admit smooth structures, so we equip Z with a smooth structure. In particular ∂Z is a smooth manifold.

On the other hand, we already know that ∂Z is homeomorphic to $\partial X_0(K)$, which is homeomorphic to $\mathbb{S}^3_0(K)$. In dimension 3, all topological manifolds admit a unique smooth structure, so ∂Z is diffeomorphic to $\mathbb{S}^3_0(K)$. Now define

$$R = Z \sqcup_\varphi X_0(K)$$

where $\varphi : \partial_Z \to \mathbb{S}^3_0(K)$ is a diffeomorphism. This is a smooth manifold, and by Mayer-Vietoris and Seifert-van Kampen, can be shown to be homeomorphic to $\mathbb{R}^4$.

In particular, $X_0(K)$ embeds smoothly in R. Since K is *not* slice, $X_0(K)$ cannot embed smoothly in $\mathbb{R}^4$. Therefore the smooth structure on R must be distinct from that on $\mathbb{R}^4$. This completes the proof. $\square$

5.10 FGMW strategy to disprove SPC4

In the previous proof, it was crucial that Z was open. This is because Freedman's proof of the existence of smooth structures (on an arbitrary manifold) works everywhere except for a single point. Can we modify the approach to find exotic smooth structures of non-open manifolds? What about shedding light on the smooth Poincaré conjecture in dimension 4?

We now describe an equivalent formulation of the smooth Poincaré conjecture in 4 dimensions, and show how we can attempt to understand it using Khovanov homology as we did above.

Proposition 5.10.1. The smooth Poincaré conjecture in dimension 4 (SPC4) is equivalent to the statement that if W^4 is smooth with $\partial W = \mathbb{S}^3$ and W contractible, then W is diffeomorphic to B^4.

The equivalence is immediate. To get from $\mathbb{S}^4$ to W, simply remove a 4-ball, and to get from W to $\mathbb{S}^4$, glue along a 3-sphere (since we know that the 3-dimensional Poincaré conjecture holds).

The Freedman-Gompf-Morrison-Walker (FGMW) strategy for disproving the smooth Poincaré conjecture is as follows: find a knot K such that K bounds a smooth disk in some W contractible with $\partial W = \mathbb{S}^3$, $s(K) \neq 0$. Then K is not slice, so $W \neq B^4$. Thus W is an exotic B^4, which gives us an exotic $\mathbb{S}^4$.

Example. Earlier in the class we considered potential counter-examples to SPC4. They can be revisited here: can we find knots K as above in our potential SPC4 counter examples?

Example. Suppose W has a handle decomposition with no 3-handles. The attaching spheres of 2-handles are in fact knots in $\mathbb{S}^3$, and moreover bound smooth disks in W (specifically the cores of the handles). Therefore if any of these K have non-trivial s-invariant, we are done. So far all such K have had trivial s-invariant.

Remark. There are invariants similar to the s-invariant that arise from Seiberg-Witten and Yang-Mills gauge theory, along with Floer homology theories. However, none of these can distinguish between sliceness in B^4 vs sliceness in homotopy B^4s, so these cannot work in a similar strategy.

Whether or not this strategy has a chance of working is an open question. More precisely, the following problem is open:

Open question. Let $K \subset \mathbb{S}^3 = \partial W^4$. Suppose W is smooth and contractible. Suppose $\Sigma \hookrightarrow W$ is a smooth proper embedding, with $\partial\Sigma = K$. Do we necessarily have

$$|s(K)| \leq 2g(\Sigma)?$$

This is of course true if $W = B^4$. If it is true for all W as above, then the FGMW strategy fails.

Theorem 5.10.2 (Manolescu, Marengon, Sarkar, Willis)**.** *The inequality $|s(K)| \leq 2g(\Sigma)$ holds as above if W is a Gluck twist of a sphere.*

As a corollary, the FGMW strategy fails for Gluck twists. We prove this in the last lecture.

Recall the following definition of a Gluck twist:

Definition 5.10.3. Let X be a 4-manifold, and $\mathbb{S}^2 \to X$ an embedding with image S. Then there is a neighbourhood of S diffeomorphic to $\mathbb{S}^2 \times D^2$. Then the *Gluck twist* of X by S is

$$X_S = (X - \mathrm{nbhd}(S)) \sqcup_\varphi (\mathbb{S}^2 \times D^2)$$

where $\varphi : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ is the map

$$\varphi : (z, e^{i\theta}) \mapsto (\mathrm{rot}_\theta(z), e^{i\theta}).$$

Question from class. Can Khovanov homology prove that there are infinitely many smooth exotic structures on $\mathbb{R}^4$?

Answer. Yes. Consider distinct knots which are topologically slice but not slice. There is another invariant such that if the two knots are distinguished by this invariant, then they induce different smooth structures. This cannot prove that there are uncountably many exotic smooth structures however, since there are only countably many knots! □

5.11 The FGMW strategy fails for Gluck twists (lecture 19)

In the previous lecture we introduced the Freedman-Gompf-Morrison-Walker strategy to disprove the smooth Poincaré conjecture. It is an open question whether or not the strategy can be carried out. However, today we give a proof outline to show that the strategy fails for Gluck twists.

Theorem 5.11.1 (Theorem A - Manolescu, Marengon, Sarkar, Willis)**.** *Let $K \subset \mathbb{S}^3 = \partial W$, where W is obtained as a Gluck twist of B^4. Suppose $\Sigma \hookrightarrow W$ is a smooth embedding, with $\partial\Sigma = K$. Then $|s(K)| \leq 2g(\Sigma)$.*

This means that if W is a smooth manifold homeomorphic to B^4 obtained via a Gluck twist, and K is a knot bounding a disk in W, we cannot show that K is not slice (and hence W is not diffeomorphic to B^4) by using the s-invariant. In other words, the FGMW strategy fails for such W.

It is interesting that such a result can be proven, since we expect to only know information about cylinders $\mathbb{S}^3 \times [0,1]$ based on the definition of the Khovanov homology.

Recall that Gluck twist, in our context, is the following manifold: let $\Sigma \cong \mathbb{S}^2 \to B^4$ be an embedding. Then there is a neighbourhood N of Σ diffeomorphic to $\mathbb{S}^2 \times D^2$. The *Gluck twist* of B^4 by Σ is

$$W = B^4_\Sigma = (B^4 - N) \sqcup_\varphi N$$

where $\varphi : \partial N = \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ is the map

$$\varphi : (z, e^{i\theta}) \mapsto (\mathrm{rot}_\theta(z), e^{i\theta}).$$

It is known that a Gluck twist of B^4 is homeomorphic to B^4, but not if it is diffeomorphic.

Definition 5.11.2. G_Σ denotes the Gluck twist of $\mathbb{S}^4$ by an embedding $\Sigma \hookrightarrow \mathbb{S}^4$, with $\mathbb{S}^2 \cong \Sigma$. By the following remark, there is no ambiguity in writing G_Σ.

Remark. The diffeomorphism $\varphi \in \mathrm{Aut}(\partial N)$ is a generator of $\pi_1(\mathbb{RP}^3) = \pi_1(\mathrm{SO}(3)) = \{\mathbb{S}^1 \to \mathrm{rot}(\mathbb{S}^2)\} = \mathbb{Z}/2\mathbb{Z}$. If two maps in $\mathrm{Aut}(\partial N)$ are homotopic, they give the same Gluck twists.

The proof outline for the MMSW theorem is as follows:

1. Prove a special case with $W = \overline{\mathbb{CP}^2} - B^4$.

2. Prove a special case with $W = \mathbb{CP}^2 - B^4$.

3. Use Kirby diagrams to prove a result analogous to the "stable diffeomorphism" classification of 4-manifolds. Concretely, we show that $G_\Sigma \# \mathbb{CP}^2$ is diffeomorphic to $\mathbb{CP}^2$, and $G_\Sigma \# \overline{\mathbb{CP}^2}$ is diffeomorphic to $\overline{\mathbb{CP}^2}$.

4. We combine the three results to prove the general result.

We now state and prove the first special case:

Proposition 5.11.3. "Theorem B". Let $W = \overline{\mathbb{CP}^2} - B^4$, and $K \subset \partial W = \mathbb{S}^3$. Let $\Sigma \subset W$ be smoothly properly embedded, with $\partial\Sigma = K$. Suppose $[\Sigma] = 0 \in H_2(W, \partial W) = H_2(\overline{\mathbb{CP}^2}) = \mathbb{Z}$. Then $s(K) \leq 2g(\Sigma)$.

Proof. The goal is to reduce the problem further to a surface in a cylinder. In that case we obtain a map corresponding to the surface (as described in an earlier lecture).

Consider the data of $W = \overline{\mathbb{CP}^2} - B^4$, $K \subset \partial W = \mathbb{S}^3$, and $\Sigma \subset W$ smoothly properly embedded, with $\partial\Sigma = K$ and $[\Sigma] = 0 \in H_2(W, \partial W)$. Note that $H_2(W, \partial W)$ is generated by $[\overline{\mathbb{CP}^1}]$.

Let N be a regular neighbourhood of $\overline{\mathbb{CP}^1}$. Then $\partial N = \mathbb{S}^3$. Moreover, the "radial" projection $\partial N \to \overline{\mathbb{CP}^1} \cong \mathbb{S}^2$ is the (negative) Hopf fibration. Decomposing along the boundary of N, we then have

$$\overline{\mathbb{CP}^2} = N \sqcup_{\partial N} (\mathbb{S}^3 \times [0,1]) \sqcup_{\partial W} B^4.$$

We also assume that $[\Sigma] = 0 \in H_2(W, \partial W)$. Therefore $[\Sigma] \cdot [\overline{\mathbb{CP}^1}] = 0$. That is, assuming transversality, Σ and $\overline{\mathbb{CP}^1}$ intersect at $2p$ points, p positively signed and p negatively signed. Therefore Σ intersects N along $2p$ disks, and intersects ∂N along $2p$ circles. Each of these circles is a fibre of the negative Hopf fibration mentioned above.

The collection of fibres forms a link $L_{p,p} \subset \mathbb{S}^3$ in the total space of the Hopf fibration. In fact, this is a torus link $T_{2p,2p}$ with p strands oriented in one direction and p the other way.

One can define Rasmussen's s-invariant for *links* rather than just knots. Recall that $\dim \mathrm{Lee}(L) = 2^\ell$ where L has ℓ components, and $\mathrm{Kh}(L) \Rightarrow \mathrm{Lee}(L)$. This time there are many generators, but our link has a given orientation, so there exist canonical generators $S_{\mathcal{O}}$ and $S_{\overline{\mathcal{O}}}$. We can define the s-invariant to be

$$s(L) = \frac{q([S_{\mathcal{O}}] + [S_{\overline{\mathcal{O}}}]) + q([S_{\mathcal{O}}] - [S_{\overline{\mathcal{O}}}])}{2}.$$

By the definition of Σ, its restriction to $\mathbb{S}^3 \times [0,1]$ is a cobordism inside $\mathbb{S}^3 \times [0,1]$ from K to $L_{p,p}$, of genus $g(\Sigma)$. By functoriality of the Khovanov homology under cobordisms (as in Rasmussen's proof of the Milnor conjecture), we find that

$$s(K) - 2g(\Sigma) + 1 - 2p \leq s(L_{p,p}).$$

We can compute $s(L_{p,p})$. (This takes some work and is the main content of the paper by MMSW), but these turn out to be $1 - 2p$. Therefore the inequality above gives the desired result. □

It is now straight forward to prove the result for $\mathbb{CP}^2$ instead of $\overline{\mathbb{CP}^2}$. Explicitly, we have the following proposition:

Proposition 5.11.4. "Theorem C". Let $W = \mathbb{CP}^2 - B^4$, and $K \subset \partial W = \mathbb{S}^3$. Let $\Sigma \subset W$ be smoothly properly embedded, with $\partial\Sigma = K$. Suppose $[\Sigma] = 0 \in H_2(W, \partial W) = H_2(\mathbb{CP}^2) = \mathbb{Z}$. Then $-s(K) \leq 2g(\Sigma)$.

Proof. This follows from theorem B by working with the mirror of K. $\square$

The final ingredient for proving the general theorem (theorem A) is a result reminiscent of stable diffeomorphisms.

Proposition 5.11.5. For any $\Sigma \hookrightarrow \mathbb{S}^4$, $G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2$, and $G_\Sigma \# \overline{\mathbb{CP}^2} \cong \overline{\mathbb{CP}^2}$, where G_Σ is the Gluck twist of $\mathbb{S}^4$ by Σ.

Proof. The proof makes use of Kirby diagrams. Given $\Sigma \hookrightarrow \mathbb{S}^4$, we can write Kirby diagrams for $\mathbb{S}^4$ and G_Σ are as in figure 5.3 (where the component labelled with a 0 is a 2-handle determined by Σ). We now briefly explain the origins of these Kirby diagrams.

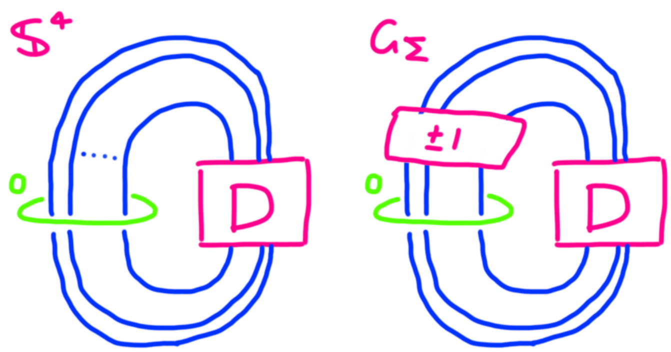


Figure 5.3: Kirby diagrams for $\mathbb{S}^4$ and G_Σ.

We can write

$$\mathbb{S}^4 = (\mathbb{S}^4 - N) \cup N, \quad G_\Sigma = (\mathbb{S}^4 - N) \sqcup_\varphi N,$$

where φ is the twisting map, and N is a regular neighbourhood of Σ. We now choose a Morse function $f : \mathbb{S}^4 \to \mathbb{R}$ such that $N = f^{-1}(-\infty, 0]$, and let $h : \mathbb{S}^2 \to \mathbb{R}$ be the standard height function. Next let $\pi : \mathbb{S}^2 \times D^2 \cong N \to \mathbb{S}^2$ be the usual projection map.

Finally we update f so that $f|_N$ is defined by

$$f|_N(x, z) = (h \circ \pi)(x, z) + |z|^2.$$

The Kirby diagram for $\mathbb{S}^4$ shown in figure 5.3 is with respect to this Morse function f, and applying a Gluck twist gives the diagram on the right.

Next we prove using Kirby calculus that

$$G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \mathbb{S}^4 \cong \mathbb{CP}^2.$$

We use the above diagrams, making only local changes at the 0-framed 2 handle shown in green. The proof is contained in figure 5.4. The proof of

$$G_\Sigma \# \overline{\mathbb{CP}^2} \cong \overline{\mathbb{CP}^2} \# \mathbb{S}^4 \cong \overline{\mathbb{CP}^2}$$

is similar, and not included. $\square$

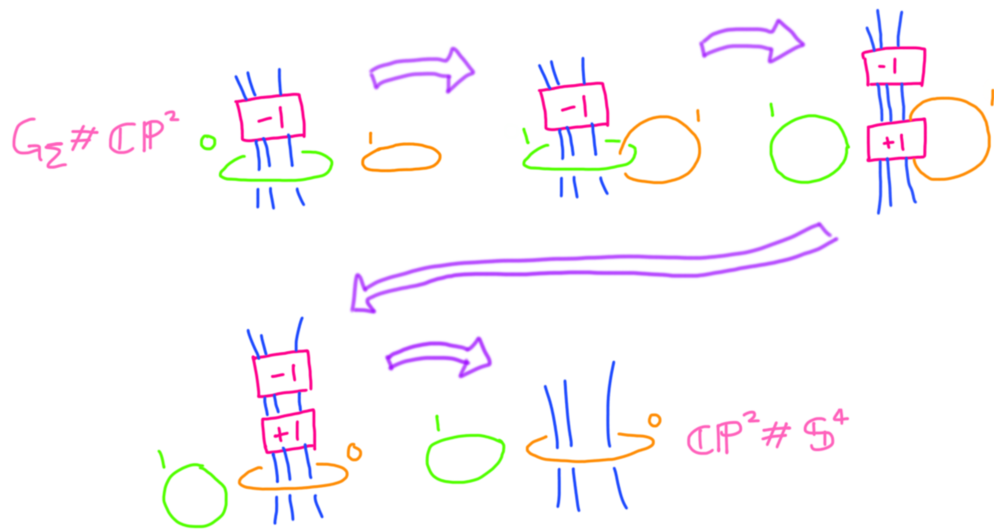


Figure 5.4: Proof that $G_\Sigma \# \mathbb{CP}^2 \cong \mathbb{CP}^2 \# \mathbb{S}^4$.

We now have all of the necessary ingredients to prove theorem A of MMSW (which we repeat here for clarity).

Theorem 5.11.6 (Theorem A - Manolescu, Marengon, Sarkar, Willis)**.** *Let $K \subset \mathbb{S}^3 = \partial W$, where W is obtained as a Gluck twist of B^4. Suppose $\Sigma \hookrightarrow W$ is a smooth embedding, with $\partial \Sigma = K$. Then $|s(K)| \leq 2g(\Sigma)$.*

Proof. Let W be a Gluck twist of B^4, and $\Sigma \subset W$ such that $\partial \Sigma = K$. For some surface S, we have $W = G_S - B^4$. By the above result, $G_S \# \mathbb{CP}^2 = \mathbb{CP}^2$, so in particular

$$W \# \mathbb{CP}^2 = \mathbb{CP}^2 - B^4.$$

By theorem C, it follows that

$$-s(K) \leq 2g(\Sigma).$$

Moreover, we also know that $G_S\#\overline{\mathbb{CP}^2} = \overline{\mathbb{CP}^2}$, from which it follows that $W\#\overline{\mathbb{CP}^2} = \overline{\mathbb{CP}^2} - B^4$, so by theorem B,

$$s(K) \leq 2g(\Sigma).$$

Combining these two results, we find that

$$|s(K)| \leq 2g(\Sigma)$$

as required. □

Is there any hope for the FGMW strategy? A modification of the strategy which might still work is the following result:

Proposition 5.11.7. Suppose K, K' are knots with $\mathbb{S}^3_0(K) \cong \mathbb{S}^3_0(K')$, but with K slice and K' not slice. Then SPC4 is *false.*

Proof. Recall that $\mathbb{S}^3_0(K)$ denotes the result of 0-surgery on K. The above result follows from the trace embedding lemma, which we saw in the previous lecture. Let $X_0(K)$ and $X_0(K')$ denote the traces of 0-surgery along K and K' respectively. Then $\partial X_0(K) = \partial X_0(K')$ as smooth manifolds. On one hand, we know that

$$\mathbb{S}^4 = X_0(K) \cup (B^4 - \mathrm{nbhd}(D)),$$

where the union glues along the boundary. Therefore we can replace $X_0(K)$ with $X_0(K')$, and consider

$$S' = X_0(K') \cup (B^4 - \mathrm{nbhd}(D)).$$

From Mayer-Vietoris, Seifert-van Kampen, and the topological Poincaré conjecture, one can show that S' is homeomorphic to $\mathbb{S}^4$. However, since K' is not slice, it cannot be diffeomorphic to $\mathbb{S}^4$ (by the trace embedding lemma). Therefore S' is an exotic $\mathbb{S}^4$, disproving SPC4. □

So far such K and K' have not been found, but there is also no evidence that they cannot be found!

5.12 Combinatorial proof of the Thom conjecture

In 2018, Lambert-Cole[1] proved the Thom conjecture using Khovanov homology techniques and no gauge theory. We give a very brief outline of the ingredients of the proof here. But first - let us recall the statement of the Thom conjecture.

[1]Originally I had erroneously written *Lambert and Cole* instead of *Lambert-Cole*, thinking it was work by two authors. Lambert-Cole himself emailed me to correct my mistake:
"Dear Mr Fushida
I came across your excellent lecture notes from Ciprian Manolescu's 4-manifold class. However I noticed an error (see attached). I'm sure this is simply a miscommunication between yourself and Mr Hardy, but Peter Lambert-Cole is in fact a single person.
Best, Peter Lambert-Cole"

Theorem 5.12.1 (Thom conjecture)**.** *Let $\Sigma \subset \mathbb{CP}^2$ be a smoothly embedded connected (real) surface, and suppose $[\Sigma] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}$. Then the genus of Σ is bounded below by $(d-1)(d-2)/2$.*

To prove this theorem using Khovanov homology, we require three main ingredients, as follows.

- Trisections. Let Σ as in the proposition be given. We choose a trisection of $\mathbb{CP}^2$, forwhich Σ is in a "bridge" position. That is, it arches over the three components of the trisection. In particular, each $\Sigma \cap Y_i$ (where Y_i is an interface of the trisection) does not intersect either of the other two Y_j.
- Contact geometry. We make each $\Sigma \cap Y_i$ transverse to the standard contact structure in $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$.
- Khovanov homology. We use the *slice-Bennequin inequality* for transverse knots, which can be proved using Khovanov homology. (In fact, this is equivalent to Milnor's conjecture!)

This completes the course! Thank you Ciprian for an amazing class.